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The Spirited Horse, the Engineer, and the Mathematician: Water Waves in Nineteenth-Century Hydrodynamics

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Once upon a time, a spirited horse was dragging a boat along a peaceful canal of Northern Scotland. Suddenly, a ferocious dog charged from the bushes. The horse took fright and ran off, drawing the boat after him. To the horse's surprise, the boat offered almost no resistance at this high speed. A clever naval engineer, who chanced to witness the scene, pursued the matter through numerous experiments and confirmed the horse's discovery. Not knowing his limits, the engineer ventured to propose a mechanical explanation of this paradox based on the existence of a wonderful, solitary wave that carried the boat with it. A renowned mathematician, who excelled in the learned calculus of water motion, mocked this amateurish attempt. There was nothing in the engineer's equations to justify his reasoning, and much to condemn it. Yet for a few years Scottish canal travelers enjoyed the commercial exploitation of the paradox. Half a century later, the aged mathematician resumed his calculations. Thanks to his long experience, he now saw new meanings in his old symbols. From this enriched analysis the solitary wave and the possibility of vanishing ship resistance emerged as if by magic. At last, the wise man rejoiced, mathematics could do as well as a galloping horse.

This fable is an imaginary simplification of a real story of which the engineer John Scott Russell, and the mathematicians (in a broad sense) George Biddell Airy, George Gabriel Stokes, Joseph Boussinesq, and Lord Kelvin were the main actors (besides the spirited horse). It is intended to indicate a major nineteenth-century transformation of the mathematical physicist's tool kit through which practically important solutions of long-known equations became much more easily accessible. The main purpose of the present article is to analyze the nature and the water-wave circumstances of this transformation.

Waves on the surface of water were an obvious field of application of the new hydrodynamics of Jean le Rond d'Alembert, Leonhard Euler, and Joseph Louis Lagrange. The latter mathematician himself wrote the basic equations of water waves, and solved them in the simplest case of small waves on shallow water. Most of what is today known on water waves was found in the nineteenth century: the celerity of small, plane, monochromatic waves on water of constant depth, the pattern of waves created by a local action on the water surface, the shape of oscillatory or solitary waves of finite size, the effect of friction, wind, and a variable bottom on the size and shape of the waves. There is, however, a puzzling contrast between the conciseness and ease of the modern treatment of these topics, and the long, difficult struggles of nineteenth century physicists with them. A modern reader of Poisson's old memoir on waves, for example, finds a bewildering accumulation of complex calculations where he would expect some rather elementary analysis.

The reason for this estrangement is not any weakness of early nineteenth-century mathematicians, but our overestimation of the physico-mathematical tools that were available in their times. It would seem for instance, that all that Siméon Denis Poisson needed to solve his particular wave problem was Fourier analysis, which Joseph Fourier had given a few years earlier. In reality, Poisson only knew a raw, algebraic version of Fourier analysis, whereas modern physicists have unconsciously assimilated a physically "dressed" Fourier analysis, replete with metaphors and intuitions borrowed from the concrete wave phenomena of optics, acoustics, and hydrodynamics. In our mind, a Fourier component is no longer a mere coefficient in an algebraic development, it is a periodic wave that may interfere with other waves in a manner we easily imagine. As we will see, the transition from a dryly mathematical analysis to a genuinely physico-mathematical analysis occurred gradually in the nineteenth century, through reversible analogies between different domains of physics. It concerned not only Fourier analysis, but also the theory of ordinary differential equations, potential theory, perturbative methods, Cauchy's method of residues, etc. The modern recourse to such mathematical techniques involves a great deal of implicit knowledge that only becomes apparent in comparisons with older usage.

As the above fable suggests, the motivation for the introduction of more powerful tools of analysis was mainly experimental. Most water-wave phenomena were known well before they could be explained. The existence of some of them, especially Russell's solitary waves, challenged contemporary mathematics. In most cases, they were discovered in connection with sailing problems. Not surprisingly, the wave theorists after Poisson and Cauchy shared an interest in the rational development of navigation. Waves were relevant to several aspects of this science: tide prediction, ship rolling, ship resistance, harbor safety, the wearing of canals, etc. British natural philosophers such as Airy, Stokes, Rayleigh, and Horace Lamb were evidently more concerned with these questions than their continental counterparts. They did much to bring the theory of water waves to the service of sea and canal fare. There were, however, a few important French contributions in the wake of the engineer-mathematician Adhémar Barré de Saint-Venant.[1]

The first section of this paper is devoted to the theories of waves developed between 1875 and 1925 by four French mathematicians, Laplace, Lagrange, Poisson, and Cauchy, mostly for the sake of mathematics, on the basis of the new hydrodynamics. The second section is devoted to Scott Russell's many instructive experiments on waves of various kinds, including his now famous and then infamous solitary wave, in the context of British-Association sponsored researches on ship-design. The third section presents Airy's wave theory of tides and his critical analysis of Russell's results. The fourth section deals with the problem of finite waves of permanent shape, as studied by Stokes, Boussinesq, and Rayleigh. It also includes Boussinesq's treatment of the evolution of an

[1] Saint-Venant [1888] also gave the most competent and thorough history of the water-wave problem to date.

arbitrary swell, through which he arrived (in 1877) at the equation which is now attributed to Korteweg and De Vries (1895). The fifth and last section is about the application of optical or acoustic ideas of interference to the explanation of water-wave phenomena. Thanks to such intuitions, Stokes, Rayleigh, and Froude forged the concept of group velocity; Rayleigh solved the problem of waves created by a drifting fishing line; and Kelvin computed the pattern of ship waves, thereby inventing the celebrated method of stationary phase.

Two topics that would naturally fit in this paper, the production of waves through wind and their breaking process, are left aside because they would not alter the more general conclusions and because they have been discussed in two earlier papers of the author.[2] Vector notation is used anachronistically as an abbreviation for systems of Cartesian components. In conformity with nineteenth-century usage, the word "wave" refers to a ridge or swell moving along the surface of a liquid, whereas in modern physics this word tends to further imply an undulating shape of the water surface.

1. French mathematicians

Laplace's attempt

In 1775–76, Pierre-Simon de Laplace published his celebrated theory of tides, based on the hydrodynamics of Jean le Rond d'Alembert. Laplace represented the oceans as a layer of perfect liquid of variable depth on a uniformly rotating spheroid, submitted to the variable attraction of the moon and the sun. Applying d'Alembert's principle of dynamics to the fluid particles, and neglecting the vertical acceleration of the water as well as any quantity of second order with respect to the fluid velocity, he obtained the fundamental equations of tidal motion. As will appear in a moment, the former approximation requires the depth of the water to be small compared to the length over which the tidal elevation varies sensibly; the latter approximation requires the tidal elevation to be much smaller than the depth.[3]

For a modern reader, it is obvious that Laplace's equations are those for the propagation of small waves in shallow water, with an additional term corresponding to the Coriolis force and an external force density corresponding to the lunar and solar perturbations. Laplace could not state so much, since at that time the theory of water waves remained to be done. He did realize, however, that his derivation of the tidal equations opened the road to the simpler problem of the propagation of small disturbances in a large pond of uniform depth. Laplace knew Isaac's Newton analogy between water waves and the oscillations of a fluid in an inverted U-shaped tube, which gave a propagation velocity proportional to the square root of the length of the wave. But he judged it "very uncertain." His own theory [1776] rested on well-established mechanical principles.

[2] Darrigol 1998, pp. 46–51 (on Helmholtz and wind waves); Darrigol 2002c, pp. 15–24 (on the Kelvin-Helmholtz instability).

[3] Cf. Cartwright 1899, chap. 6.

Laplace focused on free propagation, which only occurs if the cause of the wave is localized in space and time. The obvious example is a stone thrown into a pond. In order to ease calculation, Laplace considered a narrow canal instead of a pond, and the emersion of a solid body instead of the impact [p. 302]:

> The simplest manner to conceive the formation of waves is to imagine an arbitrary curve, dipped into the fluid to a very small depth and held in this state until all the fluid is in equilibrium; when this curve is thereafter withdrawn from the canal, it is clear that the fluid will tend to retrieve its equilibrium state by forming successive waves.

Laplace then used the so-called Lagrangian picture, in which the fluid motion is described by giving the position $(X + \xi, Y + \eta)$ of a particle of the fluid at time t as a function of its position (X, Y) at the origin of time (the moment when the curve is withdrawn). To first order in ξ and η, the incompressibility of water implies the continuity equation

$$\frac{\partial \xi}{\partial X} + \frac{\partial \eta}{\partial Y} = 0. \tag{1}$$

According to d'Alembert's principle of dynamics, the work of the sum of inertial, gravitational, and pressure forces during a virtual displacement $\mathrm{d}(X+\xi, Y+\eta)$ of the position of a fluid particle at a given time must vanish. Taking the ordinate axis to be vertical and directed upwards, this gives

$$\frac{\partial^2 \xi}{\partial t^2}\,\mathrm{d}(X+\xi) + \frac{\partial^2 \eta}{\partial t^2}\,\mathrm{d}(Y+\eta) + g\mathrm{d}(Y+\eta) + \frac{\mathrm{d}P}{\rho} = 0, \tag{2}$$

where g is the acceleration of gravity, ρ the density of water, and P its pressure. To first order, this equation makes $(\partial^2\xi/\partial t^2)\,\mathrm{d}X + (\partial^2\eta/\partial t^2)\,\mathrm{d}Y$ an exact differential, so that

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial \xi}{\partial Y} - \frac{\partial \eta}{\partial X}\right) = 0. \tag{3}$$

As the parenthesis and its first time derivative vanish identically for $t = 0$, it must vanish at any time. Together with the continuity equation, this gives

$$\frac{\partial^2 \eta}{\partial X^2} + \frac{\partial^2 \eta}{\partial Y^2} = 0. \tag{4}$$

Laplace first took η to be a function of Y and t only, times $\cos kX$. Then the previous differential equation and the boundary condition $\eta = 0$ at the bottom $Y = 0$ of the canal further restrict η to have the form

$$\eta = a(t)\sinh kY \cos kX. \tag{5}$$

The function $a(t)$ is determined through the condition that for a virtual displacement along the water surface the pressure does not vary. Using Eq. (2), assuming the form $Y = h + \varepsilon\cos kX$ for the water surface at $t = 0$, and retaining only terms of first order in ξ, η, and ε, this implies that

$$\frac{\partial^2 \xi}{\partial t^2} + g\frac{\partial \eta}{\partial X} = \varepsilon g k \sin kX \tag{6}$$

for $Y = h$. A derivation of this equation with respect to X and the continuity equation (1) then yield

$$-\frac{\partial^2}{\partial t^2}\frac{\partial \eta}{\partial Y} + g\frac{\partial^2 \eta}{\partial X^2} = \varepsilon g k^2 \cos kX \qquad (7)$$

for $Y = h$. Injecting the form (5) of η then leads to the equation

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} k \cosh kh + a g k^2 \sinh kh = -\varepsilon g k^2. \qquad (8)$$

The only solution of this equation that agrees with the vanishing of a and $\mathrm{d}a/\mathrm{d}t$ for $t = 0$ is

$$a = \frac{\varepsilon}{\sinh kh}(\cos \omega t - 1), \qquad (9)$$

with

$$\omega^2 = gk \tanh kh. \qquad (10)$$

The corresponding elevation of the water surface above its original height h is, at the same order of approximation,

$$\sigma(X, t) = \varepsilon \cos kX + \eta(X, h; t) = \varepsilon \cos kX \cos \omega t. \qquad (11)$$

Laplace thus obtained what we would now call a stationary wave, as a consequence of his seeking a factored solution. The modern reader may wonder why he did not also find a solution of the form $\sin kX \sin \omega t$, and superpose it with the former solution to get the progressive form $\cos(kX - \omega t)$. The reason is that the initial condition of zero-velocity imposes the cosine form of the time dependence. Hence Laplace did not reach the progressive sine solution for the free propagation of small disturbances on water of finite depth, although he came very close to that from a formal point of view.

The rest of Laplace's analysis was unfortunately flawed. To proceed from a sine-shaped disturbances to a disturbance caused by local emersion, Laplace could not rely on Fourier synthesis, which was unknown at that time. Instead he truncated the sine function by taking $\sigma(X, 0) = \varepsilon(\cos kX - \cos k\alpha)$ for $|X| \leq \alpha$, and $\sigma(X, 0) = 0$ for $|X| \geq \alpha$. In "a delicate application of the calculus of partial differentials" [p. 307], he then rewrote the product $\cos kX \cos \omega t$ in the expression (11) of $\sigma(X, t)$ as $\frac{1}{2}[\cos(kX - \omega t) + \cos(kX + \omega t)]$, and replaced the latter cosines by their truncated values. This gives a propagation of the depression toward the two extremities of the X axis, and without deformation. The propagation velocity ω/k only depends on the depth of water and on the spatial period of the truncated cosine (roughly determined by the curvature of the emersed curve). As the calculus of partial differentials was still in its infancy, Laplace did not realize that the truncated wave no longer satisfied his differential equations.

Lagrangian foundations

In a memoir of [1781] Joseph Louis Lagrange addressed the problem of water waves in a most elegant manner, with no mention of Laplace's earlier flawed analysis. The purpose of this memoir was to apply the methods of analytical mechanics to hydrodynamics, and thus to solve a large class of useful problems, including the traditional efflux from a vase and the less explored water-wave problem.[4]

Lagrange's first accomplishment was a clear derivation of the basic equations, both in Eulerian and in Lagrangian form. In the Eulerian picture, the fluid motion is described by giving its velocity $\mathbf{v}$ at the point $\mathbf{r}$ and at time t. In anachronistic vector notation, the equation of continuity reads

$$\nabla \cdot \mathbf{v} + \frac{\partial \rho}{\partial t} = 0. \tag{12}$$

Euler's equation reads

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{g} - \frac{\nabla P}{\rho}. \tag{13}$$

These two equations must be supplemented with the boundary conditions. On a rigid wall, the velocity $\mathbf{v}$ is parallel to the wall. At the surface of the fluid, the pressure is equal to the atmospheric pressure, which he assumed to be constant. Moreover, the continuity of the fluid implies that a particle of the fluid originally at the surface of the fluid must remain so.

Lagrange multiplied Euler's equation by d$\mathbf{r}$ to get [p. 709]

$$\frac{\partial \mathbf{v}}{\partial t} \cdot \mathrm{d}\mathbf{r} + (\nabla \times \mathbf{v}) \cdot (\mathbf{v} \times \mathrm{d}\mathbf{r}) = \mathbf{g} \cdot \mathrm{d}\mathbf{r} - \frac{\mathrm{d}P}{\rho} - \mathrm{d}\left(\frac{v^2}{2}\right). \tag{14}$$

If the velocity $\mathbf{v}$ is the gradient of a function φ, the vector $\nabla \times \mathbf{v}$ vanishes. Further assuming that the pressure P is a function of density only, this equation can be integrated to lead to the much simpler

$$\frac{\partial \varphi}{\partial t} = \mathbf{g} \cdot \mathbf{r} - \int \frac{\mathrm{d}P}{\rho} - \frac{(\nabla\varphi)^2}{2}. \tag{15}$$

Having thus introduced the velocity potential, Lagrange wondered under what condition it existed. This led him to enunciate the following important theorem: if the fluid motion admits of a potential at a given instant, it does so at any later time (granted that the only external forces acting on the fluid are gravity and surface pressure). In particular, if the motion starts from rest, it will admit a potential [pp. 714–717].

In order to prove this theorem, Lagrange applied his favorite method, power-series development, to the functions $\mathbf{v}(t)$ and $\nabla \times \mathbf{v}(t)$. As the first member of Eq. (14) is an exact differential, the vanishing of $\nabla \times \mathbf{v}$ for $t = 0$ implies that all the coefficients of its development vanish. Therefore, this quantity vanishes at any time, and there exists a

[4] Lagrange did not mention Euler's memoirs, although they probably were a major source of inspiration. Cf. Grattan-Guinness 1990, pp. 664–665.

velocity potential.[5] Lagrange seems to have believed that the conditions of his theorem were met for most motions in nature, to the point that he felt it necessary to give one example in which they were not: tidal motion (for the Coriolis force does not derive from a potential) [p. 718].

Lagrange then proceeded to apply the hydrodynamic potential to cases when the fluid particles are confined within two mutually close parallel planes, so that a power development of the potential with respect to the perpendicular coordinate can be used. He thus solved the old problem of efflux from a narrow vase, as well as the propagation of surface disturbances in shallow water [pp. 728–748]. In the latter case, his method is simply illustrated by assuming two dimensions only; a flat, horizontal, bottom; and velocity and surface disturbances so small that terms involving their second powers can be neglected.

At the lowest non-trivial order, the expansion of the potential has the form

$$\phi(x, y, t) = \phi_0(x, t) + y\phi_1(x, t) + y^2\phi_2(x, t), \tag{16}$$

where x is the horizontal coordinate, and y the vertical one. The incompressibility of water gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{17}$$

so that $\phi_0'' + 2\phi_2 = 0$ (accents denote derivation with respect to x). Vanishing vertical velocity at the bottom $y = 0$ implies $\varphi_1 = 0$. To summarize, the potential must have the form

$$\phi = \phi_0 - \frac{1}{2} y^2 \phi_0''. \tag{18}$$

The equation of the surface is obtained by making P constant and neglecting second-order terms in Eq. (15):

$$\frac{\partial \varphi}{\partial t} + g(y - h) = 0. \tag{19}$$

The condition that a particle of the surface should remain on the surface, gives

$$\frac{\partial^2 \varphi}{\partial t^2} + \dot{x}\frac{\partial^2 \varphi}{\partial t \partial x} + \dot{y}\frac{\partial^2 \varphi}{\partial t \partial y} + g\dot{y} = 0, \tag{20}$$

where $(\dot{x}, \dot{y})$ is the velocity of the fluid particle. To first-order, this gives

$$\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial y} = 0 \quad \text{for } y = h. \tag{21}$$

Combining this condition with Eq. (18) yields

$$\frac{\partial^2 \varphi_0}{\partial t^2} - gh\frac{\partial^2 \varphi_0}{\partial x^2} = 0. \tag{22}$$

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[5] This proof only holds if the function $\mathbf{v}(t)$ is analytical. Cauchy gave the first rigorous, general proof in [1827]. Lagrange also stated [p. 721] that the potential always existed when second-order terms were neglected in Euler's equation, which is incorrect.

The general integral of this equation, as given in d'Alembert's theory of vibrating strings, is

$$\varphi_0(x,t) = f(x-ct) + g(x+ct), \tag{23}$$

where f and g are two arbitrary (differentiable) functions, and

$$c = \sqrt{gh}. \tag{24}$$

According to Eq. (19), the elevation of the water surface has the same form. The two components represent the distortionless propagation with the velocities $+c$ and $-c$ of any (small) perturbation.

Lagrange concluded his analysis with a speculative extension to waves on deep water [p. 728]. He argued that the "tenacity and the mutual adherence" of the particles of water confined the agitation to a superficial layer of water, the thickness of which would depend on the propagation velocity through formula (24).

Like Laplace, Lagrange selected physics problems according to the possibilities of mathematical analysis. Both mathematicians came to the water wave problem after realizing that mathematical procedures they had designed to solve other problems, tides or efflux, applied to this problem. They both found that their mathematics only gave limited solutions of the wave problem, stationary sine waves for Laplace, small depth for Lagrange. They both tried to overcome these limitations by speculative moves that later proved to be illegitimate.

Poisson's thorny, but thorough analysis

In the following thirty years, mathematical analysis progressed so much that the flaws of Laplace's and Lagrange's theories of waves became obvious. On December 27th, 1813, an Academic committee made of Legendre, Poinsot, Laplace, Biot, and Poisson picked "the waves at the surface of an indefinitely deep liquid" for the subject of the Academy's prize for the year 1816. Laplace wrote the announcement of the prize:[6]

> A ponderable fluid mass, primitively at rest, and indefinitely deep, is set into motion under the effect of a given cause. It is asked to determine, after a given time, the form of the external surface of the fluid and the velocity of every of the molecules situated on this surface.

This was his old problem of 1776, in a slightly more general form.

Laplace's brilliant disciple Siméon Denis Poisson, who belonged to the prize committee, wrote the first memoir on this subject that reached the Academy.[7] Poisson first

[6] Cf. the *Procès-verbaux* of the Académie des Sciences, 5(1812–1815), 262, 292, 546, 556, 595, and the statement in Cauchy 1827, p. 1.

[7] Poisson's memoir was read on October 2nd, a sequel on December 18th, 1815. It was published in 1818. A summary of the main conclusions appeared in the *Annales de chimie et de physique* [1817b]. Cf. Grattan-Guinness 1990, pp. 666–674. For a modern treatment, cf. Lamb 1932, pp. 384–398.

recalled Newton's reverted-siphon analogy, Laplace's attempt of 1776, and Lagrange's memoir of 1782. He judged Newton's analogy to be "insufficiently founded." Laplace's solution, he politely noted, only applied to an initial sine-shaped form of the water surface, and could not be truncated to yield a solution of the local-perturbation problem. Lagrange's solution was correct for small depth, but its extension to large depth was illegitimate. In order to prove the latter point, Poisson appealed to "the principle of the homogeneity of quantities," probably borrowed from Fourier's theory of heat. This early dimensional argument went as follows [Poisson 1816, pp. 71–75].

Poisson, like Laplace, assumed that the waves were produced by the sudden withdrawal of a partially immersed body. In infinitely deep water, the only "lines" of the problem are the breadth l of the original depression of the water surface, and the product gt^2, where g is the acceleration of gravity and t the time of observation. Hence the distance traveled by a wave summit at time t must be a homogenous function of l and gt^2. If this distance is independent of l, then it must be proportional to gt^2 and the wave is accelerated like a free-falling body. If the wave has constant velocity, this distance must be proportional to $t\sqrt{gl}$. Therefore, Lagrange's assumption of waves traveling at a constant velocity independent of their mode of production is impossible. Whether the waves produced by emersion travel with constant velocity, with constant acceleration, or else with variable acceleration can only be decided by calculation.

Having thus dismissed Lagrange's approach to deep-water waves, Poisson adopted Lagrange's equations for the velocity potential φ. In the two-dimensional case, and for a small perturbation of the fluid surface, these equations are

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0 \tag{17}$$

within the fluid mass, $\partial\varphi/\partial y=0$ at the bottom $y=0$, and

$$\frac{\partial^2\varphi}{\partial t^2}+g\frac{\partial\varphi}{\partial y}=0 \quad \text{for } y=h. \tag{19}$$

Poisson, now imitating Laplace, sought factored solutions of the form $\cosh ky\cos k(x-a)\sin\omega t$ or $\cosh ky\cos k(x-a)\cos\omega t$. The boundary condition (19) requires [p. 82]

$$\omega^2=gk\tanh kh. \tag{10}$$

Poisson then obtained the most general solution by superposition of the factored solutions. Using Fourier's identity (without naming Fourier)

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(a)\cos k(x-a)\,\mathrm{d}a\,\mathrm{d}k \tag{25}$$

and the equation

$$\frac{\partial\varphi}{\partial t}(x,h;t)+g(y-h)=0 \tag{21}$$

of the fluid surface, he easily found that the superposition [p. 92]

$$\varphi=-\frac{g}{\pi}\int_{-\infty}^{+\infty}f(a)\,\mathrm{d}a\int_{0}^{+\infty}\mathrm{d}k\frac{\cosh ky}{\cosh kh}\cos k(x-a)\frac{\sin\omega_k t}{\omega_k} \tag{26}$$

met the initial condition of zero velocity ($\varphi = 0$) and surface shape $y = h + f(x)$. The corresponding elevation $\sigma(x, t)$ of the water surface above the level h is

$$\sigma = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(a)\, \mathrm{d}a \int_{0}^{+\infty} \mathrm{d}k \cos k(x-a) \cos \omega_k t. \tag{27}$$

Poisson then studied the behavior of these two double-integrals in the case of large depth for which $\omega_k = \sqrt{gk}$. He did this in a purely mathematical manner, through a clever combination of changes of variables, integration by parts, and power series developments [pp. 93–107]. To give a first idea of these "rather thorny transformations" [1817a, p. 85], consider the first integral in the expression (26) of the potential [1816, pp. 93–97]. It is a linear combination of terms of the form

$$\psi = \int_{0}^{+\infty} e^{-g\gamma k} \frac{\sin \omega_k t}{\omega_k} g\mathrm{d}k = \int_{0}^{+\infty} 2e^{-\gamma\omega^2} \sin \omega t \, \mathrm{d}\omega, \tag{28}$$

where γ is a linear combination of x, y, and a with complex-number coefficients. Derivation with respect to time yields

$$\dot{\psi} = \int_{0}^{+\infty} 2\omega e^{-\gamma\omega^2} \cos \omega t \, \mathrm{d}\omega. \tag{29}$$

Integration by parts then yields

$$\dot{\psi} = \left[-\gamma^{-1} e^{-\gamma\omega^2} \cos \omega t \right]_0^{+\infty} - \gamma^{-1} t \int_{0}^{+\infty} e^{-\gamma\omega^2} \sin \omega t \, \mathrm{d}t, \tag{30}$$

or

$$\gamma\dot{\psi} + \frac{1}{2} t\psi = 1. \tag{31}$$

The integral of this equation is

$$\psi = \gamma^{-1} e^{-t^2/4\gamma} \int e^{t^2/4\gamma} \, \mathrm{d}t. \tag{32}$$

Poisson thus reached a familiar form, whose behavior for small and large times t he obtained through development in positive, respectively negative powers of t.

He then computed the corresponding expression of the potential φ and the derived velocities, mostly in the case when the profile $f(a)$ of the disturbance is very narrow.

Poisson's most detailed discussion of the wave pattern was based on the formula (27) for the surface disturbance. For a very narrow disturbance, the double integral in this formula may be replaced by the simpler

$$\sigma = \frac{A}{\pi} \int_{0}^{+\infty} \mathrm{d}k \cos kx \cos t\sqrt{gk} = \frac{2A}{\pi g} \int_{0}^{+\infty} \omega \mathrm{d}\omega \cos(\omega^2 x/g) \cos \omega t, \tag{33}$$

where A is the area of a longitudinal section of the original disturbance. Poisson astutely rewrote the last integral as $\sigma = (A/\pi g)(I_+ + I_-)$, where

$$I_\pm = \int_0^{+\infty} \omega \mathrm{d}\omega \cos(\omega^2 x/g \pm \omega t) = \int_0^{+\infty} \omega \mathrm{d}\omega \cos[(x/g)(\omega \pm gt/2x)^2 - gt^2/4x)]. \tag{34}$$

For obvious symmetry reasons, it is sufficient to consider the case $x > 0$. Putting

$$w = \sqrt{x/g}(\omega \pm gt/2x), \tag{35}$$

and

$$\alpha^2 = gt^2/4x, \tag{36}$$

we get

$$I_\pm = \frac{g}{x}\int_{\pm\alpha}^{+\infty} \mathrm{d}w(w \mp \alpha)\cos(w^2 - \alpha^2) = \mp\frac{g\alpha}{x}\int_{\pm\alpha}^{+\infty} \cos(w^2 - \alpha^2)\,\mathrm{d}w, \tag{37}$$

and

$$\sigma = \frac{2A\alpha}{\pi x}\int_0^{\alpha} \cos(w^2 - \alpha^2)\,\mathrm{d}w = \frac{2A\alpha}{\pi x}\left[\cos\alpha^2 \int_0^{\alpha} \cos w^2\,\mathrm{d}w + \sin\alpha^2 \int_0^{\alpha} \sin w^2\,\mathrm{d}w\right]. \tag{38}$$

The modern reader recognizes the Fresnel integrals that appear in the theory of diffraction. Poisson, who had no such knowledge, developed these integrals in powers of α and gave numerical estimates of the position of the first extrema of σ. As he noted, these extrema occur for well-defined values of $\alpha = \sqrt{gt^2/4x}$. Therefore, the crests of the waves move with the acceleration of gravity [pp. 108–114].

For large values of α, the two integrals in the last expression of σ differ little from their limit $\frac{1}{2}\sqrt{\pi/2}$. Hence the surface profile is approximately given by

$$\sigma = \frac{A\alpha}{x\sqrt{\pi}}\cos\left(\alpha^2 - \frac{\pi}{4}\right). \tag{39}$$

The behavior of this function is mostly given by the fast oscillations of the cosine, with an amplitude increasing linearly in time and decreasing with the distance as $x^{-3/2}$. Maxima approximately correspond to $\alpha^2 = \pi/4 + 2n\pi$, where n is an integer. The distance λ between two consecutive crests at a given time, which Poisson called wave-length, is given by $\lambda\partial\alpha^2/\partial x = 2\pi$, or $\lambda = 8\pi x^2/gt^2$. The period of the oscillations at a given place is such that $\tau\partial\alpha^2/\partial t = 2\pi$, or $\tau = 4\pi x/gt$. As Poisson noted, the period is a function of the wave-length only: $\tau = \sqrt{2\pi\lambda/g}$ [pp. 113–114, 119–120].[8]

Poisson also considered the more general case in which α is still large but the width of the original disturbance is no longer negligible. He assumed a truncated parabolic profile,

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[8] This means that in the vicinity of a distant point, the progressive sine wave solution with $\omega_k = \sqrt{gk}$ approximately represents the traveling disturbance. Poisson, who did not have the modern inclination to privilege sine wave solutions, did not make this remark.

$$f(a) = h(l^2 - a^2)/l^2 \quad \text{for } |a| \le l. \tag{40}$$

and performed the integration over a explicitly in the formula (27) for σ. This led him, after painstaking consideration of the variation rates of the various factors in the remaining integral, to the formula

$$\sigma = \tilde{f}(gt^2/4x^2)\frac{\alpha}{x\sqrt{\pi}}\cos\left(\alpha^2 - \frac{\pi}{4}\right), \tag{41}$$

where

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} f(a)e^{-ika}\,\mathrm{d}a = (4hl^4/k^3)(\sin kl - kl\cos kl). \tag{42}$$

This new factor involves the sine and the cosine of $\alpha^2 l/x$, which oscillate much slower than the $\cos(\alpha^2 - \pi/4)$ factor, as long as the distance x is much larger than the width l of the original perturbation [pp. 115–118].

As Poisson noted, the crests of the modulating envelop travel at constant velocity, since the maxima of $\tilde{f}$ occur for definite values of the dimensionless ratio glt^2/x^2. Poisson described the resulting wave pattern as *ondes dentelées* (dentate waves). This expression indicates that he regarded the envelop as physically more important than its accelerated corrugation. A dent, Poisson reasoned, corresponds to a fixed value of α and therefore decreases like $1/x$ as it moves away from the origin; whereas an anti-node corresponds to a fixed value of $gt^2l/4x^2$ and therefore decreases slower, as $1/\sqrt{x}$. This is why Poisson believed the anti-nodes to be more visible than the dents [pp. 119–126].[9]

In the last sections of his memoir, Poisson obtained similar results in the more realistic, three-dimensional case. To a modern reader, much of his lengthy essay seems uselessly complicated and overly abstract. One must recall, however, that Poisson was discovering or at least perfecting much of the calculus he needed for his problem. Most important, he could not benefit from the physico-mathematical language later developed in the context of wave optics and acoustics. At that time, Fourier analysis and synthesis still were – despite Fourier's intentions – mostly formal operations. They did not carry with them the series of images and metaphors that later physicists learned together with them. Notions such as monochromatic wave, and constructive or destructive interference were lacking. As we will see in a moment, these notions not only eased the expression of Poisson's results, they also suggested more expedient demonstrations. One author of this simplification, Horace Lamb [1904, p. 372], professed a "deep admiration" for Poisson's memoir on waves.

Cauchy's prize-winning memoir

Being himself an Academician and a member of the prize committee, Poisson could not compete for the Academy's prize on waves. A young but already important mathema-

[9] At a given distance x, only the first oscillations of the water surface are unaffected by the finite width of the generating perturbation. After a time of the order $x/\sqrt{gl}$, the modulation of these oscillations begins. Their amplitude, which originally grew linearly in time, now oscillates between limits that ultimately decrease as t^{-3}.

tician, Augustin Cauchy, won the prize.[10] The original text of his memoir was published eleven years later in the *Mémoires des savants étrangers* [1827], with a few appendices taking into account Poisson's contribution. The overlap between Poisson's and Cauchy's memoirs is considerable, even though they worked independently. They both used Lagrange's velocity potential and the relevant differential equations; they both considered a local perturbation of the fluid surface; and they both solved the equations through Fourier analysis. This last point is the most remarkable because Cauchy, unlike Poisson, was not aware of Fourier's theory of heat when he submitted his memoir. He simply reinvented the reciprocal relation between a function and its Fourier transform.[11]

From a mathematical point of view, Cauchy was more systematic, more concise, and more rigorous than Poisson. In particular, he carefully attended to the existence conditions for various kinds of solutions of his differential equations. A major novelty of his memoir was a rigorous proof of Lagrange's theorem regarding the existence of the velocity potential [pp. 35–43]. For this purpose, he used the Lagrangian form of the equations of motion. Calling x_i the coordinates at time t of the fluid particle that has the coordinates X_i at time zero, F_i the components of the force density acting within the fluid, P the pressure, and ρ the density, these equations read (in anachronistic tensor notation):

$$\rho \ddot{x}_i \, \mathrm{d}x_i = F_i \, \mathrm{d}x_i - \mathrm{d}P. \tag{43}$$

If the fluid is incompressible and if the force density $\mathbf{F}$ derives from a potential, $\ddot{x}_i \, \mathrm{d}x_i$ must be an exact differential. With respect to the coordinates X_i, this implies

$$\frac{\partial}{\partial X_i}\left(\dot{v}_k \frac{\partial x_k}{\partial X_j}\right) - \frac{\partial}{\partial X_j}\left(\dot{v}_k \frac{\partial x_k}{\partial X_i}\right) = 0. \tag{44}$$

Permutations of the partial derivatives then lead to

$$\frac{\partial}{\partial t}\left(\frac{\partial v_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \frac{\partial v_k}{\partial X_j}\frac{\partial x_k}{\partial X_i}\right) = 0, \tag{45}$$

or, integrating from time zero to time t,

$$\frac{\partial v_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \frac{\partial v_k}{\partial X_j}\frac{\partial x_k}{\partial X_i} = \frac{\partial v_j^0}{\partial X_i} - \frac{\partial v_i^0}{\partial X_j}. \tag{46}$$

Using the identity $\frac{\partial v_k}{\partial X_i} = \frac{\partial v_k}{\partial x_l}\frac{\partial x_l}{\partial X_i}$, the incompressibility condition $\left|\frac{\partial x_i}{\partial X_j}\right| = 0$, and some algebra, Cauchy finally obtained the simple relation

[10] Cf. Belhoste 1991, pp. 87–91; Grattan-Guinness 1990, pp. 674–681. In July 1815, a month before Poisson deposited his first memoir on waves, Cauchy read a note containing the main results of his theory: the constant acceleration of the waves, the decrease of the height of a wave during its propagation, and the increase of the distance between two successive waves: cf. Académie des Sciences, *Procès-verbaux*, 5 (1812–1815), p. 530; Cauchy 1827, p. 188. Bruno Belhoste notes [pp. 297–298] that Cauchy also investigated the production of waves at the interface between a compressible and an incompressible fluid. This unpublished manuscript is inserted in the *Cahier sur la théorie des ondes* belonging to Madame de Pomyers.

[11] Cf. Cauchy 1818, and Cauchy 1827, p. 291.

$$\omega_j(t) = \omega_i(0)\partial x_i/\partial X_j, \tag{47}$$

where $\omega_1 = \partial v_2/\partial x_3 - \partial v_3/\partial x_2$, etc.[12] Accordingly, if a velocity potential exists at time zero, the condition for its existence is maintained at any later time. This is Lagrange's theorem.

Another mathematical difference between Poisson's and Cauchy's memoirs was the latter's systematic recourse to dimensionless variables. For example, Cauchy rewrote Eq. (33) in terms of the variables $\mu = gt^2k$ and $\kappa = gt^2/2x$ to get [p. 88]

$$\sigma = \frac{A}{\pi g t^2}\int_0^{+\infty} \mathrm{d}\mu \cos\frac{\mu}{2\kappa}\cos\mu^{\frac{1}{2}}. \tag{48}$$

Under this form, it is immediately clear that the wave crests correspond to definite values of $gt^2/2x$, so that their motion is uniformly accelerated. In general, Cauchy sought universality beyond the specific physics problems he was studying. He tried to extract formulas and structures that had intrinsic mathematical value and could eventually serve in other physical situations.

Regarding the physical discussion of waves, the scope of Cauchy's differed from Poisson's. Like Laplace, Poisson confined his analysis to disturbances created by the sudden emersion of a solid body. He briefly indicated how the case of an impulsive pressure applied on a portion of the fluid surface could be included in his general formulas [p. 92]. But he did not pursue the analysis of this case any further. In contrast, Cauchy showed how the initial fluid velocity depended on the impulsive pressure and thus reached a physical interpretation of the velocity potential as the internal impulsive pressure resulting from the external impulsion (for unit density)[p.15]. He also proved that the motion of the fluid at any instant could be regarded as created from rest by impulsive forces, a result important to later British hydrodynamicists [p. 14].

In other respects, Cauchy's physical discussion was less complete than Poisson's. Cauchy confined his discussion to waves independent of the shape of the original disturbance, [13] whereas Poisson regarded this effect as the most perspicuous aspect of wave motion. After reading Poisson, Cauchy investigated this question more thoroughly than Poisson had done [note XVI]. He showed that for any symmetric profile of the immersed body, the modulating envelop of the fast oscillations was the Fourier transform of the profile (Eqs. (41)–(42). He confirmed Poisson's result for the parabolic profile and expressed is with the slightly apter metaphor of *ondes sillonnées* [p. 196]. But he showed that for convex profiles (as in Fig. 1), the modulating factor did not oscillate [p. 220]. In response to this nice theorem, Poisson [1829] argued that the only case of physical interest was the small-depth parabolic profile, because other profiles would not be consistent with the continuity of the fluid during the first instants of the motion.[14]

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[12] This is the Lagrangian expression of the fact, established by Helmholtz in 1858, that the convective derivative of the vorticity vanishes in an incompressible, Eulerian fluid. A much easier proof of the theorem [Lamb 1932, p. 17] is obtained by noting that $\dot{\mathbf{v}}\cdot\mathrm{d}\mathbf{r} = \partial(\mathbf{v}\cdot\mathrm{d}\mathbf{r})/\partial t - \mathrm{d}(v^2/2)$ in the Lagrangian picture. As $\dot{\mathbf{v}}\cdot\mathrm{d}\mathbf{r}$ is an exact differential at any time, if $\mathbf{v}\cdot\mathrm{d}\mathbf{r}$ is an exact differential at time zero is must be so at any later time.

[13] Cauchy [pp. 92–94] gave the validity condition for that.

[14] Fourier [1818] had pleaded for the investigation of a non-parabolic profile.

Fig. 1. Disturbed water surface with a convex profile, as imagined by Cauchy

The comparison between Cauchy's and Poisson's memoir leaves us with the impression that Poisson was more concerned with physical meaning, and Cauchy with mathematical meaning. Poisson's physics, however, was idealized physics. As we will see in a moment, his and Laplace's emersion method for producing waves does not work in practice. Poisson did not perform any experiment. He contented himself with calling for an experimental confirmation of his theory in the introduction to his memoir [Poisson 1816, p. 78].

Apparent confirmations

In 1820, the Torino based hydraulician George Bidone claimed to have confirmed Poisson's most striking prediction, the uniformly accelerated motion of the first waves created by a local perturbation of the water surface, as well as the numerical values of the accelerations of the two first waves (0.3253 g, and 0.1183 g). Bidone operated with a 24-inch wide and 24-inch deep canal. He did not say how he measured the velocity of the waves. But he dwelt on the difficulty he encountered in applying the Laplace-Poisson emersion method for the production of waves. The immersed body did not instantly leave the water surface upon withdrawal as the two mathematicians had imagined. On the contrary, the water adhered to the body and followed it to a certain height until it violently fell down (Fig. 2). Bidone believed he could circumvent this difficulty by attending to the two first waves only, which in his opinion were created before the fall of the raised water column. Apparently, he did not realize that Poisson's calculations did not apply to such an impulsive excitation. It is not clear why he reached such "a marvelous agreement between theory and experiment" [Bidone 1820, p. 25][15].

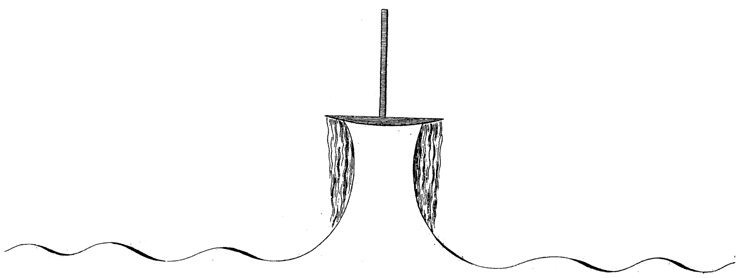

Fig. 2. The emersion of a parabolic solid according to Bidone [1820, plate]

[15] Poisson [1829, p. 571] noted Bidone's confirmation of the accelerated waves. Strangely, he did not comment on the failure of the emersion method, even though his memoir was about the permissible profiles of the initial water surface.

In 1825, the Leipzig professor Ernst Heinrich Weber and his brother Wilhelm published a very thorough *Wellenlehre*, which summarized all previous theories of waves and provided many astute, quantitative experiments on this matter. Their motivation was the recent development of wave physics in acoustic and optical contexts, owing to the works of Ernst Chladni, Félix Savart, Thomas Young, and Augustin Fresnel [1825, p. V]. They wanted to provide the subject with a solid empirical basis, using water waves as an archetype of wave motion. In their most extended series of experiments, they used two long, narrow water-tanks (see Fig. 3). They disturbed the water at one end of the tank, and obtained "self-drawn" wave profiles by the sudden withdrawal of a vertically and longitudinally immersed board [pp. 105–117]. They also measured the time a wave took to travel across the tank [pp. 166–199], and visualized the internal fluid motion through suspended dust particles [pp. 117–155].

The Weber brothers became aware of Poisson's "very important" theory of waves after they performed their experiments, but before the final redaction of their treatise [p. 377]. As they believed their observations to confirm some aspects of this theory, they included a commentary in French of Poisson's paper [pp. 377–434]. They approved his general description of the wave pattern, with faint accelerated waves at the front, followed by constant-velocity waves with a "dentate" surface. They also confirmed the proportionality between the period of oscillation and the square root of the wave length.

These conclusions would not have resisted a more accurate reading of Poisson and more adequate experiments. As Scott Russell later commented [1844, p. 25n], the Webers' tank was too narrow, too shallow, and too short to approximate the ideal conditions of frictionless deep-water wave motion far from the source. In order to create their waves, the Webers dipped a glass tube vertically into water, drew up the water by suction, and

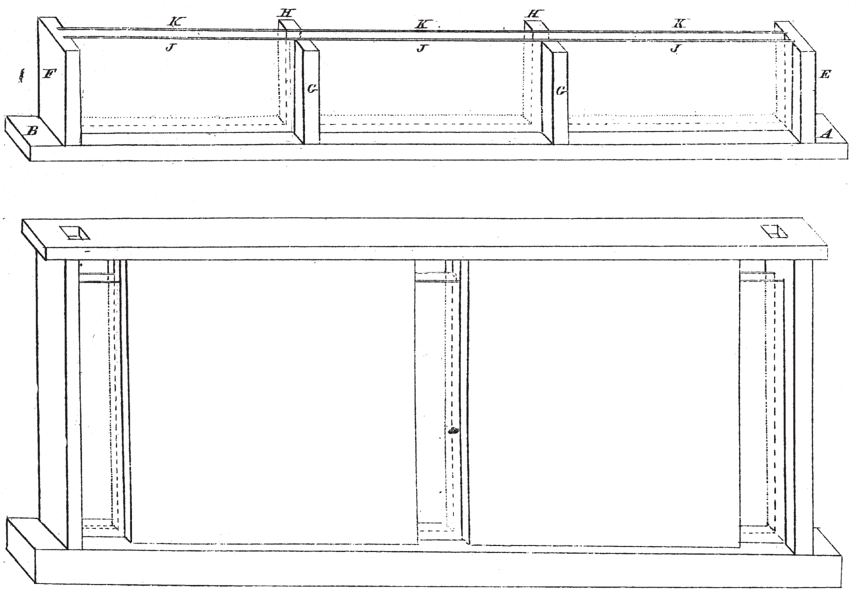

Fig. 3. The experimental tanks of the Weber brothers [1825, plate]

let it fall back [Weber 1825, p. 106]. This method widely differs from the static surface deformation imagined by Poisson. Most fatally, the two brothers mistook Poisson's *ondes dentelées* to mean large waves with a ruffled surface, whereas Poisson's formulas show that he meant what we would now call modulated waves. What they actually observed probably was capillarity ripples superposed with gravity waves. The lack of figures and concise summaries in Poisson's memoir favored the confusion. As Thomson, put it in 1871: "A great part of what they [Poisson and Cauchy] have to say would be much shortened even by the addition of graphic representations, and it would be much easier for any one (the authors I believe included) to understand the whole character of the phenomena investigated, with illustration like this of the chief function on which the expression of these depends" [Thomson to Stokes, 20 Nov 1871, *ST*]. But what had become a common practice in the 1870s would had been judged costly bad taste by a French mathematician of the early 19th century.

2. Scott Russell, the naval engineer

A horse's discovery

In 1833 the Cambridge astronomer James Challis reviewed the present state of hydrodynamics for the British Association [Challis 1833]. He praised Poisson's and Cauchy's theories of waves, and pointed to their verification by Bidone and the Webers. He concluded with a pessimistic note on the current understanding of a more pressing problem: fluid resistance. As a flagrant case of the impotency of theory in this case, he referred to a "singular fact" observed in canal navigation: for a speed of four or five miles per hour the hauled boat rose out of the water and the resistance was suddenly diminished [p. 155].

John Scott Russell, a young Glasgow engineer specialized in steam power and naval architecture, knew of this striking anomaly.[16] He later described it in vivid terms [Russell 1839, p. 79]:

> As far as I am able to learn, the isolated fact was discovered accidentally on the Glasgow and Ardrossan Canal of small dimensions. A spirited horse in the boat of William Houston, Esq., one of the proprietors of the works, took fright and ran off, dragging the boat with it, and it was then observed, to Mr. Houston's astonishment, that the foaming stern surge which used to devastate the banks had ceased, and the vessel was carried on through water comparatively smooth, with a resistance very greatly diminished. Mr. Houston had the tact to perceive the mercantile value of this fact to the Canal Company with which he was connected, and devoted himself to introducing on that canal vessels moving with this high velocity.

There was indeed, in the 1830s, a system of fly-boats carrying passengers on two Scottish canals. A pair of horse drew each boat at a speed of about 10 miles/hour.[17]

[16] For a biography, cf. Emmerson 1977. On ship hydrodynamics in the 19th century, cf. the excellent study by Thomas Wright [1983].

[17] Cf. Thomson 1887, pp. 418–420. Thomson lamented: "Is it possible not to regret the old fly-boats between Glasgow and the Ardrossan and between Glasgow and Edinburgh, and their

Stimulated by Challis' interest in this paradox of fluid resistance, Scott Russell submitted his own simple solution at the Edinburgh meeting of the British Association in 1834. The motion of a boat through water, he reasoned, raised the pressure of the water at the bottom of the ship above its static value. This caused a partial emersion of the boat, and the observed decrease of resistance. Calling S and S' the transverse sections of immersion for velocities zero and v respectively, Russell wrote the strange non-dimensional equation $S'v = S(v - v^2/2g)$, and inserted the resulting value of S' in the Newtonian resistance formula $R = S'v^2\rho/2$. Of this departure of the resistance law from a quadratic form, he said to have found ample evidence in towing experiments [Russell 1834][18].

A fuller version of this argument [1839, p. 57] displays Russell crude misunderstanding of the laws of mechanics. There he derived the bottom pressure from the well-known front pressure of the Newtonian theory of resistance, combined with the isotropy of pressure. In the rest of his reasoning, he seems to have confused the Archimedean displacement with the dynamic displacement ρSv.

The great, solitary wave

Russell remained unaware of these infractions of the science of mechanics. He did however recognize that his law only gave a gradual correction to the Newtonian resistance, not the desired Houston jump. In order to understand this anomaly, he attended to the fluid motion induced by the boat. One day, "the happiest of [his] life" [1865, vol. 1, p. 217] something unexpected happened [1839, p. 61]:

> In directing my attention to the phenomena of the motion communicated to a fluid by the floating body, I early observed one very singular and beautiful phenomenon, which is so important, that I shall describe minutely the aspect under which if first presented itself. I happened to be engaged in observing the motion of a vessel at a high velocity, when it was suddenly stopped, and a violent and tumultuous agitation among the little undulations which the vessel had formed around it, attracted my notice. The water in various masses was observed gathering in a heap of a well-defined form around the center of the length of the vessel. This accumulated mass, raising at last to a pointed crest, began to rush forward with considerable velocity towards the prow of the boat, and then passed away before it altogether, and retaining its form, appeared to roll forward alone along the surface of the quiescent fluid, a *large*, *solitary*, *progressive wave*. I immediately left the vessel, and attempted to follow this wave on foot, but finding its motion too rapid, I got instantly on horseback and overtook it in a few minutes, when I found it pursuing its solitary path with a uniform velocity along the surface of the fluid. After having followed it for more than a mile, I found it subside gradually, until at length it was lost among the windings of the channel. This phenomenon I observed again and again as often as the vessel, after having been put in rapid motion, was suddenly stopped; and the accompanying circumstances of the phenomenon were so uniform, and some consequences of its existence so obvious and important, that I was induced to make *The wave* the subject of numerous experiments.

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beautiful hydrodynamics, when, hurried along on the railway, we catch a glimpse of the Forth and Clyde Canal still used for slow goods traffic; or of some swampy hollows, all that remains of the Ardrossan Canal on which the horse and Mr. Houston and Scott Russell made their discovery?"

[18] Of course, Russell intended his formula to be used with fixed, foot and pound units.

Russell soon suspected a connection between the existence of solitary waves and Houston's resistance paradox. A few trials confirmed that "*the velocity of the motion of the solitary wave had a peculiar relation to a certain well-defined point of transition in the resistance of the fluid*" [1839, p. 61]. Russell performed the necessary experiments "during the leisure of two summers," 1834 and 1835, with the support of canal, naval, and academic authorities, and with the help of "two scientific friends" and "a dozen hired assistance" [1839, p. 47]. Four different vessels were towed in canals of various depths at velocity ranging between 3 and 15 miles per hour. Horses provided the towing force, directly in 1834, through a suspended-weight regulator in 1835 (Fig. 4). The resistance was measured by a dynamometer. Russell found it to increase regularly until a certain critical velocity depending on the depth was reached, then to suddenly diminish, and lastly to increase again (Fig. 5). The critical velocity turned out to be identical to the velocity of the solitary wave for the given depth h. With a gun-shooting friend and a chronometer, Russell measured the time that this wave took to travel between two distant points. This gave him Lagrange's velocity formula $\sqrt{gh}$, more precisely $\sqrt{g(h+\sigma)}$, where σ is the height of the wave crest above the undisturbed water surface [1835a, 1837b, 1839, pp. 49–50].

Russell also described how the shape of the water surface around the moving vessel evolved with the velocity (Fig. 6). For velocities inferior to the critical value, the water level is raised around the prow, thus forming "the great primary wave of displacement." The resulting inclination of the vessel, Russell reasoned, increases its effective transverse section of immersion and the corresponding resistance. When the velocity of the vessel reaches the critical value, this wave has the velocity of a solitary wave. The push from the vessel is no longer necessary for its progression. If the velocity is further increased, the vessel catches up with its own wave, so as to be "poised on its summit." The effective transverse section is much smaller, and so is the resistance [1835a, 1839, p. 50].

For subcritical velocities, Russell also noted the "posterior wave of displacement," namely: the depression of the water surface at the stern that necessarily accompanies its rise at the prow. As water rushes into this depression from both sides, Russell reasoned, it induces a series of oscillations of the water behind the vessel (Fig. 6a). The violence of these oscillations increases until the critical velocity is reached. They subside beyond this velocity, because the posterior wave no longer exists [pp. 65–67].

Wave-lined vessels

Whatever be the value of this intuitive reasoning, it convinced Russell that the accumulation of water at the prow of a vessel was a major obstacle to its progression.[19] In canals of small depth this obstacle could only be overcome by exceeding the critical velocity. For maritime navigation, this cause of resistance could only grow with increased

[19] According to the modern understanding of ship resistance, the wave component derives from the waves that propagate away from the ship, not from a direct action on the prow.

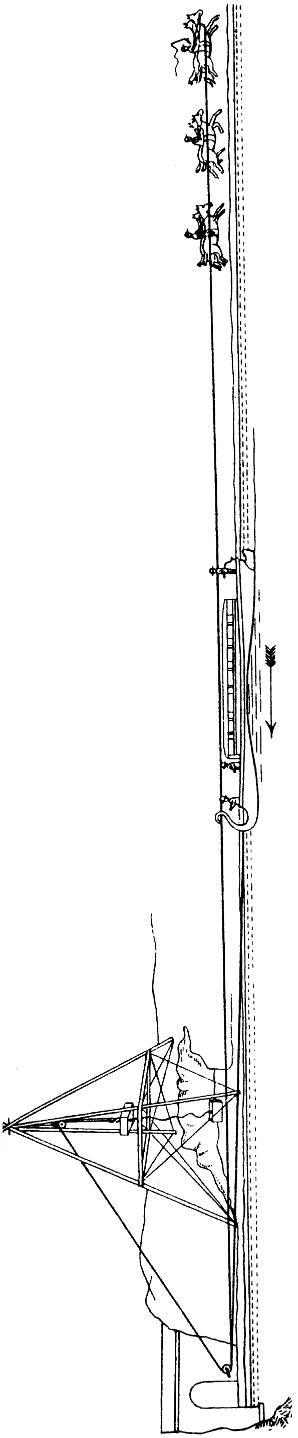

Fig. 4. Russell's towing mechanism of 1835 [plate of Russell 1839, redrawn by Thomson 1887a]

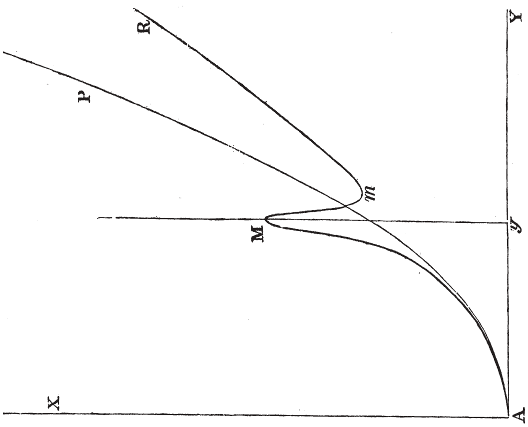

Fig. 5. Resistance as a function of towing velocity according to Russell [1839, p. 49]

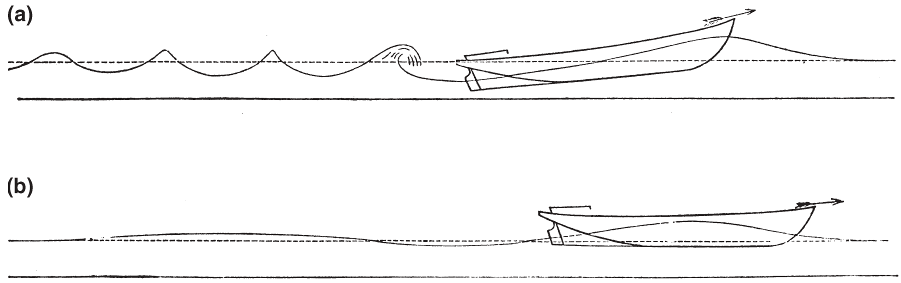

Fig. 6. Positions of a canal boat towed at a velocity inferior to the critical velocity **(a)**, superior to the critical velocity **(b)** [Russell 1839, p. 70]

velocity. It was an obstacle to the high-speed, steam-powered navigation in which the city of Glasgow had the highest stakes. Russell soon suggested a remedy: to shape the prow of the vessel according to hollow lines, so that it could enter the water without ruffling its surface [1835b, 1837a, 1839, p. 51]. Specifically, he recommended lines made of two arcs of parabola, for this shape would induce a uniformly accelerated motion of the water along the lines. As he later put it [1865, vol. 1, p. 161], "There is a way of setting about the removal of the water from the place the ship wants to enter, which is pleasant and profitable to both." Russell noted that hollow lines had long been used by pirates, to whom speed was essential. They occur spontaneously in the most primitive mode of ship construction: binding the extremities of two planks, and separating their middle part through a transverse beam. Russell only claimed to be first in showing their theoretical superiority [ibid.].[20]

In 1835, Russell built *The Wave*, a model with a 75 foot keel and a 6 foot beam to test this new principle of ship construction [1835b](Fig. 7). The following year he went on with a series of more important wave-lined vessels: the Storm, the Skiff, and... the Scott Russell. In the early 1840s, he steered a British Association committee "on the form of vessels." He then performed some twenty thousand observations with models and full-scale vessels, ranging from 30 inches to 1300 tons. The wave profile came out best, though with some modification [1841, 1842a, 1843b]. Russell found the hollowness of parabolic lines to be excessive, and ultimately adopted sine-shaped lines for their analogy with the harmonic waves of Lagrange's theory [1865, vol. 1, pp. 210–211]. For the rest of his career, he pressed for the systematic use of the "wave profile," and repeatedly denounced British conservatism in matters of ship design [1852, 1862, vol. 1, p. XXX]. In the mid-1850s he applied it to the Great Eastern, a monster metal vessel built for the Eastern Navigation Company [1854, 1857][21].

[20] Cf. Wright 1983, pp. 71–80.

[21] Cf. Emmerson 1977; Wright 1983, p. 80, who claims that Russell applied far less hollow lines to the Great Eastern than required by his theory, despite a lot of propaganda. In the later conceptions of ship resistance developed by William Rankine and William Froude, wave formation still played a role, though with mechanisms different from Russell's and in competition with two other forms of resistance: skin friction, and eddy formation. Cf. Wright 1883, chaps. 5–7.

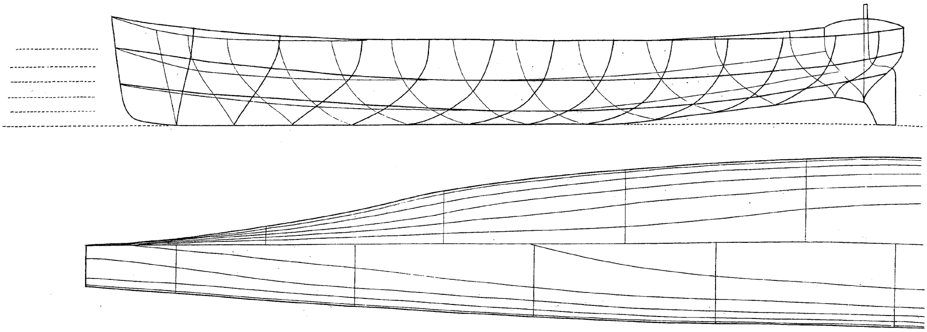

Fig. 7. Russell's first hollow-line model: "The Wave" [1839, plate]

Taming water waves

Russell's investigation of the best form of ships went along with further studies of water waves. A the Bristol meeting of 1836, the British Association appointed a "Committee on Waves" directed by Russell and John Robison. Russell gave a first report of these researches at the Liverpool meeting of 1837 [1837c]. A section of this report was devoted to an attempt at explaining tides in terms of solitary waves, to be discussed in a moment. In most of his report, Russell described experiments he made in canals and in a 20 foot long and one foot broad experimental reservoir [p. 423]. Through a clever optical method, he established the $\sqrt{g(h+\sigma)}$ velocity formula for solitary waves. He found that these waves had a quasi-cycloidal form independent of the way they were produced [p. 424]. He described the induced motion of the fluid particles: "By the transit of the wave the particles of the fluid are raised from their places, transferred forwards in the direction of the motion of the wave, and permanently deposited at rest in a new place at a considerable distance form their original position," in opposition to "second-order," oscillatory waves in which the particles oscillate around fixed point [p 423]. He found that two solitary waves "cross[ed] each other without change of any kind" [p. 425]. He observed that some sea waves, originally of second order, evolved into solitary waves when approaching the shore [p. 426]. He found that the highest possible wave had a relative height σ equal to the depth h. Lastly, he performed a few measurements on sea waves. Owing to unfavorable weather conditions, these hardly showed more than the independence of the waves on the depth of the sea [p. 426].

In a later report [1845], Russell confirmed the singular properties of solitary waves, extended his investigation to other sorts of wave, and compared his results with previous mathematical theories. As we will see in a moment, the Astronomer Royal George Biddell Airy had denied the existence of solitary waves and reduced Russell's observations to a confirmation of Lagrange's shallow-water waves [Airy 1845]. Russell, who had time to see Airy's text just before sending his report to the printer, was naturally disappointed [pp. 27, 30]:

> This paper I have long expected with much anxiety, in the hope that it would furnish a final solution of this difficult problem [the discrepancy between wave theory and wave phenomena], a hope justified by the reputation and position of the author, as well as by

> the clear views and elegant processes which characterize some of his former papers. . ..
> It is deeply to be deplored that the methods of investigation employed with so much knowledge, and applied with so much tact and dexterity, should not have led to a better result.

Russell insisted that his waves, unlike Lagrange's, had a definite shape for a given height, with a length about six times their height [pp. 33–34]. New experiments performed "after the best methods employed in inductive philosophy" confirmed this point [p. 27]. The disturbance produced by the injection of additional water at one end of his tank soon evolved, while propagating along the channel, into the perfectly stable form of the solitary wave (see Fig. 8)[pp. 45–46]. When the injection was irregular, a compound wave was produced which evolved into separate solitary waves (Fig. 9). Using the Weber's self-drawing method, Russell showed that the shape of the solitary height was perfectly determined for a given height, and tended to a cusped shape when the maximal height was reached (Fig. 10). Lastly, Russell confirmed his velocity formula for this wave, $\sqrt{g(h+\eta)}$ instead of Lagrange's or Airy's formulas.

The four orders

No one, Russell argued, had predicted or observed his great solitary wave before him [pp. 23–25]. Lagrange's waves were too small. The mode of production of Poisson's and Cauchy's calculated waves precluded solitary waves. The Weber brothers insisted that a positive wave never went without a correlative negative wave. In order to avoid confusion of his great wave with others' waves, Russell introduced four order of waves [p. 9] (Fig. 11):

1) *Waves of translation*. They involve mass transfer. Positive waves of this kind can be solitary. Negative ones are always accompanied with an undulating series of secondary waves (see Fig. 12).
2) *Oscillatory waves*. Those do not involve mass transfer. They appear as groups of successively positive and negative waves. They are the most commonly visible waves, created by wind for instance. They can be progressive or standing.
3) *Capillary waves*. Those only involve a minute-depth agitation of the water. They depend on the surface tension of the water.
4) *Corpuscular waves*. Those are rapid successions of solitary waves. Sound waves are the prime example.

Although Russell focused on the first order, he also performed careful experiments on the second and third kind. For instance, he showed that the Kelland-Airy formula $c^2 = (g/k)\tanh kh$ correctly represented the velocity of progressive oscillatory waves, even when their amplitude was not small [p. 67]. He illustrated the evolution of such waves when approaching a shore (Fig. 13). He drew the shape of steady waves produced by an obstacle in the bed of the stream (Fig. 14). He obtained a beautiful pattern of capillary wave by plunging a rod vertically in a stream of water (Fig. 15).[22]

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[22] Russell [p. 78] was aware of similar observations by Poncelet [1831, p. 78].

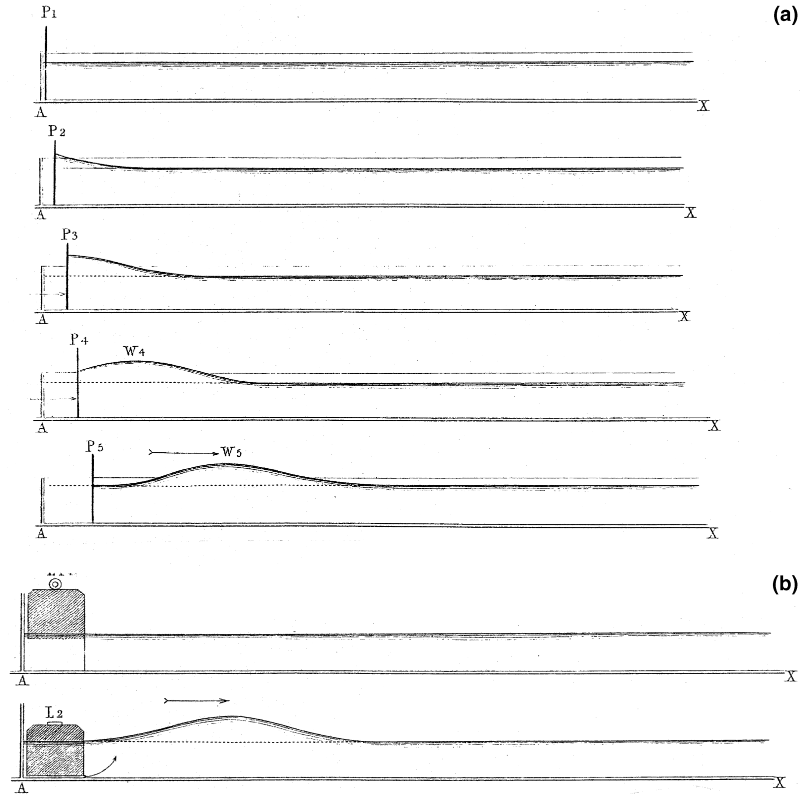

Fig. 8. Two ways of producing a solitary wave: through the displacement of a wall **(a)**; through the immersion of a solid **(b)** [Russell 1845, plate]

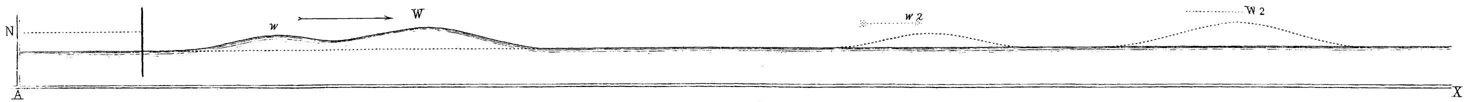

Fig. 9. The separation of two solitary waves [Russell 1845, plate]

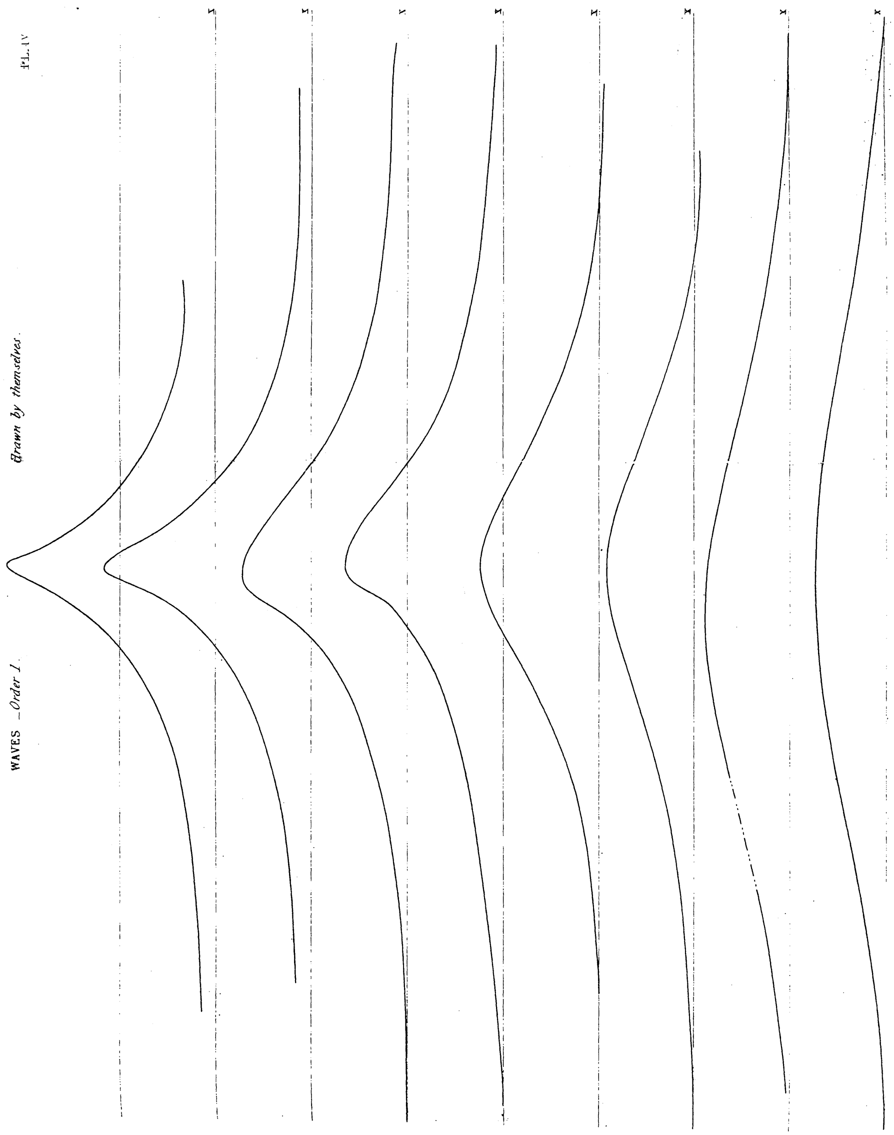

Fig. 10. Self-drawn solitary-wave profiles of various heights [Russell 1845, plate]

System of Water Waves.

Orders.	First.	Second.	Third.	Fourth.
Designation.	Wave of translation	Oscillating waves.	Capillary waves.	Corpuscular wave.
Characters...	Solitary	Gregarious.........	Gregarious.......	Solitary.
Species ...	Positive	Stationary.........	Free.	
	Negative..................	Progressive	Forced.	
Varieties	Free	Free.		
	Forced	Forced.		
Instances	The wave of resistance.	Stream ripple	Dentate waves...	Water-sound wave.
	The tide wave	Wind waves.......	Zephyral waves.	
	The aërial sound wave.	Ocean swell.......		

Fig. 11. Russell's wave orders [1845, p. 9]

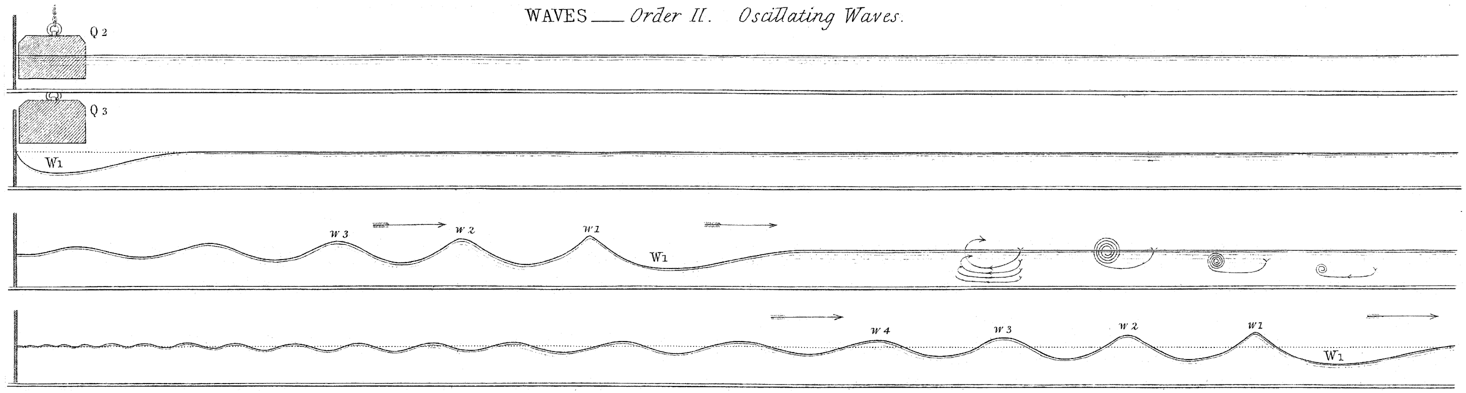

Fig. 12. Negative wave of translation and the accompanying oscillatory wave [Russell 1845, plate]

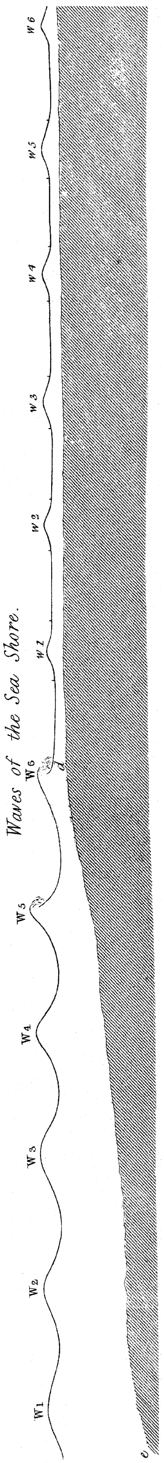

Fig. 13. Waves approaching a shore and evolving into solitary waves [Russell 1845, plate]

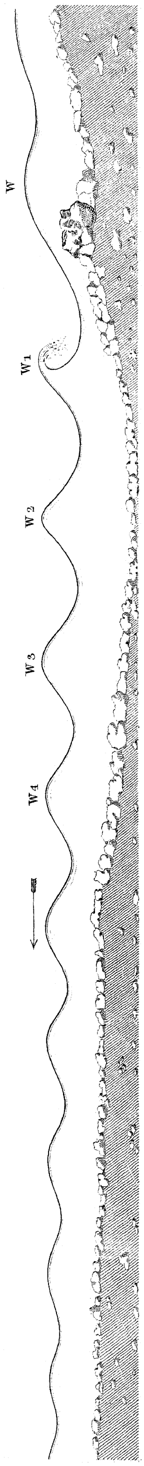

Fig. 14. Standing wave created by an obstacle in running water [Russell 1845, plate]

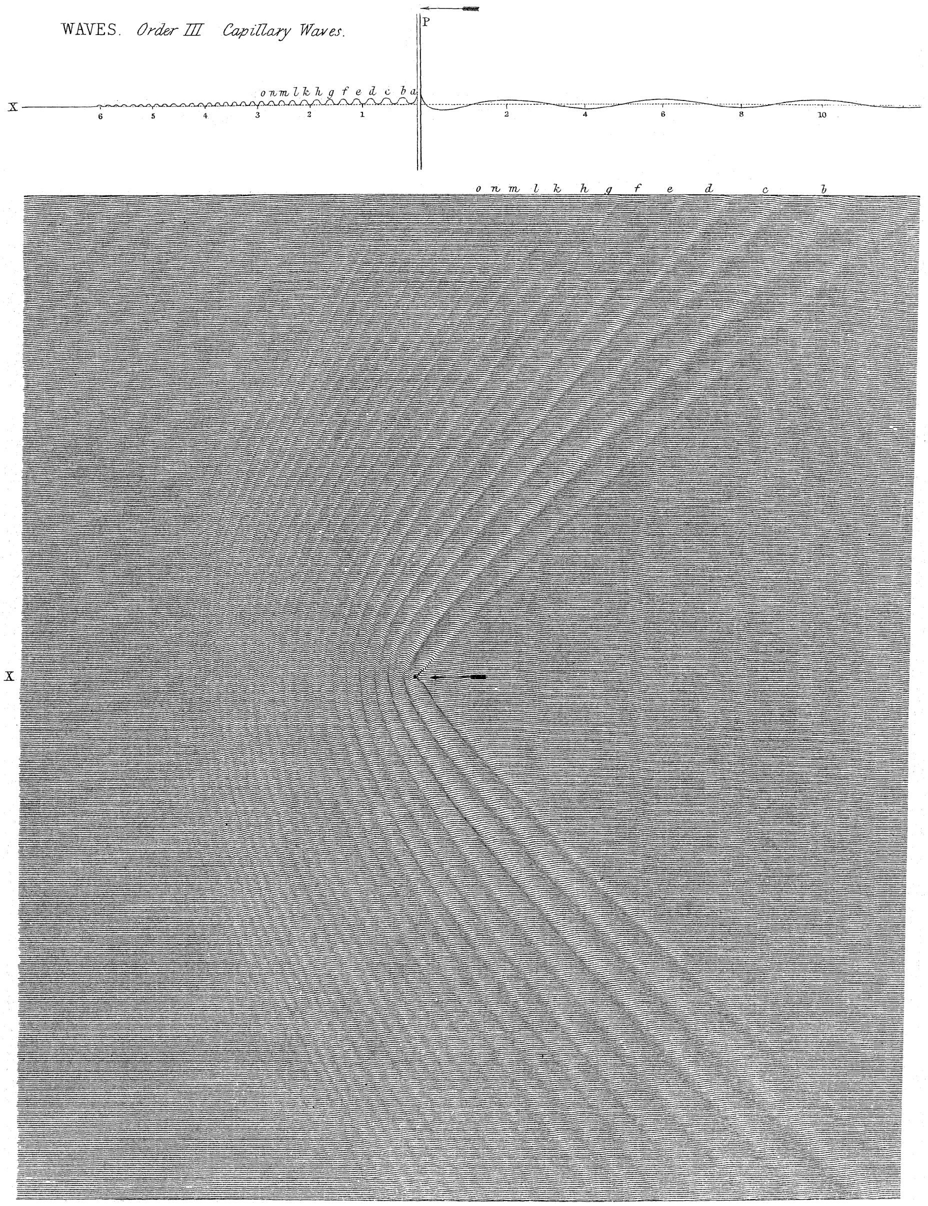

Fig. 15. Waves generated by a vertical rod (Ø = 1/16 inch) moving along the water surface with a uniform velocity. The smaller waves in front of the rod are capillarity waves [Russell 1845, plate]

Strangest to Russell's readers must have been the fourth order of waves, supposed to represent sound waves. For contemporary physicists sound corresponded to the propagation of small-amplitude vibrations through an elastic medium. No special kind of wave was needed. As appear from a posthumously published manuscript [1885], Russell rejected this explanation for he believed it could not explain the ability of sound to propagate far from its source. The sound of a tuning fork or the vibrations of a string, he observed, could be heard at a non-negligible distance only if the fork or the string was attached to a hollow case with an aperture. Sound, he inferred, was not the harmonic vibration of the fork and the surrounding air, but the repeated emission of solitary waves through the aperture of the case. As solitary waves are surface waves, Russell needed to imagine an open surface for the medium of propagation. For sound in water, the free water surface did the job. For sound in air, he imagined an ocean of air of large but finite depth around the earth. Most daringly, he proposed that light was a wave of fourth order in an even larger ocean of ether.

These suggestions only show Russell's ignorance of some elementary principles of mechanics. The Royal Society never published the series of manuscript it received from him on this theme. Yet the elite of British natural philosophers often praised Russell's early works on waves and ship forms, for they admired the quality of his experiments and the frequent validity of his intuitions.

3. Tides and waves

Russell's illumination

Between Russell's careful experiments on water waves and his hair-raising speculations on corpuscular waves, there was a middle ground which seems to have perplexed his learned supporters: the notion that tides were essentially solitary waves of very large extent. As Russell recounts, he submitted this idea to William Whewell in 1835 together with a plan for observations. Whewell had then be working for several years on tidal observations and prediction, and was with John Lubbock the leading British expert on this topic. He approved Russell's project, which thus became part of the duties of the "Committee on Waves" [Russell 1837c, 420][23].

At the Liverpool meeting of 1837, Russell reported the tidal observations the committee had made on the rivers Dee (Cheshire) and Clyde (Scotland). He also promoted his own theory of tides. The general idea was to divide the problem in two parts: the general elevation of water in the Pacific and Atlantic Ocean as ruled by celestial mechanics, and the propagation of this elevation in smaller basins, channels, and rivers as ruled by terrestrial hydrodynamics. Russell described the latter mechanism as follows [1837c, p. 426]:[24]

> The *Tide Wave* appears to be. . .identical with the great primary wave of translation; its velocity diminishes and increases with the depth of the fluid, and appears to approximate closely to the velocity due to half the depth of the fluid...–The tide appears to be a compound wave, one elementary wave bringing the first part of the flood tide, another the high water, and so on: these move with different velocities according to the depth.

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[23] In 1838 (*BAR*, p. 20), Whewell praised Robison and Russell for "highly valuable materials, likely to assist us in the further prosecution of the subject [the theory of tides]."

[24] Although Russell's identification of the tidal wave with a compound solitary wave makes little sense from a modern point of view, his theory appears to be similar to Partiot's more correct theory, discussed below in par. 4.

> On approaching shallow shores the anterior tide waves move more slowly in the shallow water, while the posterior waves moving more rapidly, diminish the distance between two successive waves. The tide wave becomes thus dislocated, its anterior surface rising more rapidly, and its posterior surface descending more slowly than in deep water. – A tidal bore is formed when the water is so shallow at low water that the first waves of flood tide move with a velocity so much less than that due to the succeeding part of the tidal wave, as to be overtaken by the subsequent waves, or wherever the tide rises so rapidly, and the water on the shore or in the river is so shallow that the height of the first wave of the tide is greater than the depth of the fluid at that place.

Russell thus explained a few basic facts: that for river tides, the time of ebb is larger than the time of flood, with a difference increasing with the distance from the mouth of the river; that tides can be very different in nearby locations, that they depend on the bottom of the sea or the form of channels and rivers, that strong river tides are often accompanied with a breaking surge or tidal bore (*mascaret* in French) [1837c, 1838]. In his later water-tank experiments, Russell verified that "compound solitary waves" evolved during their propagation so that the front became steeper than its rear (Fig. 16)[1845].

From Newton to Whewell

Although the idea that tides were a wave phenomenon was not as new as Russell suggested, it departed from the then current approaches to tide theory and prediction. Some history of these approaches will now be recalled.[25]

In his *Principia* Newton gave the correct expression of the force that is responsible for tides: the combined action of the moon's and the sun's attractions. His derivation of the resulting deformation of the surface of the oceans was only tentative and retrospectively erroneous. He seems to have adopted an equilibrium theory, with retardation due to friction. According to the pure equilibrium theory that Colin MacLaurin, Leonhard Euler, and Daniel Bernoulli developed in their competition for the 1740 prize of the French Academy, for every instantaneous configuration of the moon and the sun, the water surface takes the form it would have if the corresponding forces were permanently acting. Retaining only the lunar action in a first approximation, the net force acting on oceanic water is the Newtonian gravitational force, which is proportional to the inverse squared distance of the water from the moon, minus the inertial force due to the acceleration of the earth toward the moon, which is proportional to the inverse squared distance of the center of the earth from the moon. Therefore, the net force is a maximum at the points closest to and furthest from the moon. For a uniform ocean covering the whole earth, the resulting equilibrium surface (obtained by making the total potential of the terrestrial and lunar forces a constant), has the form indicated in Fig. 17a. Unfortunately, observed tides more nearly correspond to the form indicated in Fig. 17b.

For this reason, in 1776 Laplace proposed a dynamic theory of tides. Assuming that the horizontal velocity of the water was the same on a vertical line, and neglecting second order quantities, he obtained the equations of motion (in modern notation)

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[25] The following account is based on Cartwright 1999.

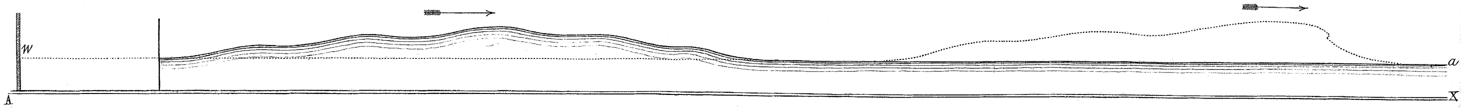

Fig. 16. The evolution of a compound solitary wave according to Russell [1845, plate]

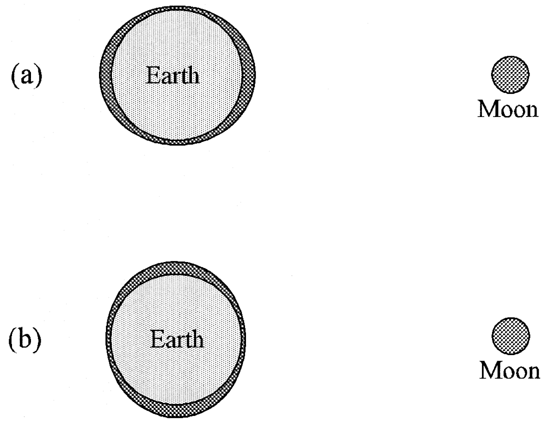

Fig. 17. Tides on an ocean of uniform depth as given the equilibrium theory **(a)**; as inferred from observed tides **(b)**

$$\frac{\partial u}{\partial t} - 2\Omega v \cos\theta = -\frac{g}{R}\frac{\partial}{\partial\theta}(\zeta - U - \delta U),$$
$$\frac{\partial v}{\partial t} + 2\Omega u \cos\theta = -\frac{g}{R\sin\theta}\frac{\partial}{\partial\phi}(\zeta - U - \delta U), \tag{49}$$

where u and v are the velocity component along the meridians and the parallels respectively, θ and ϕ the colatitude and the longitude, Ω is the angular velocity of the earth, R the radius of the earth, ζ the elevation of the water surface, U the combined gravitational potential from moon and sun, δU the gravitational self-potential of the water. The first terms on the left side of these equations correspond to the acceleration of the water particles, the second to the Coriolis force (not yet named so, of course). The right side corresponds to the sum of pressure forces (depending on the elevation of the surface) and gravitational forces. These equations are to be solved in combination with the continuity equation

$$\frac{\partial}{\partial\theta}(uh\sin\theta) + \frac{\partial}{\partial\phi}(vh) + R\sin\theta\frac{\partial\zeta}{\partial t} = 0, \tag{50}$$

where h is the depth of water.

Laplace decomposed the potential U from the moon and sun into three terms that had monthly, diurnal, and semi-diurnal variations respectively, and then solved his equations through perturbative methods in the analytically simple case for which the depth h varies as the sine-squared of the latitude. As he himself realized, this assumption could not pass for a realistic representation of the oceans. In the last section of his memoir, he switched to a semi-empirical method in which the elevation of the water at one point was represented as a sum of sine functions with the frequencies of the perturbing forces. In modern terms, we would say he realized that the forced oscillations of the water surface necessarily had the same spectrum as the perturbing forces, owing to the linearity of the basic equations.[26]

Laplace's memoir looked and still looks forbiddingly complex, not only because of the idiosyncratic notation and the elliptic style, but also because most of his developments were purely algebraic. Physical discussion was confined to the first assumptions and to the final results, whereas a modern tide-theorist would anticipate and comment the intermediate algebraic steps by appealing to general notions of forced oscillations and wave propagation. That Laplace's equations in fact describe a wave motion modified by the Coriolis force, is easily seen by combining them to get, for $\Omega = 0$ and constant h,

$$\frac{\partial^2\zeta}{\partial t^2} - gh\Delta\zeta = -h\Delta(U + \delta U), \tag{51}$$

where Δ is the two-dimensional laplacian. Although Laplace must have recognized d'Alembert's equation of vibrating strings, he did not exploit this analogy in his theory of tides. Instead, he appended to this theory the water-waves calculations with which our story began.

Laplace's theory was completely alien to contemporary British physics, which remained dependent on older Newtonian methods and professed to ignore French mathematical physics. In 1813 the founder of the wave theory of light, Thomas Young, judged

[26] Kelvin's later tide-predicting machine was based on the same principle.

that the theory of tides was too practically important to be treated with Laplace's abstruse methods. Instead of the learned calculus of partial differentials, he offered a simple analogy between the sea and a pendulum [Young 1823, p. 307]:

> The oscillation of the sea and of lakes, constituting the tides, are subject to laws exactly similar to those of pendulums capable of performing vibrations in the same time, and suspended from points which are subjected to compound regular vibrations, of which the constituent periods are completed in half a lunar and half a solar day.

In modern words, he assimilated tides with the forced oscillations of harmonic oscillators subjected to the superposition of two periodic forces.

In order to justify this analogy (perhaps suggested by Laplace's equations), Young first showed that in a canal of constant depth h, long waves of small amplitude were propagated with the Lagrangian velocity $c = \sqrt{gh}$. If the canal was terminated by a wall at one end, stationary waves occurred. If it had the finite length L, the period of the oscillations could only be an integral multiple of a fundamental period L/c, as in closed organ pipes. Further assuming that the sea was equivalent to a canal along its greatest length, Young replaced it with a set of pendulums that had the same periods. He was thus left with the elementary problem of determining the response of a damped harmonic oscillator to a sinusoidal excitation [1813, 1823].

As is now known to any physics undergrad, the general solution to this problem is the sum of a free oscillation that exponentially decreases in time owing to the damping force, and a forced oscillation whose amplitude varies as $(\omega_0^2 - \omega^2)^{-1}$ if the eigenfrequency ω_0 is not too close to the excitation frequency ω. When the former frequency exceeds the latter, the forced oscillations are in phase with the exciting ones. In the opposite case, the two oscillations are in opposition. This result has an immediate, fruitful application: for the known order of magnitude of the depth and size of the oceans, their fundamental period of oscillation is much larger than half a day, so that the phase of tidal oscillation is opposed to the phase of the inducing luminary (as in Fig. 17b). Through equally elementary reasoning, Young explained several other well-known properties of the tides.

Russell was apparently unaware of Young's earlier insights when he proposed his wave conception of tides. But he knew about Whewell's successful program of tide observation and prediction. As befits the author of *The history of inductive sciences*, Whewell's approach was inductive [1834, p. 19]:

> I believe the instances are comparatively few in the history of philosophy, in which the general laws of the phenomena have been pointed out by the theory before they had been gathered by observation. The law of the tides, thus empirically obtained, may be used either as tests of the extant theories, or as suggestions for the improvement of those portions of mathematical hydraulics on which the true theory must depend.

Like a Ptolemean astronomer, Whewell tried to fit the results of measurements into simple harmonic formulas. He thus reduced the tides in a given port. In order to connect tides observed in different locations, Whewell [1833, p. 148] followed Young's suggestion to draw "cotidal maps" that represented lines of high water at successive hours on a day of full moon (Fig. 18). According to Young [1823, p. 293], "these lines would indicate. . .the directions of the great waves, to which that of the progress of the

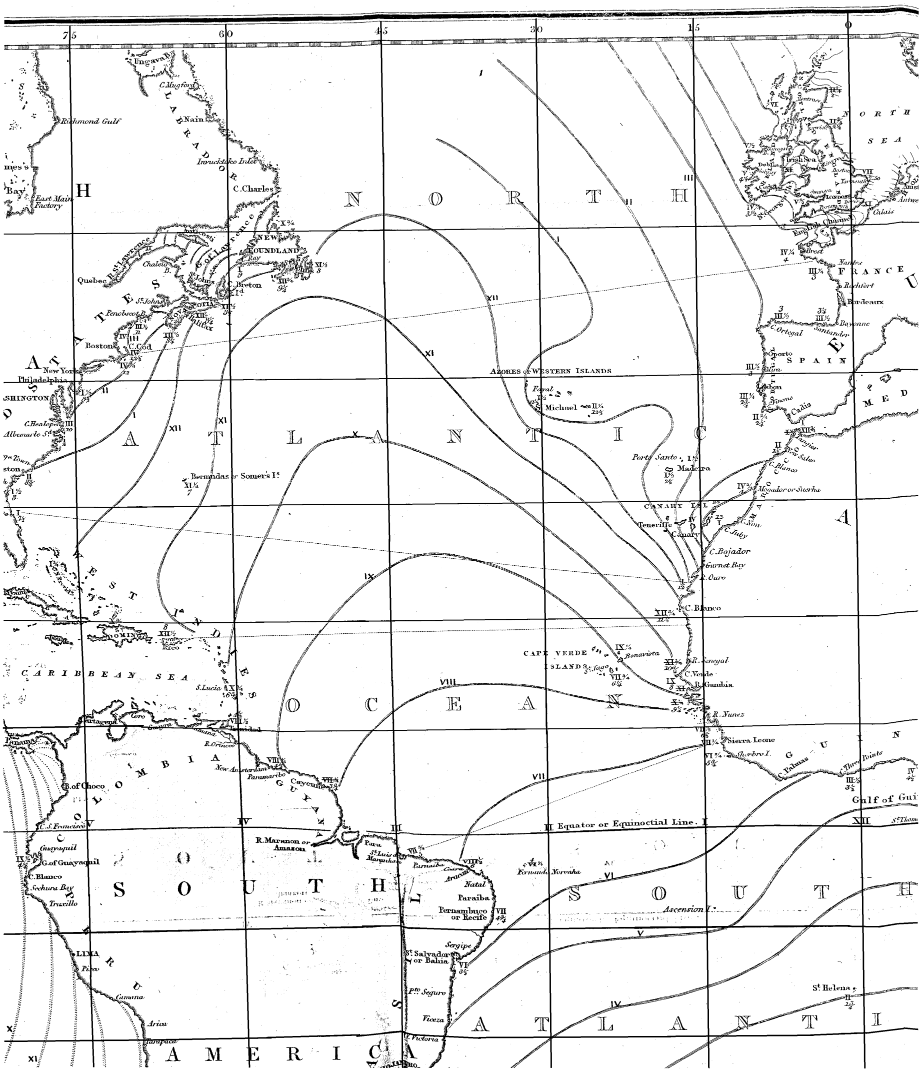

Fig. 18. A portion of Whewell's first cotidal map [1833, plate]

tides in succession must be perpendicular." Although Whewell did not refer to Young and doubted the possibility of theoretically deriving these lines, he allowed himself to identify the cotidal line of a given time with "the summit or ridge of the *tide-wave* at that time" [1833, p. 149]. He described the global forced wave that followed the motion of the moon and the sun, as well as the freely propagating waves in smaller open seas, basins, channels, and rivers. These waves progressed with the depth-dependent velocity that Lagrange had derived and the Weber brothers had verified [p. 212].

Airy's wave theory of tides

The Astronomer Royal George Biddell Airy was also unaware of Young's wave theory when he wrote the article "Tides and waves" for a 1845 volume of the *Encyclopaedia Metropolitana*. But he was familiar with Whewell's and Russell's tide studies. He did not cite Russell as a stimulus for his own theory, presumably because he had a poor opinion of Russell's theories in general. After noting the "great value" of Russell's experiments, he warned the reader "against attaching any importance to the theoretical expressions which are mingled with them in the original account" [1845, p. 350].

As an eminent representative of the new generation of British natural philosophers who had thoroughly assimilated the methods of French mathematical physics, Airy was not only able to condemn Russell's loose theorizing but also to precisely assess the merits of Laplace's formidable calculations. Whereas he found obscurities and even mistakes in this theory, his overall judgment was admiring [p. 279]:

> We must allow [Laplace's theory] to be one of the most splendid works of the greatest mathematician of the past age. To appreciate this, the reader must consider, first, the boldness of the writer who, having a clear understanding of the gross imperfection of the methods of his predecessors, had also the courage deliberately to take up the problem on grounds fundamentally correct...; secondly, the general difficulty of treating the motions of fluids; thirdly, the peculiar difficulty of treating the motions when the fluid covers an area which is not plane but convex; and, fourthly, the sagacity of perceiving that it was necessary to consider the Earth as a revolving body, and the skill of correctly introducing this consideration. This last point alone, in our opinion, gives the greater claim for reputation than the boasted explanation of the long inequality of Jupiter and Saturn.

Airy's main reason for abandoning Laplace's theory was not its mathematical difficulty nor any fundamental incorrectness in its assumptions, but the practical impossibility to solve the tidal equations for the actual form of the bottom of the sea [p. 280]. His own approach was based on the properties of canal waves. These directly informed the behavior of river tides. They also shed light on oceanic tides, as far as an ocean could be replaced by a series of adjacent canals. Accordingly, Airy began with a thorough analysis of wave propagation in a canal [pp. 281ff]. Lagrange's theory was too restrictive since it only applied to small, long waves. Cauchy's and Poisson's theories were even less relevant, since they supposed a mode of production of the waves that was never encountered in tide theory.

Airy's analysis was based on the Lagrangian picture of fluid motion, as was Laplace's theory of 1776. Call X and Y the coordinates of the fluid particles when the fluid is at rest, and $X+\xi$ and $Y+\eta$ their coordinates when the fluid is in motion. As before, the X axis lies along the bottom of the canal, and the Y axis is vertical. Like Laplace, though with more elementary methods, Airy proved that the harmonic expressions

$$\xi = \varepsilon \cosh kY \cos kX \cos \omega t, \quad \eta = \varepsilon \sinh kY \sin kX \cos \omega t \tag{52}$$

with $\omega^2 = gk \tanh kh$ satisfied the continuity equation, the equations of motion, and the boundary conditions as long as the motion was small. Unlike Laplace, he combined this solution with the other solution

$$\xi = -\varepsilon \cosh kY \sin kX \sin \omega t, \quad \eta = \varepsilon \sinh kY \cos kX \sin \omega t \tag{53}$$

to get the solution [p. 290]

$$\xi = \varepsilon \cosh kY \cos(kX - \omega t), \quad \eta = \varepsilon \sinh kY \sin(kX - \omega t) \tag{54}$$

that propagates with the velocity $c = \omega/k$ such that

$$c^2 = (g/k) \tanh kh. \tag{55}$$

In this state of motion, the fluid particles perform elliptical oscillations that tend to circular ones for infinite depth. As Airy noted, this result agrees with the earlier observations of suspended solid particles made by the Webers and by Russell [pp. 344, 347].

From river tides to ocean tides

In the case of tides, the wave-length is much larger than the depth. Then the previous equations imply that the horizontal motion is sensibly the same from the surface to the bottom, and the vertical motion is comparatively very small [p. 294]. Airy assumed this property to hold even in the case of river tides, for which the elevation of the water was no longer negligible compared to the depth. This enabled him to reach more exact, non-linear equations of motion. He reasoned as follows.

The volume of the vertical slice of fluid comprised between the planes X and $X + \delta X$ is $h\delta X$ in the undisturbed condition, and $[X + \delta X + \xi(X + \delta X) - X - \xi(X)](h + \sigma)$ in the disturbed condition (σ denotes the elevation of the surface above its original height h). Therefore, the continuity of the fluid implies

$$\left(1 + \frac{\partial \xi}{\partial X}\right)\left(1 + \frac{\sigma}{h}\right) = 0. \tag{56}$$

The pressure on each side of the slice varies hydrostatically, since the vertical acceleration is neglected. Therefore, its longitudinal gradient only depends on the slope of the surface:

$$\frac{\partial P}{\partial X} = \rho g \frac{\partial \sigma}{\partial X}. \tag{57}$$

Newton's second law applied to the fluid slice then gives:

$$\rho h \delta X \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial P}{\partial X}\delta X(h + \sigma) = -\rho g \delta X(h + \sigma)\frac{\partial \sigma}{\partial X}. \tag{58}$$

Eliminating σ through the continuity equation Airy finally reached [p. 297]

$$\frac{\partial^2 \xi}{\partial t^2} = gh \frac{\partial^2 \xi}{\partial X^2}\left(1 + \frac{\partial \xi}{\partial X}\right)^{-3}. \tag{59}$$

He solved this equation perturbatively. The motion being $\xi_0 = \varepsilon \cos(\omega t - kX)$ in the lowest approximation, he obtained the next approximation ξ_1 by integrating the equation

$$\frac{\partial^2 \xi_1}{\partial t^2} - gh \frac{\partial^2 \xi_1}{\partial X^2} = -3gh \frac{\partial^2 \xi_0}{\partial X^2}\frac{\partial \xi_0}{\partial X} \tag{60}$$

with the condition that for $X = 0$ the oscillation should still be $\varepsilon \cos \omega t$. This gives, for the corresponding elevation of the surface [p. 300],

$$\sigma_1 = -a\sin(\omega t - kx) + \frac{3}{4}\frac{a^2}{h}kx\sin 2(\omega t - kx) \tag{61}$$

with $a = kh\epsilon$ and $x = X + \xi_1$.

This solution represents the evolution of a tidal wave as it propagates from the mouth $x = 0$ along a flat, prismatic river without intrinsic current.[27] As is seen from Fig. 19, the front of the waves becomes steeper than the rear. This explains why the rise of the water takes more time than its descent at a station far from the mouth [p. 300]. Airy further derived the velocity of the wave crests (for which $\mathrm{d}\sigma_1/\mathrm{d}x = 0$) at the same approximation [p. 301]:

$$c = \sqrt{gh}\left(1 + \frac{3}{2}\frac{a}{h}\right). \tag{62}$$

He found this formula to be compatible with the velocity measurement of high waves by the Webers and Russell, despite Russell's claim that the velocity of a solitary wave of height σ obeyed the formula $c = \sqrt{g(h+\sigma)}$.[28]

In the case of oceanic tides, the height of the waves is negligible compared to the depth, so that the continuity equation (56) and the equation of motion (58) can be linearized. However, the direct action of the moon and the sun is no longer negligible. The vertical component of this action amounts to a negligible modification of gravity. But the equation of motion now includes the horizontal component F of this action:

$$\frac{\partial^2\xi}{\partial t^2} - gh\frac{\partial^2\xi}{\partial x^2} = F. \tag{63}$$

F is the superposition of harmonic components with a latitude-dependent phase. Airy determined the resulting forced oscillations for circular canals running along a parallel and along a meridian, and for canals interrupted at both ends. In each case, there are free oscillations at frequencies that are integral multiples of a fundamental frequency. The amplitude of the forced oscillations depends on how close these eigenfrequencies are to the frequencies of the tidal force F. Airy also introduced friction proportional to the velocity, and discussed the resulting damping of free oscillations [pp. 310–339].[29]

In conclusion to this analysis [p. 363], Airy admitted that his assumption of tidal canals of uniform depth and breadth was no more realistic than Laplace's assumption of an earth-covering ocean with a special law of depth. The main advantage he saw in his method was that it permitted a more detailed consideration of the interplay of the lunar, solar, and frictional forces, since all the equations could be solved in finite terms through elementary analysis. In brief, his theory failed as much as Laplace in quantitative tide prediction; but it offered more qualitative insights.

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[27] Airy believed the solution to be still valid far from the mouth. In fact, the consistency of the approximation requires that $x \ll h/ka$.

[28] As Stokes, Saint-Venant, and Boussinesq later made clear, Airy's formula applies to the crest of long, non-permanent waves, whereas Russell's formula applies to permanent waves whose length is comparable to the depth of water. Simple derivations of Airy's formula are found in Lamb 1932, pp. 261–262, 278–280.

[29] For a concise account of Airy's theory of oceanic tides, cf. Lamb 1932, pp. 267–273.

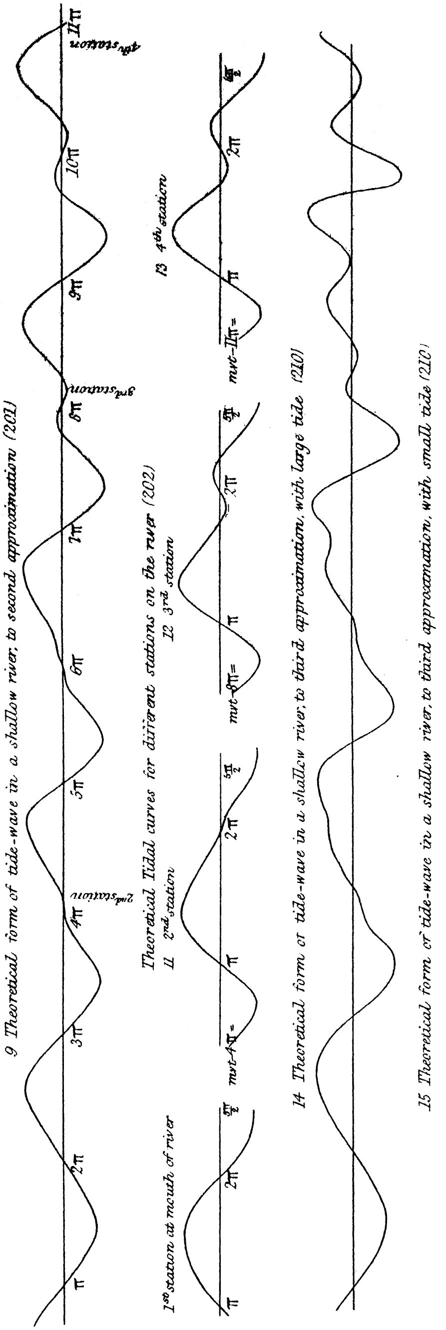

Fig. 19. The evolution of a sine wave along a canal to second and third order [Airy 1845, plate]

The inverse method, for and against Russell

Airy did not confine his study of waves to aspects relevant to tide theory. He also explained commonly known properties of water waves, and some of Russell's more surprising results. As he was generally unable to integrate his hydrodynamic equations for the actual forces that produced the wave motion, he ingeniously inverted this procedure: he sought to compute, for an hypothetical form of fluid motion, the forces that would maintain this motion. This is much easier to do, since differentiations are involved instead of integrations. From the knowledge of these forces, he then inferred what the actual motion would be in their absence; or what additional action on the water could produce the hypothetical motion.

As a first example, consider the breaking of waves on a sloping shore. Airy computed the forces necessary to maintain a constant shape of the waves when they approach the shore [p. 314]. The result is forces that pull the tip of each wave in the direction opposite to that of their progression. As in reality these forces do not act, the tips of the waves must bend forward, as expected at the beginning of the breaking process. Another example is the swelling of waves under wind. Airy injected a swelling motion in the equations of motion. The resulting forces turn out to be pressures applied to the rear of the waves, as would naturally be expected for waves before the wind.

A third example is the "great primary wave," or forced wave that accompanies a canal boat in its motion [pp. 349–350]. In this case, the horizontal disturbance ξ and the surface disturbance σ are functions of $x - vt$ only, where v is the velocity of the boat. In the small-long-wave approximation, the equation of motion (63) gives $F = (v^2 - gh)\xi'$, while the continuity equation (56) gives $\sigma = -h\xi'$. Therefore, the force that is necessary to maintain this motion has the same sign as the slope of the surface when the velocity of the boat is inferior to that of free waves; it has the opposite sign in the reverse case; and it vanishes when the two velocities are equal. This conclusion agrees with the relative position of a canal boat and its forced wave, and with the drop of resistance in the critical case.

Airy thus explained Russell's observations but implicitly rejected his intuitive theory of solitary wave riding. Through the same kind of argument, he dismissed Russell's solitary wave. For waves of finite height, the equation of motion is the non-linear equation (59). Without additional force and for a disturbance propagating without any change of shape, it can only hold if the slope ξ' of the disturbance is a constant. As this slope vanishes at infinity, there is no such disturbance. Airy's verdict was clear [pp. 346–347]: the solitary wave is mathematically impossible. What Russell had observed was a wave small enough for Lagrange's theory to apply approximately [p. 346][30]:

> We are not disposed to recognize this wave [Russell's] as deserving the epithets "great" or "primary"... and we conceive that, ever since it was known that the theory of shallow waves of great length was contained in the equation $\partial^2\xi/\partial t^2 = gh\partial^2\xi/\partial x^2$ ...the theory of the solitary wave has been perfectly well known.

[30] As Stokes later noted, Airy overlooked the fact that his equation of motion applied to waves longer than those observed by Russell.

As we already saw, this authoritative judgment failed to disturb Russell's belief in the novelty of his solitary waves.

4. Finite waves

Stokes' BA report

In 1846 the new leader of British hydrodynamics, the Cambridge professor George Gabriel Stokes, reviewed the state of this field for the British Association. Since the previous report by Challis, there had been much British work on waves, in a good part stimulated by Russell's experiments. Stokes played down the importance of Poisson's and Cauchy's memoir: "The mathematical treatment of such cases [waves produced by emersion] is extremely difficult; and, after all, motions of this kind are not those which it is most interesting to investigate." In the wake of Russell's and Airy's works on waves, tides, and navigation, what had become most important was the study of "simpler cases of wave motion, and those which are more nearly connected with the phenomena which it is most desirable to explain" [p. 161].[31]

Among the simpler cases of motion, Stokes retained waves with a length much longer than the depth [pp. 161–164]. As Lagrange [1781], George Green [1838], Philip Kelland [1840], and Airy [1845] had shown, these waves propagated without deformation in a canal of constant section as long as their height was much smaller than the depth. Their velocity obeyed a simple formula. Green and Airy had computed their deformation for a slowly varying canal depth or breadth. Airy had shown how finite height affected their propagation. Stokes also dwelled on the fruitful application that Airy had given of this sort of waves to the theory of tides [pp. 171–175].

Another case of special interest was given by "waves which are propagated with a constant velocity and without change of form, in a fluid of uniform depth, the motion being in two dimensions and periodical." In an implicit analogy with monochromatic plane waves in optics, he regarded these waves as "the *type* of oscillatory waves in general"[p. 164]. Green [1839] had given the expression $\sqrt{g/k}$ of the velocity of such waves in the case of infinite depth, and Kelland [1840] had anticipated Airy's results in the case of finite depth.[32]

Stokes then turned to the controversial issue of solitary waves [pp. 168–169]. Stokes admitted that Russell's experiments made the *sui generis* character of solitary waves

[31] This opinion echoed an earlier remark by Kelland [1840, p. 497]: "I doubt much. . .whether such men as Laplace and Lagrange would have been induced, with the expectation of joining experiment on her lower and more trodden fields, to reconsider and remodel their investigations; nor have I any reason to hope, that such men as Poisson and Cauchy will quit the delectable atmosphere in which they are involved, of abstruse analysis, for the more humble, but not less important task of endeavouring to treat the simpler problems in a manner not made general arbitrarily to lead to the most elegant formulae, but general to that extent, and in that mode, in which the problem in nature is so."

[32] Kelland believed the motion to have a form independent of the height of the waves, for he used erroneous boundary conditions.

probable. But he denied that friction was the only cause of decay of such waves. To sustain this opinion he did not use Airy's objection, which only excluded solitary waves of arbitrarily long length. He rather referred to recent calculations by Samuel Earnshaw [1849; read in Dec 1845]. The Reverent mathematician had integrated the equations of motion for a wave of permanent shape that met a condition experimentally verified by Russell: that fluid particles originally in the same vertical plane remained so during the passage of the wave. In Earnshaw's opinion this result confirmed the existence of solitary waves. Stokes drew the opposite conclusion from the same calculation, for he noted that Earnshaw waves could not be connected to the surrounding fluid at rest without an absurd discontinuity of the velocity. As Stokes did not question the experimental truth of parallel-plane motion, he concluded to a necessary non-frictional decay of solitary waves [pp. 169–170].

Not only Stokes denied the properties of solitary waves that Russell judged most essential, but he also condemned – without naming Russell – applications of solitary waves to tides and to sound [p. 170]:

> With respect to the importance of this peculiar wave. . .it must be remarked that the term *solitary wave*, as so defined [as a phenomenon *sui generis*] must not be extended to the tide wave, which is nothing more. . .than a very long wave, of which the form may be arbitrary. It is hardly necessary to remark that the mechanical theories of the solitary wave and the aërial sound wave are altogether different.

Stokes on finite oscillatory waves

In 1846 Stokes believed permanent, solitary waves of finite height to be impossible. But the existence of permanent, oscillatory waves of finite height remained plausible. Also, Russell had found that the (phase) velocity obeyed the Kelland-Airy formula (55) (for infinitely small waves) even when the waves were no longer small with respect to the depth. Stimulated by this result and its apparent contradiction with Airy's velocity formula (62) for finite waves, Stokes sought a perturbative solution of Euler's equations that made the fluid velocity the gradient of a potential and a function of $x - ct$ and y only [Stokes 1847].

As in Lagrange's theory of waves, the potential must satisfy

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0. \tag{17}$$

The equation of the surface is

$$\frac{\partial \varphi}{\partial t} + \frac{(\nabla \varphi)^2}{2} + g(y - h) = 0. \tag{64}$$

The boundary condition at the bottom of the channel is $\partial\varphi/\partial y = 0$ when $y = 0$. The condition that a particle on the surface should remain on the surface is

$$\left(\frac{\partial}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial y}\right)\left(\frac{\partial \varphi}{\partial t} + \frac{(\nabla \varphi)^2}{2} + g(y - h)\right) = 0 \tag{65}$$

at any point of the surface. The general integral of Eq. (17) that meets the first boundary condition is

$$\varphi = Cx + \sum_k \cosh ky(A_k \cos kx + B_k \sin kx). \tag{66}$$

The first term may be dropped as it represents a constant velocity. Then, to first order in φ the second boundary condition and the condition that the velocity is a function of $x - ct$ and y only imply that

$$c^2 = (g/k)\tanh kh \tag{55}$$

for every term of the sum over k. As there is only one value of k that meets this condition, the sum is reduced to a sine wave [pp. 199–204].

As a corollary, the propagation of a solitary wave without change of form is impossible at first order. In modern terms, we would say that infinitely small monochromatic water waves are submitted to a dispersion (dependency of celerity on wave-length) that implies a spread of wave packets. Stokes commented [p. 204]:[33]

> Thus the degradation in the height of such waves, which Mr. Russell observed, is not to be attributed wholly, (nor I believe chiefly,) to the imperfect fluidity of the fluid. . .but is an essential characteristic of a solitary wave. It is true that this conclusion depends on an investigation which applies strictly to indefinitely small motions only: but if it were true in general that a solitary wave could be propagated uniformly, without degradation, it would be true in the limiting case of indefinitely small motions; and to disprove a general proposition it is sufficient to disprove a particular case.

After this new blow to Russell's interpretation of the solitary wave, Stokes proceeded to a theoretical justification of Russell's experimental results on oscillatory waves [pp. 205–208]. To second order in the amplitude a of the wave, and for infinite depth, he found

$$y = h + a\cos kx - \frac{1}{2}ka^2 \cos 2kx \tag{67}$$

for the equation of the surface, and the same expression $\sqrt{g/k}$ of the wave velocity as in the first approximation. To third order, the equation of the surface becomes

$$y = h + a\cos kx - \frac{1}{2}ka^2 \cos 2kx + \frac{3}{8}k^2a^3 \cos 3kx; \tag{68}$$

and the velocity formula is modified to

$$c = \sqrt{g/k}\left(1 + \frac{1}{2}k^2a^2\right). \tag{69}$$

Hence the celerity depends very little on height, as found by Russell. Airy's formula (62) does not apply in this case, Stokes explained [p. 209], because it assumes waves much

[33] This objection is invalid, because it assumes that the length of the waves is kept constant in the zero-amplitude limit, whereas for a solitary wave the length grows indefinitely when the amplitude goes to zero.

(a)

(b)

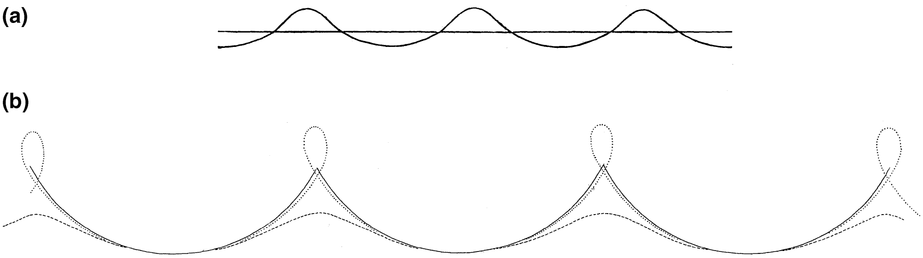

Fig. 20. Wave of finite height according to Stokes's theory [1847, p. 212] (**a**); according to Russell's cycloidal interpretation of ocean waves (**b**)

longer than the depth, whereas the smallness of Stokes' perturbations is easily seen to exclude this assumption.[34] About the form of the waves, Stokes agreed with the cycloid that Russell had inferred from observations of high sea waves (Fig. 20). Lastly, Stokes found that for high waves, the propagation of the waves was accompanied with a net flux of water. He even suggested to take into account this flux in the dead reckoning of the position of ships [pp. 198–199, 208–209].

Gerstner's waves and ship rolling

Stokes returned to water waves in the 1870s, when he had to write a memorandum on the measurement of waves for the Meteorological Council.[35] A good knowledge of the height and length of sea waves, he argued, was necessary for a proper control of ship rolling. This preoccupation might have led him to improve his theory of high waves and to reflect on the highest possible wave. Another incentive must have been his discussions with William Thomson, who had become involved in similar questions. In 1880 the publication of the first volume of Stokes' collected papers gave him the opportunity to update his views on this topic.

In the first place, Stokes expressed his opinion on an old theory of finite, oscillatory waves on infinitely deep water that had become popular among naval engineers. This theory, published in 1802 by the Prague mathematics professor Franz Joseph Gerstner, assumed a circular motion of the fluid particles with a radius diminishing with the distance from the surface:[36]

$$x = X + k^{-1}e^{kY}\cos k(X - ct), \quad y = Y - k^{-1}e^{kY}\sin k(X - ct), \tag{70}$$

where x and y are the coordinates at time t of the particle whose mean coordinates are X and Y, $2\pi/k$ is the wave-length, and c is the celerity of the wave. This motion is easily

[34] Moreover, Airy dealt with a different problem: the deformation of a wave that has a sine shape near the origin.

[35] Cf. Froude to Stokes, 17 Jan 1873, in Stokes 1907.

[36] This motion has the same form as the large depth limit of Airy's Eq. (54) for infinitesimal oscillatory waves. The only difference is that for Airy the surface of the water could only correspond to a large negative value of Y, whereas for Gerstner any negative value would do.

seen to satisfy the continuity condition and the equations of motion. The pressure for a given fluid particle is independent of time if and only if $c = \sqrt{g/k}$. It then is a function of Y only, so that the wave surface can be any of the lines for which Y is a negative constant. Figure 21 pictures the resulting waves for different values of this constant. The highest waves, for which the constant vanishes, have an infinitely sharp edge. Their surface is a cycloid generated by a circle of radius k^{-1} rolling on the under side of the line $y = k^{-1}$. The other waves are trochoids with an eccentricity decreasing with their amplitude. Gerstner believed his waves to derive from the general principles of mechanics. In fact, as the Leipzig mathematician Ferdinand Moebius noted some twenty years later, Gerstner's derivation relied on the specific assumption that the pressure around any particle of the fluid remains the same in the course of time (whereas general principles require this to be true only for the particles at the open surface of the fluid).[37]

The Webers' *Wellenlehre* [1825, pp. 338–372] included a detailed analysis of Gerstner's waves. They found reasonable agreement with the observed motion of suspended particles, although the radius of the circular motions did not quite vary as Gerstner predicted. Their overall judgment was laudatory: "Even if these conditions [for Gerstner's calculation to apply] are not completely met in reality, Gerstner's investigation remains not only interesting but also useful" [p. 368]. Russell, who became acquainted with Gertsner's waves through the Webers' book, found even better agreement with observation than the Webers had found. His judgment was enthusiastic: "Gerstner's theory is characterized by simplicity of hypothesis, precision of application, its conformity with the phaenomena, and the elegance of its results" [1845, p. 368n].

In the 1860s, British and French interest in ship rolling led to three rediscoveries of Gerstner's waves, by the Edinburgh engineering professor William Rankine [1862], by the naval engineer William Froude [1862], and by the Director of the Ecole du Génie Maritime Frédéric Reech [1869]. When, in the early 1870s, the leading French expert on applied mechanics Adhémar Barré de Saint-Venant and his disciple Joseph Boussinesq became aware of Gerstner's theory, they fully endorsed it [Saint-Venant 1871b; Boussinesq 1877, p. 345–346]. As they noted, Gerstner's waves imply a rotational motion of the water and therefore cannot be regarded as generated by pressures acting on a perfect liquid originally at rest. In their eyes, this fact did not preclude the application to sea waves, for the latter usually had a long history in which the imperfect fluidity of water plausibly played a role. Stokes judged differently [1880a]. In his view, only irrotational waves could be produced by natural causes. Consequently, these waves were worth analytical efforts despite the much greater simplicity of Gerstner's waves.

From wedge-shaped waves to solitary waves

Next to his dismissal of Gerstner's waves, Stokes inserted a supremely elegant proof that if the crest of an irrotational wave had a sharp edge, this edge necessarily made

[37] Cf. Stokes 1880a; Lamb 1932, pp. 421–423; Weber 1825, p. 368 (for Moebius' remark). In Gestner's original reasoning [1802], steady waves are investigated first, and a uniform translation is superposed to these waves to yield progressive waves.

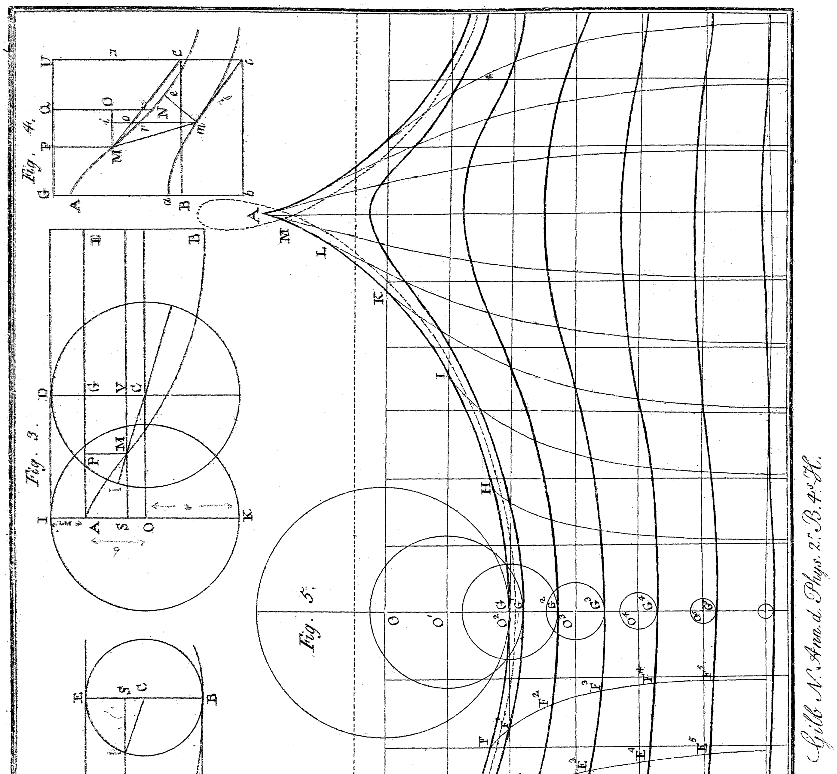

Fig. 21. Gerstner's waves [1802, plate]. The lines $B_iA_iB_iC_iD_i$ represent possible wave profiles; the circles represent the orbits of fluid particles; the remaining lines represent the successive forms of a line of particles that is vertical when passed by a crest or a through

an angle of 120° [1880b]. As a fluid particle travels along the surface, its velocity (in a reference system in which the wave is stationary) must vanish at the angular points. At a short distance r from such a point, the velocity must vary as $\sqrt{r}$ according to Bernoulli's law. The irrotational character of the wave implies the existence of a velocity potential. As this potential is harmonic, it is the real part of a function of the complex variable $x + iy$ that can be developed in whole powers of this variable. Taking the origin of coordinates at an angular point, this implies that near this point the potential behaves as the real part of a power of $x + iy$. In polar coordinates, this gives the form $\varphi \propto r^n \cos n\theta$. On the vertex, the normal velocity $\partial\varphi/r\partial\theta$ must vanish, and the tangential velocity $\partial\varphi/\partial r$ must be proportional to $\sqrt{r}$. The latter condition implies $n = 3/2$. The former then requires that the angle of the vertex should be 120°.

By 1880 Stokes believed that he highest possible wave (for a given wave length) had this 120° cusped shape. Yet his correspondence with Thomson shows that a few months earlier he still hesitated [20 Sep 1879, *ST*]. It also shows that he sought opportunities to verify this prediction: "I have in mind when I have occasion to go to London to take a run down to Brighton if a rough sea should be telegraphed, that I may study the forms of waves about to break. I have a sort of imperfect memory that swells breaking on a sandy beach became at one phase very approximately wedge-shapes" [11 Oct 1879, *PST*]. During the next summer, Thomson invited him "to see and *feel* the waves" on his yacht [14 Jul 1880]. In the fall, Stokes wrote to his friend [15 Sep 1880]:

> You ask if I have done anything more about the greatest possible wave. I cannot say that I have, at least anything to mention mathematically. For it is not a very mathematical process taking off my shoes and stockings, tucking up my trousers as high as I could, and wading out into the sea to get in line with the crest of some small waves that were breaking on a sandy beach.

These brave observations seemed to confirm the 120° edge for the highest possible waves.

From a theoretical point of view, what convinced Stokes of the existence of wedge-shaped waves was a new perturbation method that enabled him, in the fall of 1879, to pursue the calculation of finite oscillatory (irrotational) waves to a much higher order than he had done before. The trick was to simplify the expression of the boundary conditions by using the potential φ and Lagrange's stream function ψ (the harmonic conjugate of φ) as independent variables instead of the coordinates x and y. The calculations clearly indicated that in the limiting case for which the series started to diverge, the 120° cusp shape was reached. In the case of infinite depth this occurred for a definite value of the amplitude/wavelength ratio; in the case of finite depth, for a definite value of the amplitude/depth ratio. Most interestingly, Stokes found that in the latter case the waves "tend[ed] to assume the character of a series of disconnected solitary waves." [1880c, p. 325]. On 6 October 1879, this finding prompted him to write to Thomson [*ST*]:

> Contrary to an opinion expressed in my [BA] report [of 1846], I am now disposed to think there is such a thing as a solitary wave that can be theoretically propagated without degradation.

Thomson disagreed [10 Oct 1879, *ST*]: "The more I think of it the more I am disposed to conclude that there is no such thing as a steady free periodic series of waves in water of any depth. I can't believe in the solitary wave." This divergence of opinion came from Thomson's suspicion that Stokes series for finite waves never converged and only indicated *approximately* steady waves. In the following years, there was indeed much controversy about the convergence of these series. It only ended in 1925 with a rigorous existence proof by Levi-Civita [see Lamb 1932, p. 420].

Boussinesq on solitary waves

Unknown to Stokes and Thomson, the mathematical existence of solitary waves had already been argued twice: in 1871 by a remote French theorist, and in 1876 by a rising star of British natural philosophy. The French investigator, Joseph Boussinesq, had been working on open-channel theory for some time. In the steps of his mentor Saint-Venant, he submitted every aspect of the motion of water in rivers and canals to mathematical analysis.[38] He was aware of Russell's observations, and also of more precise measurements of solitary waves performed by the French hydraulician Henri Bazin [1865]. He had already written a long memoir on water waves of small height on water of constant depth [1872; read in 1869]. Most of the results could be found in earlier memoirs by Green, Kelland, and Airy, of which he was unaware. Boussinesq, however, offered a few preliminary considerations on waves of finite height that may have led him to the solitary wave.

In his first derivation of the solitary wave, published in the *Comptes Rendus* for 1871, Boussinesq sought an approximate solution of Euler's equations that propagated at the constant speed c without deformation in a rectangular channel. His success in this difficult task depended on his special flair in estimating the relative importance of the various terms of his developments. His basic strategy was to develop the velocity components u and v in powers of the vertical distance y from the bottom of the channel, and to determine the coefficients of this development through the boundary conditions. Lagrange had already tried this route, and written the resulting series of differential equations. But he found their integration to exceed the possibilities of contemporary analysis, unless non-linear terms were dropped. A century later, Boussinesq managed to include the non-linear terms.

To second order in y, Lagrange's expression (18) of the velocity potential implies the form

$$u = \alpha - \frac{1}{2}\alpha'' y^2, \quad v = -\alpha'' y \tag{71}$$

of the velocity components, where α is a function of x only and the accents denote derivation with respect to x. Call σ the elevation of the surface above its original height. Boussinesq introduced the condition

$$\int_0^{h+\sigma} u \,\mathrm{d}y = c\sigma, \tag{72}$$

[38] Cf. Darrigol 2002b, pp. 228–235.

which is easily obtained by expressing the conservation of flux in a reference system bound to the wave. The resulting constraint on the unknown function α is

$$\alpha(h+\sigma) - \frac{1}{6}\alpha''(h+\sigma)^3 = c\sigma. \tag{73}$$

Boussinesq solved this equation perturbatively. At the lowest order of approximation, the cubic term is neglected on the left side, and σ is neglected with respect to h so that $\alpha = c/h$. At the next order of approximation, the latter value of α is injected in the cubic term, and σ is neglected with respect to h in this term only. This gives

$$\frac{\alpha}{c} = \frac{\sigma}{h+\sigma} + \frac{1}{6}\sigma'' h, \tag{74}$$

and

$$\frac{u}{c} = \frac{\sigma}{h+\sigma} + \frac{1}{6}\frac{\sigma''}{h}(h^2 - 3y^2), \quad \frac{v}{c} = -\frac{\sigma'}{h}y. \tag{75}$$

Boussinesq then obtained the equation of the surface by injecting these expression in the boundary condition[39]

$$u^2 + v^2 - 2\frac{\partial\varphi}{\partial t} + 2g(y-h) = 0 \quad \text{for } y = h+\sigma. \tag{76}$$

As the potential φ is a function of $x - ct$ only, $\partial\varphi/\partial t$ is the same as $-cu$. In order to clarify subsequent approximations, we introduce the dimensionless variables $\varepsilon = \sigma/h$, $\varepsilon' = \sigma'$, $\varepsilon'' = h\sigma''$. As Boussinesq assumed the wave to be small and gently sloped, he treated ε, ε'/ε, and $\varepsilon''/\varepsilon$ as small quantities. The resulting equation of the surface is

$$c^2 = gh\left(1 + \frac{3}{2}\sigma/h + \frac{1}{3}\sigma''/\sigma\right), \tag{77}$$

where terms in ε^2, $\varepsilon'^2/\varepsilon$, ε'' and still smaller terms have been neglected.[40] This equation may be rewritten as

$$\varepsilon'' = 3K\varepsilon - \frac{9}{2}\varepsilon^2, \quad \text{with } K = c^2/gh - 1. \tag{78}$$

A first integration yields

$$\varepsilon'^2 = 3\varepsilon^2(K - \varepsilon). \tag{79}$$

The maximum $\varepsilon' = 0$ of the corresponding curve is reached when $\varepsilon = K$. Consequently the velocity of the wave is related to the height σ_M of its summit through

$$c = \sqrt{g(h+\sigma_M)}, \tag{80}$$

[39] The other boundary condition, that a particle of the surface should remain on the surface, is a consequence of Eq. (72).

[40] In fact, Boussinesq kept the $\varepsilon'^2/\varepsilon$ terms, but neglected them while integrating the equations.

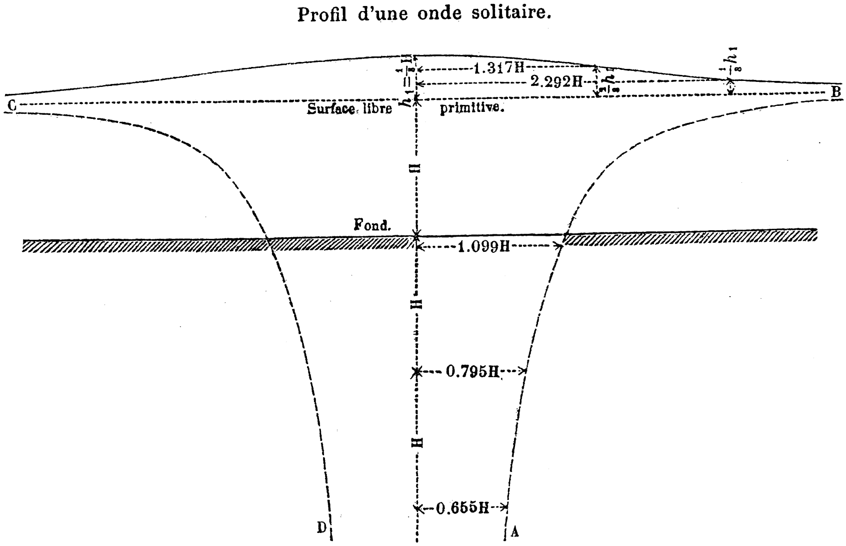

Fig. 22. The profile of a solitary wave (curved solid line) according to Boussinesq [1872, p. 90]

which is Russell's formula. Boussinesq then integrated a second time to reach

$$\frac{\sigma}{h} = \frac{2K}{1 + \cosh[\sqrt{3K}(x - ct)/h]}. \tag{81}$$

His plot of this curve is seen in Fig. 22.

A couple of months later, Boussinesq submitted to the French Academy a more general theory that gave the deformation of a small, gently sloped but otherwise arbitrary wave during its progression in a channel of constant depth [1872].[41] His calculation was still based on Lagrange's development of the velocity potential in powers of y. To fourth order, this development has the form[42]

$$\varphi = \beta - \frac{1}{2}\beta'' y^2 + \frac{1}{24}\beta'''' y^4. \tag{82}$$

The vanishing of pressure at the free surface gives

$$g\sigma + \frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 = 0 \quad \text{for } y = \sigma(x, t). \tag{83}$$

The condition that a particle originally on the surface should remain on the surface gives

[41] For a brief, but accurate discussion of this memoir, cf. Miles 1981. Miles notes that the memoir implicitly contains the KdV equation, but seems to be unaware of the fact that Boussinesq explicity gave this equation in 1877 (see below).

[42] The reader may wonder why Boussinesq now includes the fourth-order term, which he seems to have neglected in his earlier determination of the solitary profile. The reason is that the use of the differential condition (84) instead of the integral condition (72) requires a higher approximation of the potential.

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \sigma}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \sigma}{\partial x} \quad \text{if } y = \sigma(x, t). \tag{84}$$

At the lowest order of approximation, using dots for time derivatives and accents for derivatives with respect to x, these two conditions yield (in the reverse order)

$$\dot{\sigma} = -\beta'' h, \quad \dot{\beta} = -g\sigma. \tag{85}$$

Eliminating β, this gives Lagrange's wave equation

$$\ddot{\sigma} = gh\sigma''. \tag{86}$$

Hence at this order, σ is the sum of a function of $x - c_0 t$ and a function of $x + c_0 t$, with $c_0 = \sqrt{gh}$. Boussinesq retained only the first component, which represents a perturbation traveling at the constant speed c_0 in the direction of increasing x.

At the next order of approximation, the two conditions give

$$\dot{\sigma} = -\beta'' h - \beta''\sigma - \beta'\sigma' + \frac{1}{6}\beta'''' h^3, \quad \dot{\beta} = -g\sigma + \frac{1}{2}\dot{\beta}'' h^2 - \frac{1}{2}\beta'^2, \tag{87}$$

where $h + \sigma$ has been replaced with h in terms that have a derivative of third order or higher in factor. In order to eliminate β, Boussinesq derived the first equation with respect to time and the second equation twice with respect to x. This gives

$$\ddot{\sigma} = -\dot{\beta}'' h - (\dot{\beta}'\sigma)' - (\beta\dot{\sigma}')' + \frac{1}{6}\dot{\beta}'''' h^3, \quad \dot{\beta}'' = -g\sigma'' + \frac{1}{2}\dot{\beta}'''' h^2 - \frac{1}{2}(\beta'^2)''. \tag{88}$$

In the terms that follow the first, dominant term in each of these equations, $\dot{\sigma}$ and $\dot{\beta}$ can be replaced by their first approximation (85), and the operator $\partial/\partial t$ can be replaced with $-c_0^{-1}\partial/\partial x$. This gives

$$\ddot{\sigma} = -\dot{\beta}'' h + g(\sigma^{2'})'' - \frac{1}{6}gh^3\sigma'''', \quad \dot{\beta}'' = -g\sigma'' - \frac{1}{2}gh^2\sigma'''' - \frac{1}{2}gh^{-1}(\sigma^2)''. \tag{89}$$

Hence follows Boussinesq's equation for the evolution of the perturbation [p. 74]:

$$\ddot{\sigma} = gh\sigma'' + \frac{3}{2}g(\sigma^2)'' + \frac{1}{3}gh^3\sigma''''. \tag{90}$$

In order to ease the integration of this equation, Boussinesq imagined a series of fictitious vertical planes moving in such a manner that the volume of liquid between two consecutive planes should remain constant [p. 76]. The velocity w of these planes is easily seen to depend on their abscissa x in such a way that

$$\dot{\sigma} = -(\sigma w)'. \tag{91}$$

With the notation

$$\gamma = \frac{3}{2}g\sigma^2 + \frac{1}{3}gh^3\sigma'', \tag{92}$$

Eq. (90) leads to

$$\partial\sigma w/\partial t + c_0^2\sigma' + \gamma' = 0. \tag{93}$$

Granted that the operator $\partial/\partial t$ can be replaced with $-c_0^{-1}\partial/\partial x$ in front of the small quantity γ, the quantity

$$\psi = \sigma(w - c_0) - \gamma/2c_0 \tag{94}$$

is easily seen to obey the equation

$$\dot{\psi} = c_0\psi'. \tag{95}$$

Hence ψ is a function of $x + c_0 t$ only. As it is a combination of quantities that are functions of $x - c_0 t$ vanishing at infinity, it must vanish. This implies

$$w = c_0 + \gamma/2c_0\sigma, \tag{96}$$

and, approximately [p. 78],

$$w^2 = gh\left(1 + \frac{3}{2}\sigma/h + \frac{1}{3}h^2\sigma''/\sigma\right). \tag{97}$$

Boussinesq then injected his expression of w into Eq. (91) to get the convective variation of the height of the fluid slices as [p. 79]

$$\dot{\sigma} + w\sigma' = -c_0\left(\frac{3}{2}\sigma/h + \frac{1}{3}h^2\sigma''/\sigma\right)'. \tag{98}$$

He also expressed the volume, momentum, and energy of a wave in terms of integrals of functions of h. Most importantly, he identified a fourth invariant of the motion: the "moment of instability" [p. 87]

$$M = \int_{-\infty}^{+\infty} (\sigma'^2 - 3\sigma^3/h^3)\mathrm{d}x. \tag{99}$$

The constancy of this quantity is a consequence of Eq. (98). Boussinesq probably came to suspect its existence while studying the condition of permanent shape.

Remembering that w is the velocity of constant-volume slices of the swell, the shape of a swell will be permanent if and only if w is a constant c representing the celerity of the wave:

$$c^2 = gh\left(1 + \frac{3}{2}\sigma/h + \frac{1}{3}\sigma''/\sigma\right). \tag{77}$$

This condition is identical to that earlier reached by Boussinesq by a more direct method. For any one familiar with the calculus of variations, this equation obviously derives from the condition that the integral M should be a minimum for a fixed value of the integral $\int_{-\infty}^{+\infty}\rho g\sigma^2\mathrm{d}x$ which approximately gives the energy of the wave. As a corollary to this theorem, the moment of instability must be a constant for a solitary wave. Boussinesq presumably guessed that it should be a constant for any swell. Having verified this property, he further inferred that M measured the difference between the departure of a given swell from a solitary wave, or the speed at which its shape varied in time. This remark justified the name "moment d'instabilité." It also led to an explanation of the ease with which Russell and Bazin had produced solitary waves [p. 100]:

> If the moment of instability of a wave slightly exceeds the minimum value, the shape of the swell will oscillate about that of a solitary wave with the same energy, without ever

> differing much from the latter wave: indeed a notable difference would imply an increase of the moment of instability, which is impossible, since this moment does not vary in time; or, rather, a solitary wave will soon be formed; because frictional forces, which we have neglected so far, damp the oscillations of the effective form of the swell about its limiting form.... And we may even conceive, in the absence of any stable form about which a wave might oscillate, that any swell susceptible, by its positive and moderate volume, to form a solitary wave with a height small enough not to break, should assume this form after a certain time. Thus is explained the ease with which solitary waves are produced.

Torrents and tidal bores

Lastly, Boussinesq used the expression (96) of the velocity w of constant-volume slices to determine the evolution of an arbitrary swell. Wherever the curvature σ'' is small compared to σ^2/h^3, this velocity is given by Airy's formula $w = \sqrt{g(h + \frac{3}{2}\sigma)}$. This applies for instance to the case of the flat horizontal part of a swell produced by the continuous injection of fluid at one end of a canal [pp. 100–103]. Boussinesq interest in this case had to do with the distinction between river and torrents and with the theory of river tides.

In 1870, by elementary reasoning based on momentum conservation, Saint-Venant had shown that a step-shaped swell propagated in a prismatic canal at the Lagrangian velocity $\sqrt{gh}$ in a first approximation, at the velocity $\sqrt{g(h + \frac{3}{2}\sigma)}$ in a second approximation (σ being the height of the step). Incidentally, Saint-Venant proposed to call such velocities of wave-propagation "celerity" in order to distinguish them from the fluid velocity. Superposing a uniform flow at the velocity $-\sqrt{gh}$ to this wave motion, Saint-Venant then synthesized a hydraulic jump (*ressaut*), that is, a sudden, variation of the height of water on a constant stream. In a stream of velocity inferior to the critical value $\sqrt{gh}$, any such jump must drift in the downstream direction; in a stream of velocity superior to this critical value, jumps recede in the upstream direction. Therefore, when the water encounters an obstacle in the bed of the stream, it tends to accumulate upstream from the obstacle in the subcritical case; it tends to jump over the obstacle in the supracritical case. The former case defines a river, the latter a torrent according to Saint-Venant.[43]

A few months later, the *Ponts et Chaussées* engineer Henri Partiot gave a theory of river tides based on Bazin's idea [1865, pp. 633–635] that the tidal flux entered the river through a succession of small step-swells propagating at the Lagrangian velocity for the height of the water they encountered during their progression.[44] Following Bazin, Partiot explained the tidal bore or *mascaret* by the fact that successive step-swells encountered higher and higher levels of water and therefore propagated at higher and higher velocities. Consequently, the front of the tidal wave became steeper and steeper. For strong tides or quickly narrowing beds, it could reach the vertical slope for which breaking occurred.

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[43] Cf. Darrigol 2002b, pp. 218–219.

[44] Bazin was himself inspired by Brémontier, who in 1809 analyzed rivers tides in terms of successive laminae of water (though without recourse to Lagrange's formula).

After reading Partiot, Saint-Venant showed that the same evolution of the level of water along the river could be derived from the general equation of non-permanent, gradually varied flow which he obtained by momentum conservation.[45] For small step-swells, this equation leads back to the celerity formula $\sqrt{g(h+\frac{3}{2}\sigma)}$. This result seemed to contradict Russell's and Bazin's $\sqrt{g(h+\sigma)}$ formula. Whereas in his former communication Saint-Venant held friction responsible for the discrepancy, he now understood that the formula of Russell and Bazin applied to situations in which his approximation of gradually varied flow was not allowed. For Russell, σ represented the height of a solitary wave. For Bazin, it represented the height of the surging head of a step-swell, which happened to be fifty percent higher than the step itself.

When Boussinesq wrote on solitary waves, he made clear that Saint-Venant's formula only applied to a portion of a wave in which the curvature could be neglected. In the curved part of the swell, convexity decreases the velocity w, concavity increases it. Through this simple remark, Boussinesq justified the oscillatory shape of the front of Bazin's swell, as well as the oscillations behind Russell's negative waves [1872, pp. 103–108]. In the end there was nothing, in the multifarious wave phenomena that Bazin had observed in open channels, that Boussinesq could not explain through his powerful analysis. Saint-Venant applauded:[46] "These numerous results of high analysis, founded on a detailed discussion and on judicious comparisons of quantities of various orders of smallness, sometimes to be kept, sometimes to be neglected or abstracted, and their constant conformity with the results obtained by the most careful experimenters and observers, appear most remarkable to me."

Rayleigh on the solitary wave

Five years after Boussinesq's note in the *Comptes Rendus*, Lord Rayleigh independently reached the solitary wave equation and profile [1876, pp. 256–261]. With Lagrange and Boussinesq he shared the idea of developing the fluid velocity in powers of the vertical coordinate y. His implementation of this idea was remarkably elegant, thanks to two subterfuges: he analyzed the fluid motion in a reference system bound to the wave; and he conjointly used Lagrange's potential φ and the stream function ψ such that $-v\mathrm{d}x + u\mathrm{d}y = \mathrm{d}\psi$ [Lagrange 1781, p. 720]. The required developments are

$$\varphi = \beta - \frac{y^2}{2!}\beta'' + \frac{y^4}{4!}\beta'''' - \cdots, \quad \psi = y\beta' - \frac{y^3}{3!}\beta''' + \cdots. \tag{100}$$

The stream-line $\psi = 0$ forms the bottom of the channel. In Rayleigh's reference system, the motion is stationary, and the condition that a particle of the fluid surface should remain on this surface is replaced by the condition that this surface should be the stream line $\psi(x, y) = -ch$. The condition of uniform pressure at the free surface is

$$u^2 + v^2 = c^2 - 2g(y - h). \tag{101}$$

[45] Saint-Venant was apparently unaware of Airy's earlier theory.

[46] In his report on Boussinesq's *Eaux courantes* reproduced in [Boussinesq 1877, p. XXI].

Rayleigh then inserted the power developments of φ and ψ in these two conditions, and neglected terms that involved higher orders of derivations with respect to x. This led him to the differential equation[47]

$$\frac{1}{y^2} + \frac{2}{3}\frac{y''}{y} - \frac{1}{3}\frac{y'^2}{y^2} = \frac{1}{h^2} - \frac{2g(y-h)}{c^2h^2} \tag{102}$$

for the function $y(x) = \sigma(x) + h$. The first integral of this equation is the same as Boussinesq's Eq. (79). Rayleigh discussed it and integrated it with results equivalent to Boussinesq's.

The so-called KdV equation

In a note to his monumental *Eaux courantes* [1877, p. 360n], Boussinesq noted that his second-order equation

$$\ddot{\sigma} - c_0^2\sigma'' = \gamma'' \quad (\text{with } c_0^2 = gh \text{ and } \gamma = \frac{3}{2}g\sigma^2 + \frac{1}{3}gh^3\sigma'') \tag{90}$$

for the deformation of a swell during its propagation could be integrated without recourse to the constant slice-motion, by rewriting it as

$$\left(\frac{\partial}{\partial t} - c_0\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c_0\frac{\partial}{\partial x}\right)\sigma = \gamma'' \approx -\frac{1}{2c_0}\left(\frac{\partial}{\partial t} - c_0\frac{\partial}{\partial x}\right)\gamma'. \tag{103}$$

A reasoning similar to that given for the vanishing of the quantity ψ of Eq. (94) leads to the first-order equation

$$\left(\frac{\partial}{\partial t} + c_0\frac{\partial}{\partial x}\right)\sigma = -\frac{\gamma'}{2c_0}, \text{ or} \tag{104}$$

$$\dot{\sigma} = -\sqrt{gh}\left(\sigma + \frac{3}{4}\sigma^2/h + \frac{1}{6}h^2\sigma''\right)'. \tag{105}$$

This is the so-called KdV equation, which Boussinesq wrote some twenty years before his Dutch followers. Rather than this equation, Boussinesq used the equivalent equations (97) for w, and (98) for the convective variation of height, because they represented the deformation of the swell in a more direct manner.

In 1895, the Dutch mathematician Diederik Johannes Korteweg and his doctoral student Gustav de Vries extended Rayleigh's method of 1876 to include oscillatory waves, arbitrary long waves of evolving shape, the effect of capillarity, and an investigation of higher order terms in the Lagrange-Rayleigh expansion. They thus rediscovered the "very important equation" [1895, p. 428] that now bears their name, apparently unaware of Boussinesq's relevant work.[48]

[47] Cf. Lamb 1932, pp. 424–426.

[48] Korteweg and de Vries gave the evolution of the wave in a reference system moving together with the wave. Hence their equation involved an undetermined constant depending on the celerity of the wave. Strictly speaking, they did not write Boussinesq's Eq. (105), which is now called the KdV equation. On the precise connection between their equation and the KdV equation, cf. Miles 1981.

They also extended Rayleigh's derivation to periodic waves of permanent shape, not knowing that Boussinesq had already solved this problem in his *Eaux courantes*[1877, pp. 390–396]. In this case, the condition of constant pressure at the surface involves an undetermined constant, since the disturbance no longer vanishes at infinity. Consequently, Eq. (75) for the slope of the wave is replaced with

$$\varepsilon'^2 = 3(\varepsilon - a)(\varepsilon - b)(k - \varepsilon), \tag{106}$$

where a, b, and k are three positive constants. The integral can be expressed in terms of the elliptic function "cn," which explains the name "cnoidal" that Korteweg and de Vries gave to these periodic waves [p. 424]. As these authors showed, Stokes's finite oscillatory waves are large-depth approximations of the cnoidal waves. Solitary waves correspond to the limit of infinite period.[49]

Korteweg and de Vries believed that the permanence of the shape of their cnoidal waves was preserved at large orders, and gave a tentative proof of this fact [pp. 438–443]. Although there was no consensus on this point before Levi-Civita's proof of 1925, Stokes believed since 1879 in the existence of waves of permanent type, both solitary and oscillatory. In [1891], he identified the false step which had earlier led to the widespread belief in the impossibility of permanent solitary waves. It was assumed that for a given height a solitary wave could be so long that the horizontal velocity was the same on a vertical line. Then Airy's deformation applied and the wave could not be permanent. But the assumption was wrong, since the length of a solitary wave is determined by its height. Stokes could have added that an argument of his own, according to which for a given wave-length the height of a solitary wave could be so small as to undergo finite-depth dispersion similarly fails. As modern soliton theorists know, the possibility of solitary waves rests on the exact compensation between a linear dispersive term and a non-linear term in the equation of motion. For a given height of the wave, this compensation only occurs for a definite shape and length.

5. The principle of interference

Group velocity

In his report on waves of 1844, Russell wrote [1845, p. 67]:

> One observation which I have made is curious. It is that in the case of oscillating waves of the second order, I have found that the motion of propagation of the whole group is different from the apparent motion of wave translation along the surface.

The remark went largely unnoticed, until William Froude privately made a similar observation to Stokes and to Rayleigh in the early 1870s.

At that time Stokes was working on the measurement of sea waves for the Meteorological Council. In particular, he was asked to determine the origin of the strong

[49] Cf. Lamb 1932, pp. 426–427; Miles 1981, p. 137, who notes that KdV's expression of the relation between cnoidal waves and Stokes' waves is not quite correct.

swells sometimes observed in fine weather. Stokes immediately explained these swells by wave propagation from distant storms, and commented: "It is curious to see that captains seem to have so little idea of the propagation of waves excited in a stormy region into a region where as regards the wind, it is comparatively calm" [Stokes to Captain Toynbee, 5 Sep 1878, *SM* 2, p. 141]. According to the formula $c = \omega/k = \sqrt{g/k}$ of small deep-water waves, Stokes explained, the velocity c of a periodic wave is related to its time period $\tau = 2\pi/\omega$ through $c = g\tau/2\pi$. A measurement of τ would thus provide information on the location of the storm [Stokes to Colonel Sabine, 22 Sep 1870, *SM* 2, p. 136].

In 1873 William Froude read the relevant section of Stokes' memorandum. He commented to the author [17 Jan 1873, *SM* 2, pp. 156-157]:

> *Primâ facie*, the speed of such waves would determine the duration of their passage over a given distance. But this is not really so: because the foremost waves are perpetually dying out, as they invade the undisturbed water, and are undergoing metempsychosis in the ranks behind them.

For example, Froude went on, if the wheels of a paddle ship are stopped while its speed is kept constant by other means, the waves remain stationary with respect to the ship but their front moves away from the ship. From the perspective of an observer at rest, this means that the undulations within the train of waves advance faster than the front of the train. Froude had seen a lot of that in his towing tanks.

In January 1876, Stokes reported to Airy [5 Jan 1876, *SM* 2, p. 177]:

> I have lately perceived a result of theory which I believe is new – that the velocity of propagation of roughness on water is, if the water be deep, only half of the velocity of propagation of the individual waves. This is of importance in connecting records of long swells which may be found in ships' logs with records like those of Ascension or St. Helena.

The following month he proposed the following problem for the Smith prize examination papers at Cambridge University [1876]:

> Find the expression for the velocity of propagation of a series of simple periodic waves in water of uniform depth, the motion being small and in two dimensions. – If two such series, of equal amplitude and nearly equal wave-length, travel in the same direction, so as to form alternate lulls and roughness, prove that in deep water these are propagated with half the velocity of the waves; and that as the ratio of the depth to the wave-length decreases from ∞ to 0, the ratio of the two velocities increases from 1/2 to 1.

Calling k and $k + \mathrm{d}k$ the wave numbers of the two superposed waves, and ω and $\omega + \mathrm{d}\omega$ the corresponding pulsations, the amplitude of the superposition varies as $\cos \frac{1}{2}(x\mathrm{d}k - t\mathrm{d}\omega)$. The resulting modulation travels with the velocity $\mathrm{d}\omega/\mathrm{d}k$. For small waves in water of depth h according to Kelland and Airy, $\omega^2 = gk \tanh kh$. The corresponding ratio between the group and phase velocities,

$$(\mathrm{d}\omega/\mathrm{d}k)/(\omega/k) = \frac{1}{2}(1 + kh \tanh^{-1} kh - kh \tanh kh), \tag{107}$$

varies from 1/2 to 1 when kh varies from ∞ to 0, as Stokes asked the Smith prize competitors to demonstrate.[50]

The following year, the Manchester engineering professor Osborne Reynolds reported his own observations of wave groups produced by throwing a stone into a pond, by interference of sea waves, or by the motion of a ship [1877]. Like Russell and Froude, he noted that groups of waves in deep water traveled faster than the individual waves of which they were made. To explain this result, he first noted that the velocity of a wave group obviously represented the velocity of propagation of energy. He then showed that the latter velocity differed from the phase velocity. For instance, the waves produced by wind in a corn field obviously do not propagate any energy, since the motions of the individual corn stems are independent. In the more complex case of a sine wave on deep water, the particles of water move on circles with constant velocity, so that no kinetic energy is transmitted by the wave. In contrast, the potential energy is transmitted at the phase velocity. Since the potential energy of such waves is half their total energy, the speed of energy propagation is half the phase velocity. Therefore, the group velocity is half the phase velocity.

In his influential *Theory of sound* [1877–78], Rayleigh included Stokes' derivation of the group velocity, which he had independently obtained under Froude's stimulus. In a contemporary article [1877], he proved Reynolds' equality between energy and group velocity in a precise mathematical manner. In the case of small waves on water of finite depth, he did this by computing the ratio between the work of pressure forces on a transverse section of the water and the energy density of the waves. In the general case of waves in an arbitrary dispersive medium, he astutely introduced a fictitious friction proportional to the absolute velocity of the parts of the medium. Assuming vibrational energy to be created at $x = 0$ and to propagate in the direction of increasing x, he computed the damping effect of the frictional force by noting that it turned the operator $\partial^2/\partial t^2$ into $\partial^2/\partial t^2 + \mu\partial/\partial t$, wherein μ is the friction coefficient divided by the fluid density ρ. This is nearly equivalent to changing the pulsation ω into $\omega - \frac{1}{2}i\mu$. The corresponding change of k is $-\frac{1}{2}i\mu\mathrm{d}k/\mathrm{d}\omega$. Consequently, the oscillating factor $e^{i(\omega t-kx)}$ of a forced oscillation at the pulsation ω is turned into $e^{-\frac{1}{2}\mu x\mathrm{d}k/\mathrm{d}\omega}e^{i(\omega t-kx)}$. The dissipated energy in the region $x > 0$ is the integral of $\mu\rho v^2$. It is therefore equal to 2μ times the kinetic energy, or else μ times the total energy in this region (according to a well-known theorem for harmonic oscillations). Calling E the energy per unit length of the undamped wave, this remark leads to the expression $\mu E\int_0^{+\infty} e^{-\mu x\mathrm{d}k/\mathrm{d}\omega}\mathrm{d}x = E\mathrm{d}\omega/\mathrm{d}k$ of the dissipated energy. By energy conservation, this dissipation must be compensated by the energy flux Ec_E through the section $x = 0$ of the water. Therefore, the velocity c_E of energy propagation must be identical to the group velocity $\mathrm{d}\omega/\mathrm{d}k$.

The concept of group velocity could plausibly have emerged in the fields of physics where dispersion was first known: optics or in acoustics.[51] In reality it did not. As we just saw, observations made on deep-water waves played a crucial role. Stokes and Rayleigh first introduced this concept for water waves, although their reasoning

[50] A more general argument with a continuous distribution of k is found in Rayleigh 1881.

[51] On a possible anticipation in William Rowan Hamilton's optics, cf. Lamb 1832, p. 381n.

supposed familiarity with interference and beat phenomena in optics and acoustics. Froude's own inspiration came from his engineering concern with the energy carried by the waves.

Thomson's fishing line

In early 1871, the catastrophic sinking of the H.M.S Captain prompted the British Admiralty to name a "Committee on designs for Ships of War." In the name of this committee William Thomson asked his friend Stokes a few questions about waves: "The longest waves that have been observed? – by whom? – their length from crest to crest? – and height from hollow to crest?" [3 Mar 1871, *ST*] The following summer, while sailing on his personal yacht the Lalla Rookh, he observed a gentler but no less interesting phenomenon: a fishing line hanging from the slowly cruising yacht caused very short waves or "ripples" directly in front of the line, and much longer waves in its wake [1871b]. The whole pattern was steady with respect to the line, so that the celerity of both kinds of waves was equal to the velocity of the line's progression through the water. Unknown to Thomson, the French military engineer-mathematician Jean Victor Poncelet [1831] had already described the phenomenon with his colleague Joseph Aimé Lesbros, and Scott Russell had already identified capillarity as the cause of the ripples. Thomson was first, however, to solve the hydrodynamic equations in this case.[52]

In Poisson's manner, Thomson [1871a] sought solutions of the form $\cos(kx - \omega t)$ for the linearized equations of motion. The only difference with the Lagrange-Poisson conditions for the velocity potential is the substitution of $g\sigma - T\sigma''$ for $g\sigma$ in the pressure equation at the free surface, where T is the superficial tension per unit density. Therefore, g must be replaced with $g + k^2T$ in the dispersion formula $\omega^2 = gk$ for waves on deep water. The corresponding celerity is

$$c = \sqrt{g/k + Tk}. \tag{108}$$

Hence for a given value of the celerity there are two possible wave-lengths $2\pi/k$, as observed toward the front and the rear of the fishing line. When c^2 is large compared to $2\sqrt{gT}$, the smaller waves approximately obey $c = \sqrt{Tk}$ as capillarity waves would exactly do, and the larger waves approximately obey $c = \sqrt{g/k}$ as gravity waves would exactly do. Moreover, there should be a minimum velocity below which the waves can no longer be formed. Thomson verified this point on his yacht with the help of an eminent guest, Hermann von Helmholtz [1871c, p. 88].

Rayleigh's solution

Thomson only reasoned on free waves and did not try to put in equations the process through which the fishing line caused the waves. In [1883] Rayleigh accomplished this much more difficult task. Using one of his favorite tricks, he first turned the problem of

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[52] In the same papers, Thomson treated wave-generating instability of a water surface under wind: cf. Darrigol 2002c.

progressive waves into a steady-wave problem by selecting a reference system bound to the perturbing cause (the fishing line). Then he computed the distribution of surface pressure that corresponded to a sinusoidal potential in the restricted two-dimensional problem. Although he only treated the case of infinite depth, the finite-depth formulas are given here to allow a parallel discussion of later similar works.

The assumed expressions of the potential φ and the stream function ψ are

$$\varphi/c = x + \alpha \cosh ky e^{ikx}, \quad \psi/c = y + i\alpha \sinh ky e^{ikx}, \tag{109}$$

where α is a small constant (extraction of the real part of complex expression is understood). The unperturbed motion ($\alpha = 0$) is a uniform flow at the velocity c in the direction of increasing x. The stream line $\psi = 0$ corresponds to the bottom $y = 0$ of the water. The free surface fits the stream line $\psi = ch$. The corresponding surface deformation is

$$\sigma = -i\alpha \sinh kh e^{ikx}. \tag{110}$$

The pressure on the free surface obeys Bernoulli's law modified by capillarity,

$$P/\rho = -g\sigma + T\sigma'' - \frac{1}{2}(u^2 + v^2 - c^2). \tag{111}$$

To first order in the small quantity α, this gives

$$P/\rho = i\alpha[(g + Tk^2)\sinh kh - c^2 k \cosh kh)]e^{ikx}. \tag{112}$$

Consequently, the surface deformation that corresponds to the pressure point

$$P = F\delta(x) = (F/2\pi)\int_{-\infty}^{+\infty} e^{ikx}\mathrm{d}x \tag{113}$$

of intensity F is

$$\sigma = \frac{F}{2\pi\rho g}\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1 + Tk^2/g)(c^2/c_k^2 - 1)}\mathrm{d}k, \tag{114}$$

with

$$c_k^2 = (gk^{-1} + Tk)\tanh kh. \tag{115}$$

When the wave number k is such that the velocity of the corresponding free wave is equal to the velocity c of the stream, this integral is ill-defined. In order to circumvent this difficulty, Rayleigh included a small, fictitious frictional force $\mu(\mathbf{c} - \mathbf{v})$ that damped any free oscillation of the uniform stream. As Rayleigh had already shown in his *Theory of sound* [1877–78, §239], Lagrange's theorem for the existence of the potential remains true in the presence of this force. Its only effect on the previous calculation is an additional term $\mu(cx - \varphi)$ in the pressure equation.

From a formal point of view, Rayleigh thus anticipated the adiabatic turning on of the perturbing force that is commonly used in modern scattering theory. Indeed, a slow variation of the coefficient α implies and additional term $-\partial\varphi/\partial t = (\dot{\alpha}/\alpha c)(cx - \varphi)$ in the pressure equation. This term has exactly the same form as Rayleigh's frictional term.

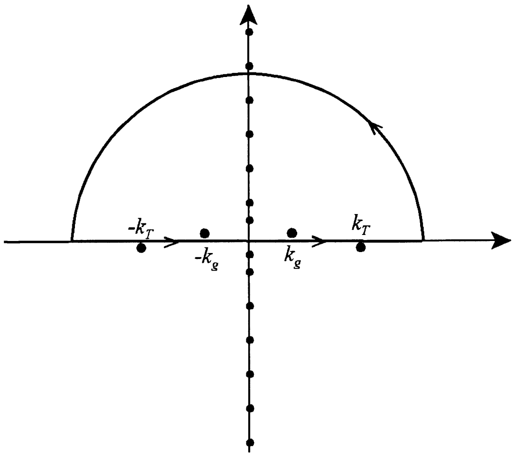

Fig. 23. Integration curve and poles in the complex k-plane for evaluating a certain integral

Taking into account the frictional term, Eq. (114) is replaced with

$$\sigma = \frac{F}{2\pi\rho g} \int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1 + Tk^2/g)(c^2/c_k^2 - 1 - i\varepsilon_k)} \mathrm{d}k, \tag{116}$$

wherein ε_k is a small quantity that has the same sign as k.

In the case of infinite depth, Rayleigh expressed this integral in terms of elementary or already tabulated functions (the sine-integral “Si”). It is more convenient, however, to retain a large but finite depth (for the integrand to be meromorphic) and to make use of Cauchy's theorem of residues.[53] For positive x, the integration path can be closed in the complex k-plane by the upper half of an infinite circle centered on the origin, as shown on Fig. 23. Hence the integral is given by the sum of the residues in the upper half of the complex k-plane. Symmetrically, for negative x the integral is given by the sum of the residues in the lower half of the complex k-plane. The poles of the integrand are represented on the figure. The four poles close to the real axis correspond to the two wave-lengths for which the celerity of free waves is equal to the velocity of the stream. The two poles marked by thick dots on the imaginary axis correspond to the wave-length for which the free waves have minimum velocity. The remaining poles on the imaginary axis correspond to the infinite sequence of imaginary wave-lengths for which the celerity of free waves is equal to the velocity of the stream. Their distance $|k|$ from the origin is approximately given by the successive zeros of the function $\tan |k|h - (c^2/gh)|k|h$.

The contribution of the imaginary poles is a series of terms that decrease exponentially with x. The physically important terms are the oscillatory terms given by the quasi-real poles. For positive x, the two symmetric poles of larger wave-length contribute an oscillation at this wave-length; for negative x, the contributing poles are those of smaller wave-length. Concretely, the pressure point induces shorter capillary waves upstream, and longer gravity waves downstream, in conformity with Thomson's observations.

[53] Cf. Lamb 1895, pp. 396–397; 1932, pp. 406–410.

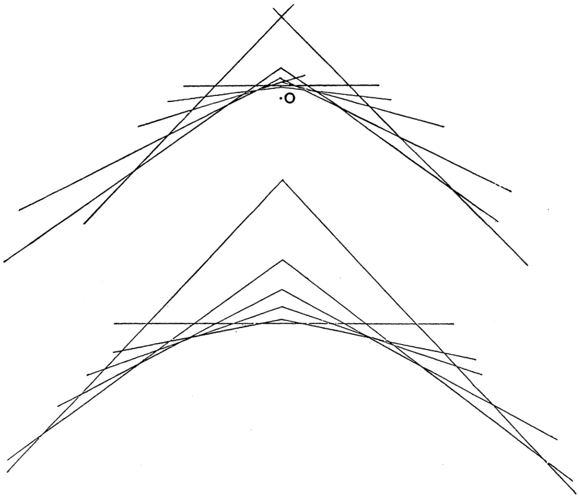

Fig. 24. Rayleigh's construction of the waves created by a drifting fishing-line o [1883, p. 267]

A fuller analysis of the wave pattern created by a fishing line requires a three-dimensional analysis. For this purpose, Rayleigh [1883] superposed the disturbances produced by pressures constantly applied on straight horizontal lines passing through a fixed point of the water surface, the direction of the line being uniformly distributed. The individual wave patterns are those of the two-dimensional problem. Their wave-lengths $2\pi/k$ are such that the corresponding celerity c_k is equal to the projection $c \cos \psi$ of the velocity of the stream on their wave normal. The successive wave crests of each components thus form continuous systems of straight lines whose distance from the origin is a given function of their orientation. Presumably inspired by an analogy with caustic surfaces in optics, Rayleigh obtained the crests of the resultant disturbance as the envelopes of the successive systems of straight lines (see Fig. 24).

Houston's paradox solved

Three years later, Thomson studied the similar problem of the waves produced by a moving boat. In an address to mechanical engineers, he eloquently justified his interest in this topic [1887, p. 410]:

> Of all the beautiful forms of water waves that of Ship Waves is perhaps most beautiful, if you can compare the beauty of such beautiful things. The subject of ship waves is certainly one of the most interesting in mathematical science. It possesses a special and intense interest, partly from the difficulty of the problem, and partly from the peculiar complexity of the circumstances concerned in the configuration of the waves.

In the two-dimensional canal case, Thomson pushed the analysis far enough to explain Houston's old towing paradox. He enthusiastically reported to Stokes [8 Nov 1886, *ST*]:

> I have been getting out some very curious things about waves (water), among them complete confirmation of Scott Russell's doctrine of sudden diminution of force, in towing a *boat* in a canal, when the velocity is got to exceed $\sqrt{gh}$. I find (which is now quite obvious) that if water were inviscible, zero force would suffice to keep a boat moving at any constant speed $> \sqrt{gh}$, whether in a canal or in open water.

As Thomson explained in an evening lecture for a popular audience [1887a], his theory relied on the group-velocity concept, and on balancing the work produced by the towing force and the energy emitted by the boat in the form of waves. The procession of waves behind a boat, he began, is known to be steady with respect to the boat. Therefore the phase velocity of this procession must be equal to the velocity of the boat. According to the Kelland-Airy formula (50), the former velocity cannot be larger than the velocity $\sqrt{gh}$ of infinitely long waves. Therefore, the procession can only exist if the boat moves slower than this critical velocity, in conformity with Russell's "accurate observations and well devised experiments" [pp. 415–420].

As the boat must have started from rest, the wave procession necessarily has a finite length. Its end moves with the group velocity, which is smaller than the phase velocity. Therefore, the length and the energy of the procession increase in time, and an equivalent work must be spent to propel the boat. If the boat moves faster, the procession lengthens at a slower rate but the waves are much higher, so that the resistance grows. A crisis occurs when the velocity of the boat approaches that of infinitely long waves. "Once that crisis has been reached," Thomson declared, "away the boat goes merrily" [p. 418]. Thomson then recalled how "the discovery [had been] made by a horse" and had been exploited for a few years by Scottish canal authorities in a system of fly-boats between Edinburgh and Glasgow on the Forth and Clyde canal, until, in the early 1840s, the development of railways had rendered this poetical notion of speed obsolete [pp. 418–419].

The previous year Thomson had published abundant, complex calculations that justified this theory [1886]. The basic mathematical problem was to determine the disturbance of a uniform flow caused by a local pressure on the water surface of a canal. Thomson first solved the similar problem of the waves produced by a bump at the bottom of the canal when water flows at constant velocity. These two problems resemble Rayleigh's fishing-line problem, except that capillarity is now neglected and depth is finite. Like Rayleigh, Thomson obtained the solution of this problem by superposition of sinusoidal solutions. To a modern reader, his execution of this plan seems awkward. Instead of obtaining a pressure peak by direct superposition of sine functions, he used the mathematical intermediate of a periodic succession of Lorentzian peaks, and then had the distance between too successive peaks go to infinity. He encountered enormous difficulties in evaluating the resulting integral. Yet his results were the same as those of the following calculation based on the method of residues.

For the two-dimensional problem of a local pressure disturbing a uniform flow, the disturbance is given by the particular case

$$\sigma = \frac{F}{2\pi\rho g} \int_{-\infty}^{+\infty} \frac{e^{ikx}}{c^2/c_k^2 - 1 - i\varepsilon_k} \mathrm{d}k \tag{117}$$

of Eq. (116) in which capillarity is neglected and $c_k^2 = gk^{-1}\tanh kh$. The lowest possible value of c_k^2 is its infinite wave-length limit gh. Therefore, when c exceeds $\sqrt{gh}$, the integrand only has imaginary poles, and the integral is an exponentially decreasing function of x. There is no wave production, and the boat can "travel merrily." In the opposite case, the integrand has two symmetric, quasi-real poles $\pm k_g + i\varepsilon$ in the upper half of the complex k-plane that yield a downstream undulation of the water surface, with a period equal to the length $2\pi/k_g$ of free waves traveling at the speed c. This explains why waves are produced by a boat at subcritical speed, and why these waves always *follow* the boat. When the speed c is slightly below the critical velocity $\sqrt{gh}$, the two residues are $(3/2k_g h^2)e^{\pm ik_g}$. The amplitude of the resulting oscillations diverges together with their period $2\pi/k_g$ when the pressure point reaches the critical velocity, in conformity with the "crisis" described in Thomson's popular lecture. In the limit of infinite depth, the two residues are $k_g e^{\pm ik_g x}$, so that the amplitude of the oscillations is inversely proportional to the wave-length. Lastly, Thomson computed the necessary propelling force by balancing the energy flux of the waves with the work done by the propelling force.

Echelon waves

Thomson's greatest achievement in this area was to derive the ship-wave pattern in the three-dimensional case [1887a, 1906].[54] Following Rayleigh, Thomson superposed the disturbances produced by pressures constantly applied on straight horizontal lines passing through a fixed point O of the water surface to be identified with the location of the boat.[55] Call r and θ the polar coordinates of the point P with respect to the origin O and to the velocity $\mathbf{v}$ of the boat, ψ the angle that the rearward wave normal of one of the component waves makes with the axis $\theta = 0$, and $\lambda = g/v^2$ the wave-length of a plane free wave whose celerity is equal to the velocity v of the boat. Then the distance x of the point P from the pressure-line of the component ψ is $r\cos(\psi - \theta)$. The wave-number k of this component must be such that the corresponding wave velocity $\sqrt{g/k}$ be equal to the projection $v\cos\psi$ of the velocity of the boat on the wave normal. Hence the phase kx of this component at point P is

$$\phi = (2\pi r/\lambda)\cos(\psi - \theta)/\cos^2\psi. \tag{118}$$

[54] In 1887, Thomson only gave the formulas for the configuration of the wave crests, which he claimed to have obtained by Stokes' principle of group velocity [p. 423]. In the following it is assumed that the relevant calculations were similar to those of Thomson [1906]. On could speculate that Thomson reasoned in the more elementary manner given at the end of the present paper. That manner, however, does not seem to yield the height of the waves, which Thomson claimed to have computed in 1887.

[55] Thomson could have followed Rayleigh even further to obtain the wave pattern as the envelope of the crests of the component wave, as Lamb did in [1895]. The equation of the envelope is easily seen to be identical with the condition of stationary phase that Thomson presumably used in 1887.

Its amplitude is proportional to $k = 2\pi/\lambda\cos^2\psi$. The angle ψ is uniformly distributed between $\theta - \pi/2$ and θ, since P must belong to the oscillating half-plane of the component. The resultant disturbance has the form [1906, p. 409]

$$\sigma \propto \int_{\theta-\pi/2}^{\pi/2} \frac{\cos\phi}{\lambda\cos^2\psi}\mathrm{d}\psi. \tag{119}$$

In order to evaluate this integral, Thomson appealed the "the principle of interference, as set forth by Prof. Stokes and Lord Rayleigh in their theory of group-velocity and wave-velocity" [1887b, p. 303]. At a distance from the boat much larger than the characteristic wave-length λ, the phase ϕ is very large and therefore $\cos\phi$ oscillates very quickly between positive and negative value when ψ varies. This oscillations imply a destructive interference, unless there are particular values of ψ for which the phase is stationary, that is, $\mathrm{d}\phi/\mathrm{d}\psi = 0$.

If x and y denote the Cartesian coordinates of the point P, and τ the tangent of the angle ψ, we have

$$\phi = (2\pi/\lambda)(x + y\tau)(1+\tau^2)^{1/2}. \tag{120}$$

The condition of stationary phase then gives

$$x\tau + y(1+2\tau^2) = 0. \tag{121}$$

This quadratic equation has real roots only if $(y/x)^2 < 1/8$. Hence the disturbance is confined between the two lines through the (point-like) boat that make an angle $\tan^{-1}\sqrt{1/8} \approx 19°28'$ with the mid-wake of the boat. The curves of constant phase obey the parametric equations [1906, p. 413; 1887a, p. 424][56]

$$x = a(1+2\tau^2)(1+\tau^2)^{-3/2}, \quad y = -a\tau(1+\tau^2)^{-3/2}, \tag{122}$$

where $a = (\lambda/2\pi r)\phi$. They have the "beautiful" shape represented on Fig. 25. Thomson further determined the amplitude of the waves by summing the contribution of the two roots of Eq. (121) to the integral (119). He even had a clay model made to represent the wave pattern [1887a, p. 424].

The stationary-phase method

Thomson's success in completing the theoretical analysis of ship waves depended on the stationary-phase method. An anticipation of this method is found in a mathematical paper of 1850 by Stokes. In his theory of spurious rainbows, Airy [1838] had obtained the amplitude of light waves near a caustic under the form

[56] The formulas of Thomson 1887 have a different parameter, $w = (1-2t^2)/(1+2t^2)$, but represent the same curves despite Larmor's contrary statement in a footnote to Thomson 1906 [p. 413n].

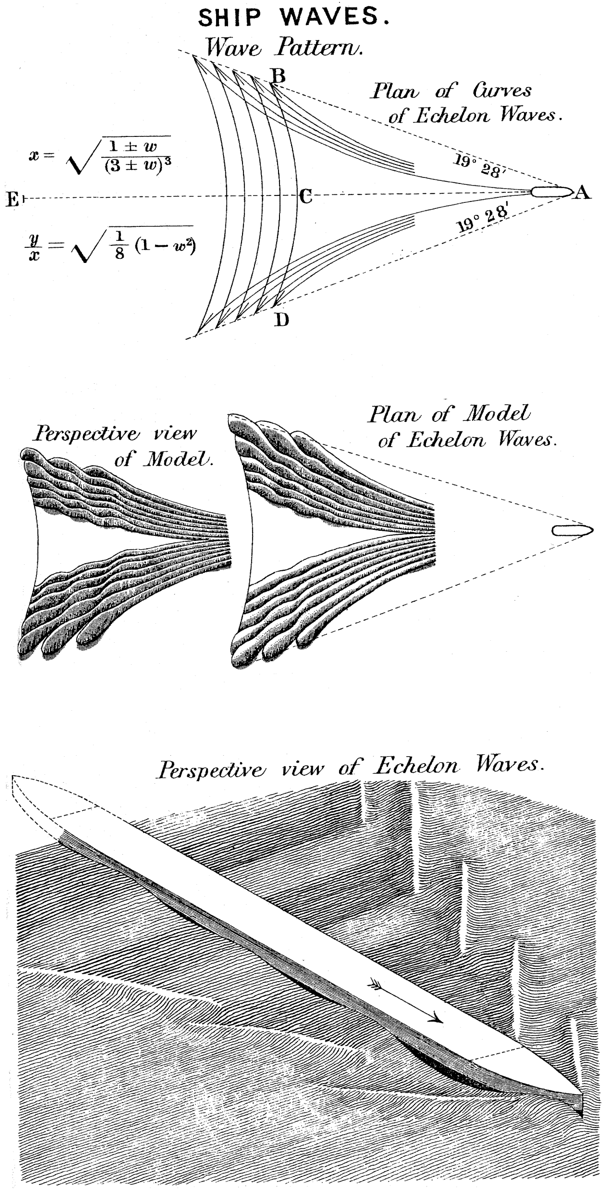

Fig. 25. Kelvin's ship-waves [Thomson 1887a, plate; perspective view borrowed by Kelvin from R.E. Froude]

$$f(x) = \int_0^{+\infty} \cos(w^3 - wx)\mathrm{d}w, \tag{123}$$

where x is proportional to the distance from the caustic. Airy's evaluation of this integral was insufficient for accurate comparison with William Miller's excellent data of [1842]. As a first step toward a better estimate, Stokes [1850] derived the differential equation

$$f'' + (x/3)f = 0. \tag{124}$$

For large values of x, he reasoned, the relative variation of the coefficient $x/3$ is small when x increases by a small increment δx. Consequently, an approximate solution of the previous equation is

$$f(x+\delta x) \approx f(x)e^{i\sqrt{x/3}\delta x} \approx f(x)e^{i(2/3)\delta\sqrt{x^3/3}}. \tag{125}$$

Hence Stokes guessed the form

$$f = e^{i(2/3)\sqrt{x^3/3}} \sum_0^{+\infty} \alpha_n x^{-n/2}, \tag{126}$$

The coefficients α_n are determined by substitution in Eq. (124) up to two integration constants. In order to calculate the latter constants, Stokes transformed the original integral (123) into the integral of a real exponential, and applied what we would now identify as the steepest descent method. He thus determined the 15 first roots of $f(x) = 0$, and found them to match Miller's data. He also showed how similar integrals could be treated in a similar manner, by deriving the associated differential equation, and by seeking approximately exponential solutions of this equation.

In a footnote (p. 341n), Stokes briefly noted that the procedure he had applied to a real exponential could also be applied to a complex exponential. He thus pointed to the method of stationary phase, though in a purely formal manner. He did not explain why the procedure worked, and did not provide a clear criterion for judging which integral was amenable to it. In contrast, Thomson [1887b] reasoned according to the "principle of interference" and visualized in his mind the destructive interference of the quickly oscillating integrand. He promptly realized that the method applied to the integral of any quickly oscillating function, and published a striking application to the Poisson-Cauchy integral

$$\sigma = \frac{A}{\pi} \int_0^{+\infty} \mathrm{d}k \cos kx \cos \omega_k t \tag{33}$$

that represents the water-surface disturbance caused by a local deformation around $x = 0$.

The progressive part of this disturbance is the real part of the integral

$$\sigma_+ = \frac{A}{2\pi} \int_0^{+\infty} \mathrm{d}k\, e^{i(kx-\omega_k t)}. \tag{127}$$

The phase is stationary when $x\mathrm{d}k - t\mathrm{d}\omega_k = 0$, that is, when the group velocity $\mathrm{d}\omega_k/\mathrm{d}k$ is equal to x/t. For gravity waves on infinitely deep water, $\omega_k = \sqrt{gk}$. The phase is stationary for $k = \kappa \equiv gt^2/4x^2$. Its value around this stationary point is

$$\phi \approx -gt^2/4x + \beta(k-\kappa)^2, \tag{128}$$

where $\beta = x^3/gt^2$ is the value of $\frac{1}{2}\mathrm{d}^2\phi/\mathrm{d}k^2$ for $k = \kappa$.

The resulting approximation of the integral for large values of $gt^2/4x$ is

$$\begin{aligned}\sigma_+ &= \frac{A}{2\pi} e^{-igt^2/4x} \int_{-\infty}^{+\infty} \mathrm{d}k\, e^{i\beta(k-\kappa)^2} = \frac{A}{2\pi} e^{-igt^2/4x} \sqrt{\pi/\beta}\, e^{i\pi/4} \\ &= \frac{At}{2} \sqrt{g/\pi x^3}\, e^{i(gt^2/4x-\pi/4)},\end{aligned} \tag{129}$$

in conformity with Poisson's Eq. (39).

In the same spirit, Horace Lamb later noted that in the large-phase approximation and for relatively small variations of the distance x from the origin, the disturbance at a given time differs very little from a sine wave with the wave number $k = gt^2/4x^2$. Therefore, the disturbance created by a non-local deformation with the profile $f(a)$ results from the interference of a system of sine waves with amplitudes proportional to $f(a)e^{ik(x-a)}$. Hence this disturbance can be obtained by replacing the coefficient A with the Fourier transform of the profile in the expression (129) of the disturbance created by a point-like perturbation. Lamb thus short-circuited Poisson's delicate and lengthy derivation of Eq. (41) [Lamb 1932, pp. 392–394].

In sum, Thomson and Lamb substituted a physico-mathematical analysis to the purely formal developments of Poisson, Cauchy, and Stokes. The gain was enormous: whereas the anterior treatment required ad hoc formal tricks that required much ingenuity and worked only in particular cases, the stationary-phase method offered an intuitive strategy that automatically gave the asymptotic behavior of large-phase integrals. Under Thomson's magic wand, the enormous memoirs of Cauchy and Poisson collapsed in a few lines of physico-mathematical common sense. And the formidable problem of ship-waves received a strikingly simple solution.

After-math

This solution was not quite definitive. Although Thomson gave the result and the method of stationary waves in 1887, he published the calculations only in 1906, at age 83. As he knew, the pressure obtained by isotropic superposition of pressures localized on straight lines passing through a fixed point varies as the inverse of the distance from this point. In modern notation, this results from the identity $\int_0^{2\pi} \delta(r\cos\theta)\mathrm{d}\theta = 2/r$. Such a slowly decreasing function cannot realistically represent the pressure exerted by a boat on the water surface. The year of his death, Thomson was still working on an improved version of his theory in which the perturbation was more sharply localized [1907].

In his conference of 1887, Thomson suggested a more direct approach: the disturbance produced by the ship, he noted, may be regarded as the superposition of the disturbances produced by a succession of impulses along its path. A Newcastle lecturer in applied mathematics, Thomas Havelock, managed the corresponding calculations in [1908], [57] thanks to a repeated application of the method of stationary wave. At a point

[57] See also Lamb 1916 and Lamb 1932, pp. 433–437.

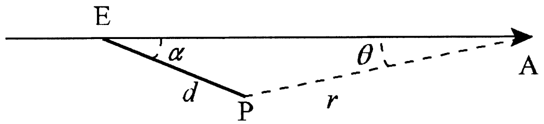

Fig. 26. Diagram for Havelock's calculation of Kelvin's ship-wave pattern

P in the wake of the ship (see Fig. 26), the wave created by each individual impulse is the superposition of monochromatic waves with the phase $\omega t - kd$ (up to a constant), where k is the wave number, d the distance EP between the impulse and the point P, ω the deep-water frequency $\sqrt{gk}$, t the time that has elapsed since the ship was at E.

In order to avoid destructive interference between the waves created by successive impulses, this phase must be stationary with respect to a variation of the time t. Hence the phase velocity ω/k must be equal to $\dot{d} = v \cos\alpha$, where v is the velocity of the boat and α the angle that EP makes with the direction of motion of the ship. This means that for a given wave-length, there is only one point E that contributes significantly to the disturbance at P. Furthermore, the phase must be stationary with respect to a variation of k. This implies that the group velocity $\mathrm{d}\omega/\mathrm{d}k$ must be equal to the ratio d/t.

The first condition of stationarity and some trigonometry leads to the expression

$$\phi = (gr/v^2)\cos(\alpha - \theta)/\cos^2\alpha \tag{130}$$

of the phase, where r is the distance from P to the ship's present location A, and θ is the angle that AP makes with the direction of motion of the ship. This expression is the same as Thomson's Eq. (118), although the angles α and ψ now have different interpretations. As the variation with respect to k is equivalent to a variation with respect to α, Havelock's calculation yields the same lines of constant phase as Kelvin's. Only the height of the wave crests differs, because the amplitude of Havelock's spherical component waves differs from the amplitude of Kelvin's plane component waves.[58] This correction does not really improve the comparison with the experimental pattern, because the latter depends on the form of the ship, and because at the cusps the theory leads to a divergent amplitude that is incompatible with the original small-wave assumption.[59]

In Kelvin's and Havelock's derivations of the echelon-shape of ship waves, there seems to be a disproportion between the simplicity of the results and the complexity of the calculations. In his popular lecture of 1887, Thomson hinted at a more elementary derivation [1887a, p. 426]. After noting that the disturbance produced by the ship could be regarded as the superposition of the disturbances produced by a succession of impulses along its path, he declared that the point E on Fig. 25a (such that EC = CA) represented the position of the ship at the time when it caused the impulse responsible for the portion of wave crest around C. His justification holds in one sentence: "Calculate out the result from the law that the group-velocity is half the wave-velocity – the velocity of a group of waves at sea is half the velocity of the individual waves." Indeed if the

[58] This agreement should be expected, because the disturbance created by a diffuse pressure is the superposition of geometrically similar echelon patterns created by symmetrically distributed pressure points.

[59] On more realistic theories of ship waves, cf. Lamb 1932, pp. 437–439.

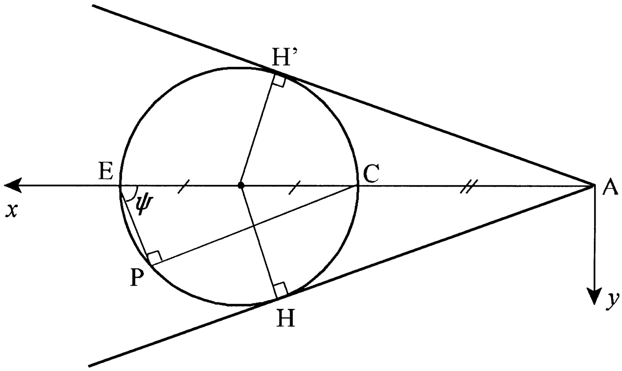

Fig. 27. Diagram for elementary calculation of Kelvin's ship-wave pattern

disturbance travels from E to C at the group-velocity, and if the phase velocity along the x axis is equal to the velocity of the ship, this law implies that the ship must move twice as fast as the disturbance. Thomson seems to have grasped these two conditions intuitively, through the picture of a train of waves made of individual waves that are steady with respect to the ship. As we just saw, they can be justified through the method of stationary phase.

Thomson's consideration may now be extended to a portion of wave crest around a point P that is no longer on the x axis.[60] The small wave-crest portion around P is caused by a disturbance that has traveled from E with the group velocity in the direction EP. For steadiness with respect to the ship, the phase-velocity in this direction must be equal to the projection of the ship velocity on this direction. Hence the angle EPC is a right angle, and the point P must be located on the circle of diameter EC (see Fig. 27). The extreme points of the wave crests belong to the tangents AH and AH', which make an angle $\sin^{-1}(\mathrm{OH}/\mathrm{OA}) = \sin^{-1} 1/3 = \tan^{-1}\sqrt{1/8}$.[61]

Calling X the diameter EC and ψ the angle that the emission line EP makes with the axis, the Cartesian coordinates of the point P are (see Fig. 27)

$$x = X(2 - \cos^2\psi), \quad y = X \sin\psi \cos\psi. \tag{131}$$

On a given wave crest, y is a function of x; or, equivalently, X is a function of ψ. This function can be determined through the condition $\mathrm{d}y/\mathrm{d}x = \cot\psi$, which means that the front of the wave emitted from E should be normal to its direction of propagation. Computing $\mathrm{d}y/\mathrm{d}x$ from the previous expressions of x and y, we get

$$\mathrm{d}X/\mathrm{d}\psi = -X \tan\psi. \tag{132}$$

The integral $X = a\cos\psi$ of this equation then gives

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60 The following reasoning is found in Lighthill 1957, pp. 21–22; Lighthill 1978, pp. 269–279; Billinghham and Konig 2000, pp. 99–105.

61 Thomson [1887a, pp. 425–425] gives this geometrical construction of the characteristic angle, without the physical interpretation.

$$x = a \cos\psi(2 - \cos^2\psi), \quad y = a \sin\psi \cos^2\psi, \tag{133}$$

which is the same as Eq. (122) with $\tau = \tan\psi$.

The extreme simplicity of this derivation strikingly illustrates the transformation of mathematical physics announced in the introduction. In 1775 Laplace already knew the equations of hydrodynamics that are needed to formulate the ship-wave problem mathematically. Had he dared to approach this problem, he would probably have fell in the same error as in the waves-by-emersion problem, for he did not know how to synthesize local perturbations from sinusoidal ones. Some forty years later, Poisson and Cauchy could have written the multiple integral that yields the water disturbance behind the ship. But they lacked efficient means to evaluate this integral. Ninety years later, Thomson succeed in this task thanks to "the principle of interference." Through the related intuition of wave groups, he even suggested a way to circumvent the integral and reason in mostly geometric terms.

This story exemplifies a symbiotic evolution of mathematical analysis and physical interpretation in the nineteenth century. The need to solve the differential equations of certain physics problems, for instance the propagation of heat, inspired new mathematical tools such as Fourier analysis. In turn, the application of these tools to a broad array of physical phenomena provided them with a physical interpretation that greatly increased their efficiency. Whereas the older, algebraic tools generated impenetrable integrals, the newer, physico-mathematical tools revealed the behavior of the integrals. A guide who has traveled much is a better guide.

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Abbreviations:

AHES: *Archive for history of exact sciences*.
BAR: British Association for the Advancement of Science, *Annual report*.
CR: Académie des Sciences, Paris, *Comptes-rendus hebdomadaires des séances*.
HSPS: *Historical studies in the physical* (*and biological*) *sciences*.
MAS: Académie (Royale) des Sciences, *Mémoires* (*de physique et de mathématiques*).
MSE: Académie des Sciences de l'Institut de France, *Mémoires présentés par divers savants*.
PM: *Philosophical Magazine*.
PRS: Royal Society of London, *Proceedings*.
PT: Royal Society of London, *Transactions*.
SMPP: George Gabriel Stokes, *Mathematical and physical papers*, 5 vols. (Cambridge, 1880–1905).
TCPS: Cambridge Philosophical Society, *Transactions*.
TMPP: William Thomson, *Mathematical and physical papers*, 6 vols. (Cambridge, 1882–1911).

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