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On three-dimensional packets of surface waves

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In this note we use the method of multiple scales to derive the *two* coupled nonlinear partial differential equations which describe the evolution of a three-dimensional wave-packet of wavenumber k on water of finite depth. The equations are used to study the stability of the uniform Stokes wavetrain to small disturbances whose length scale is large compared with $2\pi/k$. The stability criterion obtained is identical with that derived by Hayes under the more restrictive requirement that the disturbances are oblique plane waves in which the amplitude variation is much smaller than the phase variation.

1. INTRODUCTION

The evolution of progressive waves of slowly varying amplitude moving under gravity in water of finite depth has generated considerable interest in recent years. In particular, Whitham, in a series of papers beginning in 1965, has developed an attractive theory in which the motion is described in terms of a phase variable and an amplitude variable. The Whitham theory is actually applicable to a wide class of non-dissipative wave systems, especially those for which a Lagrangian is known, but in this paper we shall restrict attention to its use in the theory of water waves. Extensions of the theory have been made by Lighthill and by Hayes; the reader is referred to a recent paper by Hayes (1973) for a list of relevant articles on the Whitham theory. Perhaps the most notable success of the theory is in the study of the instability of the (uniform) Stokes wavetrain. If the depth is h and if the wavelength of this train is $2\pi/k$, Benjamin & Feir (1967) proved theoretically and Feir (1967) demonstrated experimentally that it is unstable if $kh > 1.363$ approximately. The Whitham theory implies that the evolution of a wave-packet is governed either by a hyperbolic or an elliptic equation and that the transition from the first kind to to the second occurs as kh increases through the value 1.363. When the governing equation is elliptic it appears that the mathematical problem of determining the motion is not well posed in that it is set as a Cauchy problem and the solution breaks down after a finite time.

There are a number of features of the Whitham theory which deserve attention. In the first place it is difficult to understand precisely what is meant by a slow variation. It is usual in studies of slowly varying phenomena to introduce a small parameter ϵ which in some sense is allowed to tend to zero and the notion of slow variation is made precise by reference to ϵ. No such parameter is explicitly defined in the

Whitham theory. Secondly, the appearance of ill-posed equations is rare and since the rigorous theory of such equations is in its infancy many students feel a certain unease in using their solutions widely, particularly as the leading term of some asymptotic expansion. Finally, the restriction to non-dissipative systems excludes many important fluid dynamical problems which involve bifurcation and transition.

An alternative way to study nonlinear wave-packets is to make use of the method of multiple scales wherein the small parameter ϵ is explicitly built in to the expansion scheme. This method has been used by a number of authors in various fields and has been applied to gravity waves by Hasimoto & Ono (1972); they also give a list of useful references. If we write the height ζ of the free surface above its undisturbed value in the form

$$\left.\begin{aligned} g\zeta &= \mathrm{i}\epsilon\omega A\,(\xi,\tau)\exp\{\mathrm{i}(kx-\omega t)\}+\text{c.c.}+O(\epsilon^2),\\ \text{where}\qquad \xi &= \epsilon(x-c_{\mathrm{g}}t),\quad \tau=\epsilon^2 t,\end{aligned}\right\} \tag{1.1}$$

x denotes distance in the direction of the wave motion, t is time, $\omega^2 = gk\tanh kh$, c_{g} is the velocity, g is the acceleration due to gravity, and c.c. denotes the complex conjugate, then they showed that A satisfies the nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial A}{\partial \tau}+\lambda\frac{\partial^2 A}{\partial \xi^2} = \nu|A|^2 A, \tag{1.2}$$

where λ, ν are known real functions of g, k and h. With the further substitution

$$A = R\exp\{\mathrm{i}\Theta\}, \tag{1.3}$$

where R and Θ are real functions, (1.2) is equivalent to the pair of real equations

$$\left.\begin{aligned} R\frac{\partial\Theta}{\partial\tau}+\lambda\left\{R\left(\frac{\partial\Theta}{\partial\xi}\right)^2-\frac{\partial^2R}{\partial\xi^2}\right\} &= -\nu R^3,\\ R\frac{\partial R}{\partial\tau}+\lambda\frac{\partial}{\partial\xi}\left(R^2\frac{\partial\Theta}{\partial\xi}\right) &= 0.\end{aligned}\right\} \tag{1.4}$$

The form of Whitham's equations for this problem may be obtained from (1.4) on making the assumption that the phase variations, although small, are much larger than the amplitude variations so that

$$R\left(\frac{\partial\Theta}{\partial\xi}\right)^2 \gg \frac{\partial^2R}{\partial\xi^2}. \tag{1.5}$$

Thus in a sense (1.2) includes Whitham's theory, but we must not exclude the possibility that it may have a wider application outside the range of validity of (1.2). Further (1.2) is a parabolic type of equation and as such the determination of A is a well-posed problem if A is given at $\tau = 0$ and at, for example, $|\xi| = \infty$ and these conditions are natural for the study of the evolution of wave-packets. Moreover the uniform wavetrain

$$A = R_0\exp\{\mathrm{i}\nu R_0^2\tau\} \tag{1.6}$$

may be obtained from (1.2) on assuming A to be independent of ξ. Hasimoto & Ono (1972) established that this solution is stable to relatively small disturbances only if $\lambda\nu > 0$, a condition which leads to $kh < 1.363$ and is the same as that found by Benjamin & Feir (1967). The reason why this more general equation gives the same stability criterion as the Whitham theory, in which the simplifying assumption (1.5) has been made, is that all disturbances of sufficiently long wavelengths are unstable and these include those for which (1.5) is valid.

Numerical studies by Karpman & Krushkal (1968) have established that when $\lambda\nu < 0$, the uniform wavetrain breaks up into a number of solitons and an oscillatory tail and completely loses its original structure. The solution does not terminate at a finite value of τ, as appears to be the case in the Whitham formulation, the breakdown being prevented by the extra term $\partial^2 R/\partial\xi^2$ in (1.4). Hayes (1973) refers to this term as a diffraction term and suggests that it 'may be made as small as desired through a suitable scale transformation'. In our opinion this is too narrow a view. Since the Whitham equations are correct in the sense in which they are formulated it is certainly possible to consider evolutionary systems in which the diffraction term might be removed by a suitable scale transformation. It does not, however, follow that such a transformation enables us to remove this term in general, or, for example, in all studies for which (1.2) is relevant, for the *initial* conditions may not permit such a transformation to be made.

We observe also that Stewartson & Stuart (1971), using the method of multiple scales, have developed a theory for the evolution of small two-dimensional disturbances in marginally unstable plane Poiseuille flow, where dissipative effects are important. Their governing equation is very similar to (1.2). The differences are that λ, ν are now complex and there is an additional term proportional to A to represent the linear growth of A due to the marginal instability. Perhaps it may prove possible to generalize the Whitham theory to include dissipation, but at present it does not seem obvious how to do this even when nonlinear effects are neglected. (See note added in proof on p. 110.)

An important result obtained by Hayes (1973) is the stability criterion for the Stokes wavetrain in three dimensions when subject to small disturbances consisting of oblique plane waves. Our aim in this paper is to develop a theory for three-dimensional wave-packets parallel to that of Hasimoto & Ono (1972). We shall show that for the three-dimensional problem *two* partial differential equations are needed to describe the motion, but that when the three-dimensionality is in the form of an oblique plane wave the two equations can be converted into a single equation similar to (1.2), so that the stability criterion follows very easily. Further it is shown that the Stokes wavetrain is stable to all small disturbances if it is stable to oblique plane waves.

2. Multiple scale derivation of the evolutionary equations

We choose a fixed Cartesian system of coordinates $Oxyz$, with origin O in the undisturbed free surface of the water and Oz pointing vertically upwards so that the bed of the water is defined by $z = -h$ and the plane Oxy coincides with the undisturbed free surface. We suppose that at time $t = 0$ a progressive wave is established such that the elevation of the free surface is raised to $z = \zeta$, where

$$g\zeta|_{t=0} = \mathrm{i}\epsilon\omega a(\epsilon x, \epsilon y)\exp\{\mathrm{i}kx\} + \text{c.c.} \tag{2.1}$$

In (2.1) g is the acceleration due to gravity, k and ω denote the wavenumber and frequency of the progressive wave respectively, a is a given function of $\epsilon x, \epsilon y$ and ϵ is a small positive constant. In physical terms this form corresponds to a progressive wave of wavelength $2\pi/k$ travelling in the direction of x increasing and with an amplitude slowly varying with position and on a scale inversely proportional to its height. The dispersion relation between k and ω is

$$\omega = (gk\sigma)^{\frac{1}{2}}, \tag{2.2}$$

where $\sigma = \tanh kh$.

At subsequent times, measured by t, let the velocity potential be $\phi(x, y, z, t)$, so that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \quad \text{in } -h < z < \zeta. \tag{2.3}$$

The corresponding boundary conditions are

$$\frac{\partial\phi}{\partial z} = 0, \quad \text{when} \quad z = -h, \tag{2.4}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\zeta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}, \quad \text{when } z = \zeta, \tag{2.5a}$$

and

$$2g\zeta + 2\frac{\partial\phi}{\partial t} + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 = 0, \quad \text{when} \quad z = \zeta. \tag{2.5b}$$

Since the disturbance is assumed to be a progressive wave we can look for a solution of (2.3)–(2.5) of the form

$$\phi = \sum_{n=-\infty}^{\infty} \phi_n E^n, \quad \zeta = \sum_{n=-\infty}^{\infty} \zeta_n E^n, \tag{2.6}$$

where

$$E = \exp\{\mathrm{i}(kx - \omega t)\}, \quad \phi_{-n} = \tilde{\phi}_n, \quad \zeta_{-n} = \tilde{\zeta}_n, \tag{2.7}$$

and a tilde denotes the complex conjugate. Further, we may write

$$\phi_n = \sum_{j=n}^{\infty} \epsilon^j \phi_{nj}, \quad \zeta_n = \sum_{j=n}^{\infty} \epsilon^j \zeta_{nj}, \quad (n \geqslant 0), \tag{2.8}$$

where ϕ_{nj} is a function of ξ, η, z, τ only, ζ_{nj} is a function of ξ, η, τ only and $\phi_{00} = \eta_{00} = 0$. Here

$$\xi = \epsilon(x - c_g t), \quad \eta = \epsilon y, \quad \tau = \epsilon^2 t, \tag{2.9}$$

and c_g is the group velocity of the primary progressive wave so that

$$c_g = \omega'(k) = (g/2\omega)\{\sigma + kh(1-\sigma^2)\}. \tag{2.10}$$

We substitute the expansions (2.6), (2.8) for ϕ into the partial differential equation (2.3) and use the method of multiple scales to obtain a series of ordinary differential equations for the functions ϕ_{nj}, and when the boundary condition (2.4) is satisfied we find that, in particular,

$$\left.\begin{aligned} \phi_{11} = A\frac{\cosh k(z+h)}{\cosh kh}, \quad \phi_{22} = F\frac{\cosh 2k(z+h)}{\cosh 2kh}, \\ \phi_{12} = D\frac{\cosh k(z+h)}{\cosh kh} - \mathrm{i}\frac{\partial A}{\partial \xi}\left\{\frac{(z+h)\sinh k(z+h) - h\sigma\cosh k(z+h)}{\cosh kh}\right\}, \end{aligned}\right\} \tag{2.11}$$

where A, D, F are functions of ξ, η, τ only. The solutions of the equations for the ϕ_{0j} functions are of especial interest for it emerges that their properties are strongly dependent on the value of ϵkh. We shall restrict attention here to the case $\epsilon kh \ll 1$, sometimes referred to in the literature as the case when 'the wave feels the bottom'. (The wave motion when this restriction is removed has been considered by J. M. Gardiner, who will report on his work elsewhere.) It follows that ϕ_{01}, ϕ_{02} are independent of z, while

$$\frac{\partial \phi_{03}}{\partial z} = -(z+h)\left\{\frac{\partial^2 \phi_{01}}{\partial \xi^2} + \frac{\partial^2 \phi_{01}}{\partial \eta^2}\right\}. \tag{2.12}$$

The next step is the laborious one of substituting (2.6)–(2.8) into the boundary conditions (2.5*a*), (2.5*b*), using the method of multiple scales, and equating coefficients of $\epsilon^j E^n$ to zero, seriatim, for $j = 1, 2, 3$ and $n = 0, 1, 2$. The procedure closely follows that of Hasimoto & Ono (1972) and so we shall omit the details. The results are:

$$\epsilon E^0; \qquad \zeta_{01} = 0, \tag{2.13a}$$

$$\epsilon E^1; \qquad g\zeta_{11} = \mathrm{i}\omega A, \tag{2.13b}$$

$$\epsilon^2 E^0; \qquad g\zeta_{02} = c_g\frac{\partial \phi_{01}}{\partial \xi} - k^2(1-\sigma^2)|A|^2, \tag{2.13c}$$

$$\epsilon^2 E^1; \qquad g\zeta_{12} = \mathrm{i}\omega D + c_g\frac{\partial A}{\partial \xi}, \tag{2.13d}$$

$$\epsilon^2 E^2; \quad g\zeta_{22} = k^2 A^2\left(\frac{\sigma^2 - 3}{2\sigma^2}\right), \quad \omega F = 3\mathrm{i}k^2 A^2\left(\frac{1-\sigma^4}{4\sigma^2}\right). \tag{2.13e}$$

When we consider the coefficient of $\epsilon^3 E^0$ in (2.5*a*) we have, in addition to the contribution from differentiating ζ_{02} with respect to ξ, associated with $\partial\zeta/\partial t$, a contribution from $\partial\phi_{03}/\partial z$, which is non-zero from (2.12). It then follows, on using (2.13c) to eliminate ζ_{02}, that

$$(gh-c_g^2)\frac{\partial^2\phi_{01}}{\partial\xi^2}+gh\frac{\partial^2\phi_{01}}{\partial\eta^2} = -k^2\{2c_p+c_g(1-\sigma^2)\}\frac{\partial|A|^2}{\partial\xi}, \tag{2.14}$$

together with an equation for ϕ_{03} which we shall not need; in (2.14) $c_p = \omega/k$ denotes the phase speed of the primary wave.

The result of equating the coefficients of ϵ^3E^1 in (2.5a), (2.5b) yields two algebraic equations for ϕ_{13} and ζ_{13} at $z = 0$. If we eliminate ζ_{13} (or ϕ_{13}) from these two equations we find that they are only compatible if

$$\begin{aligned}2\mathrm{i}\omega\frac{\partial A}{\partial\tau}-\{c_g^2-gh(1-\sigma^2)(1-kh\sigma)\}\frac{\partial^2A}{\partial\xi^2}+c_pc_g\frac{\partial^2A}{\partial\eta^2}\\ = \tfrac{1}{2}k^4\{9\sigma^{-2}-12+13\sigma^2-2\sigma^4\}|A|^2A+k^2\{2c_p+c_g(1-\sigma^2)\}A\frac{\partial\phi_{01}}{\partial\xi}.\end{aligned} \tag{2.15}$$

Equations (2.14), (2.15) together describe the evolution of the progressive wave, to first order in ϵ. The appropriate initial condition on A is that

$$A(\xi,\eta,0) = a(\xi,\eta). \tag{2.16}$$

On physical grounds a reasonable boundary condition is that, for any fixed τ, the wave completely dies away sufficiently far from its centre so that

$$|A|\to 0, \quad \mathbf{grad}\,\phi_{01}\to\mathbf{0} \quad \text{as} \quad \xi^2+\eta^2\to\infty. \tag{2.17}$$

In the deep-water limit $kh\to\infty$ (but preserving $\epsilon kh \ll 1$) $\sigma\to 1$, and equations (2.14), (2.15) simplify to $\mathbf{grad}\,\phi_{01} = \mathbf{0}$ and

$$2\mathrm{i}\omega\frac{\partial A}{\partial\tau}-\frac{g}{4k}\frac{\partial^2A}{\partial\xi^2}+\frac{g}{2k}\frac{\partial^2A}{\partial\eta^2} = 4k^4|A|^2A. \tag{2.18}$$

In the shallow-water limit $kh\to 0$, $c_g\to c_p$ and equations (2.14), (2.15) simplify to

$$k^2h^2\frac{\partial^2\phi_{01}}{\partial\xi^2}+\frac{\partial^2\phi_{01}}{\partial\eta^2} = -\frac{3k^2}{g^{\frac{1}{2}}h^{\frac{1}{2}}}\frac{\partial|A|^2}{\partial\xi}, \tag{2.19}$$

and

$$\frac{2\mathrm{i}k}{g^{\frac{1}{2}}h^{\frac{1}{2}}}\frac{\partial A}{\partial\tau}-k^2h^2\frac{\partial^2A}{\partial\xi^2}+\frac{\partial^2A}{\partial\eta^2} = \frac{9k^2}{2gh^3}|A|^2A+\frac{3k^2}{g^{\frac{1}{2}}h^{\frac{1}{2}}}A\frac{\partial\phi_{01}}{\partial\xi}, \tag{2.20}$$

although difficulties may now arise due to the non-uniformity of the approach of the asymptotic expansions for ϕ, ζ to the double limit $\epsilon\to 0$, $kh\to 0$. At some stage in the limit process $kh\to 0$ extra terms may enter, depending on the magnitude of ϵ, and the governing equations change to a form of the Korteweg–de Vries equation. We note that g, k and h may all be formally removed from (2.19), (2.20), for example by using the transformation

$$\xi = kh\xi^*, \quad \tau = kg^{-\frac{1}{2}}h^{-\frac{1}{2}}\tau^*, \quad A = k^{-1}g^{\frac{1}{2}}h^{\frac{3}{2}}A^*, \quad \phi_{01} = k^{-1}g^{\frac{1}{2}}h^{\frac{3}{2}}\phi_{01}^*.$$

Moreover if A and ϕ_{01} are independent of η then (2.19), (2.20) reduce to

$$\frac{2ik}{g^{\frac{1}{2}}h^{\frac{1}{2}}}\frac{\partial A}{\partial \tau}-k^2h^2\frac{\partial^2 A}{\partial \xi^2}=\frac{-9k^2}{2gh^3}|A|^2A. \tag{2.21}$$

Hasimoto & Ono (1972) have pointed out that the nonlinear plane wave solution of (2.21) corresponds to the weak cnoidal wave solution of the Korteweg–de Vries equation.

At first sight the form of the equations for A and ϕ_{01} look rather different from those obtained by Hasimoto & Ono (1972) for the two-dimensional problem but it is possible to write (2.14), (2.15) in an alternative way which makes the connexion obvious. The height of the free surface for $t > 0$ is given by

$$g\zeta = \mathrm{i}\epsilon\omega A \exp\{\mathrm{i}(kx-\omega t)\}+\text{c.c}+O(\epsilon^2), \tag{2.22}$$

and as a consequence of the passage of the progessive wave, the local height of the free surface varies slowly, in addition to the more rapid variation characterized by (2.22). Let this secular variation be

$$\epsilon^2\left[\frac{k^2}{g}Q(\xi,\eta,\tau)-\frac{k\{\sigma+2kh(1-\sigma^2)\}}{gh-c_g^2}|A|^2\right] \tag{2.23}$$

and we note in passing that it is also equal to $\epsilon^2\zeta_{02}$. Then equivalent forms for (2.14), (2.15) are

$$\mathrm{i}\frac{\partial A}{\partial \tau}+\lambda\frac{\partial^2 A}{\partial \xi^2}+\mu\frac{\partial^2 A}{\partial \eta^2}=\nu|A|^2A+\nu_1 AQ, \tag{2.24a}$$

$$\lambda_1\frac{\partial^2 Q}{\partial \xi^2}+\mu_1\frac{\partial^2 Q}{\partial \eta^2}=\kappa_1\frac{\partial^2|A|^2}{\partial \eta^2}, \tag{2.24b}$$

where

$$\left.\begin{gathered}\lambda=\tfrac{1}{2}\omega''(k)\leqslant 0,\quad \mu=\frac{\omega'(k)}{2k}\equiv\frac{c_g}{2k}\geqslant 0,\\ \nu=\frac{k^4}{4\omega\sigma^2}\left[9-10\sigma^2+9\sigma^4-\frac{2\sigma^2}{gh-c_g^2}\{4c_p^2+4c_pc_g(1-\sigma^2)+gh(1-\sigma^2)^2\}\right],\\ \nu_1=\frac{k^4}{c_g}\{2c_p+c_g(1-\sigma^2)\},\quad \lambda_1=gh-c_g^2\geqslant 0,\quad \mu_1=gh\\ \text{and}\qquad \kappa_1=ghc_g\left\{\frac{2c_p+c_g(1-\sigma^2)}{gh-c_g^2}\right\}.\end{gathered}\right\} \tag{2.25}$$

The principal equation (4.5) of Hasimoto & Ono (1972) is now recovered on assuming A to be independent of η and putting $Q = 0$.

3. Stability of a uniform wavetrain

The simplest solution of the equations for A and Q is the Stokes wavetrain in which A is a function of τ only, of constant modulus A_0. The boundary conditions (2.17) as $\xi^2+\eta^2\to\infty$ must now be relaxed since the train is unmodulated and the solution is not unique since Q can take on any constant value Q_0. The solution is

$$A = A_0\exp\{\mathrm{i}p\tau\},\quad Q=Q_0,\quad \text{where}\quad p=-\nu A_0^2-\nu_1 Q_0. \tag{3.1}$$

Associated with such a wavetrain is a change in the height of the mean free surface equal to

$$\epsilon^2\left[\frac{k^2}{g}Q_0-\frac{k\{\sigma+2kh(1-\sigma^2)\}}{gh-c_g^2}A_0^2\right] \tag{3.2}$$

and a drift speed

$$\epsilon^2\left[\frac{k^2}{c_g}Q_0-\frac{k^2\{2c_p+c_g(1-\sigma^2)\}}{gh-c_g^2}A_0^2\right]. \tag{3.3}$$

When considering the stability of (3.1), however, the choice of the value of Q_0 is immaterial as the stability criterion is independent of Q_0. For on writing

$$A = A_0\hat{A}(\xi,\eta,\tau)\exp\{\mathrm{i}p\tau\},\quad Q = Q_0+\hat{Q}, \tag{3.4}$$

we find that $\hat{A}$ and $\hat{Q}$ satisfy

$$\left.\begin{aligned} &\mathrm{i}\frac{\partial\hat{A}}{\partial\tau}+\lambda\frac{\partial^2\hat{A}}{\partial\xi^2}+\mu\frac{\partial^2\hat{A}}{\partial\eta^2} = \nu A_0^2(|\hat{A}|^2-1)\,\hat{A}+\nu_1\hat{A}\hat{Q},\\ \text{and}\qquad &\lambda_1\frac{\partial^2\hat{Q}}{\partial\xi^2}+\mu_1\frac{\partial^2\hat{Q}}{\partial\eta^2} = \kappa_1A_0^2\frac{\partial^2|\hat{A}|^2}{\partial\eta^2}.\end{aligned}\right\} \tag{3.5}$$

Now suppose that $\hat{A}-1$ and $\hat{Q}$ are sufficiently small so that squares and products of these quantities may be neglected. Then $\hat{A}-1$ and $\hat{Q}$ satisfy a pair of linear differential equations which may be solved by taking a Fourier transform with respect to ξ and η. Thus the stability problem for (3.1) may be reduced to setting

$$\hat{A}-1 = \exp\{\mathrm{j}(l\xi+m\eta)\}P(\tau),\quad \hat{Q} = \exp\{\mathrm{j}(l\xi+m\eta)\}S(\tau), \tag{3.6}$$

where P, S are functions of τ only and examining whether the ordinary differential equations satisfied by P, S have bounded solutions for all τ when l, m are arbitrary real numbers. Here $\mathrm{j} = \sqrt{-1}$ and is distinguished from i because the complex conjugate of $\hat{A}$, needed in (3.5), is obtained from $\hat{A}$ by changing the sign of i but it is still thought of as a function of the type (3.6).

An equivalent procedure is to assume that A, Q are functions of $\chi = l\xi+m\eta$ and τ only when (2.24) yield

$$Q = \frac{\kappa_1m^2}{\lambda_1l^2+\mu_1m^2}|A|^2+Q_1, \tag{3.7}$$

and

$$\mathrm{i}\frac{\partial A}{\partial\tau}+(\lambda l^2+\mu m^2)\frac{\partial^2A}{\partial\chi^2} = \left(\nu+\frac{\nu_1\kappa_1m^2}{\lambda_1l^2+\mu_1m^2}\right)|A|^2A+\nu_1AQ_1, \tag{3.8}$$

where Q_1 is a real function τ only. We note that Q_1 may be eliminated from (3.8) by an appropriate frequency shift in A. The stability of the solution of (3.8) in which A is a function of τ only, is subject to the same criterion as that found by Hasimoto & Ono (1972), namely

$$(\lambda l^2+\mu m^2)\left(\nu+\frac{\nu_1\kappa_1m^2}{\lambda_1l^2+\mu_1m^2}\right) > 0, \tag{3.9}$$

and this criterion is identical with the one obtained by Hayes (1973), who used the Whitham formulation. Hayes discusses the implication of (3.9) in some detail referring to the curve in l, m space on which the first factor vanishes as the dispersion neutral curve and the curve on which the second factor vanishes as the hardness neutral curve. He points out that except when $kh = 0$ and $kh = 0.380$ it is always possible to choose l, m so that (3.9) is violated. Strictly therefore the Stokes wave-train is unstable for all other values of kh but Hayes notes that the two curves are so close together for $0 \leqslant kh < 0.5$ that 'the predicted instability is non-existent practically'.

The reason for the two approaches to the stability problem leading to the same conclusion is as explained in the introduction for two-dimensional disturbances. The unstable modes include those with very long wavelengths for which the governing equations reduce to the Whitham formulation, the amplitude variation in space being much smaller than the phase variation. The present theory makes two new contributions to this problem. First, it establishes that if the Stokes wave-train is stable to disturbances whose length scale is large compared with the inverse amplitude of the wave (taking the wavelength of the train as the unit of length) then it is also stable to disturbances whose length scale is of the same order as the inverse amplitude. Secondly, it establishes that if the train is stable to plane-wave disturbances then it is stable to all disturbances including centred disturbances.

4. Discussion

In this paper we have presented an alternative approach to the problem of the evolution of surface waves of slowly varying amplitude by using the method of multiple scales. The principal results which we have obtained are in agreement with those found using an approach based on Whitham's ideas, when the flow properties are such that both theories are relevant, but we claim that our approach has a number of advantages even though it explicitly lacks the unifying features of the Lagrangian, which can be such a powerful aid to obtaining evolutionary properties. These advantages are, first, the explicit appearance of a small parameter ϵ, which can be used to set up an asymptotic expansion of the solution, secondly, the leading term satisfies a well-posed differential equation, and thirdly, the generalization to other systems which may include dissipation is immediate. As an example of a typical problem in which dissipation is important, and which may be readily studied by use of the method of multiple scales, the reader is referred to Davey, Hocking & Stewartson (1974) for an account of the evolution of three-dimensional disturbances in marginally unstable plane Poiseuille flow. For this problem they find, incidentally, that the evolution may be described by two coupled nonlinear partial differential equations similar to (2.14), (2.15) except that it is necessary to add a term proportional to A to (2.15), and to change the coefficients of all the terms in (2.15) to *complex* constants of known value. It is not apparent as to how the Whitham theory could be used, or even a modified form thereof, to consider this problem.

Note added in proof, 4 *March* 1974.

Whitham (1970) has discussed the equivalence of his variational method with a particular multiple scale theory for dissipationless systems. The scaling he considered ($X = \epsilon^2 x, T = \epsilon^2 t$) is, however, different from and coarser than that adopted here. It is the same scaling as used by Stewartson & Stuart (1971) in the *first* stage of their theory which led to a first order equation for A. In the second stage they used the finer scaling of (1.1) which led them to the dissipative equivalent of (1.2).

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