The generation of waves by wind

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This paper describes an investigation of the height and length of ocean waves and swell in relation to the strength, extent and duration of the wind in the generating area, and the subsequent travel of the swell through calm and disturbed water. The investigation is based on records of waves made on the north coast of Cornwall, in the Irish Sea and in Lough Neagh. It is a practical continuation of the work of Barber & Ursell (1948), who showed that the waves leaving the generating area behave as a continuous spectrum of component wave trains which travel independently with the group velocities appropriate to their periods. The spectral distribution of energy in the storm area is considered, and the relative amplitudes of the different components are deduced empirically under various wind conditions. The results indicate that the wave characteristics become practically independent of fetch after 200 to 300 miles, and that in the equilibrium condition the steepness of the highest waves is inversely proportional to the square root of the wind speed.

Some theoretical foundation can be found for the form of the empirical relationships if it is assumed that the wind acts on each wave component independently, and that the sheltering coefficient used by Jeffreys is proportional to the wave steepness.

The results provide a basis for making reasonably accurate predictions of waves and swell from meteorological charts and forecasts.

Introduction

In view of the present lack of knowledge of the fundamental processes governing wave generation, systems of wave prediction are based almost entirely on empirical data concerning wave characteristics and the speeds of the winds generating them. To predict swell—the more rounded forms into which the waves change after they leave the storm area—empirical data relating rate of decay to distance travelled are also needed.

The information available consists mainly of synoptic meteorological charts and records of wave motion usually in the form of records of the pressure fluctuations produced by the waves on the sea bed at a depth of about 50 ft. at one point on the coast.

Previous methods of analyzing wave records have consisted in determining some average value of wave period and wave height over a suitable time, and these values were compared with the relevant wind speeds. Although a prediction of average values is often sufficient for a local storm and restricted waters, a general method to take account of a combination of waves and swell must embrace more than this. The periods of the waves in the generating area cover a wide spectrum, and an investigation of this spectral distribution is necessary for further progress in understanding the processes of wave generation and propagation.

A method of deriving the wave spectrum from a wave record is described by Barber, Ursell, Darbyshire & Tucker (1946). This is a Fourier analyzing method and the wave spectrum is given in terms of peaks (see figure 1), each of which corresponds to a wave period which is a submultiple of the duration of the wave record. Barber & Ursell (1948) have shown that when waves travel through areas where the wind is light so that the conditions of the classical hydrodynamical theory are more or less satisfied, the wave components travel independently, each component having the group velocity corresponding to its period.

Object of the present investigation

The present paper investigates the spectral distribution of wave energy in the storm area; it is a subject of some importance because the rate at which the wave energy arrives at a distant coast depends on this distribution as well as the distance of the coast from the storm. The investigation is confined in the first instance to storms near the wave recording station at Perranporth so that the effect of the dispersion of the waves on leaving the generating area can be neglected. These results are supplemented by others obtained for shorter fetches on Lough Neagh and the Irish Sea. Throughout the investigation, it is assumed that there is no interaction, and therefore no interchange of energy, between waves of different periods. This assumption, although justified by Barber & Ursell for waves travelling through an area of no wind, may not be strictly applicable when the waves are continuously under the action of the wind, but it is the best assumption that can be made in view of the present lack of knowledge about the behaviour of waves under such conditions. It is also necessary to make the somewhat arbitrary assumption that the wind acts on each wave component independently.

It is assumed that in the development of a wave spectrum, the wave components all grow independently, and on this basis rules governing the relative amplitude of the various components for a given wind speed are deduced empirically from analysis of the wave spectra under various wind conditions. The rules obtained are then applied to storms a large distance away from the recording station to investigate the attenuation and rate of travel of waves after they leave the storm area.

The empirical formulae obtained for the spectral distribution of energy in the storm area imply that waves with periods covering a wide range grow together under the action of the wind, and lead to the rather surprising result that the wave characteristics become practically independent of the fetch after 200 to 300 miles. Under such equilibrium conditions, the longest wave present has a wave period which, expressed in seconds, is one-third of the gradient wind speed expressed in knots. The shorter waves which are dominant at the beginning are successively dwarfed by longer waves that can grow without breaking, till, after 200 to 300 miles, the highest waves have an equilibrium period which, expressed in seconds, is approximately one quarter of the gradient wind speed in knots; the steepness of the highest waves appears to be inversely proportional to the square root of the wind speed.

An attempt is made to explain some of these conclusions in terms of the normal and tangential stresses exerted by the wind on the water surface.

Method of investigation

The investigation was started by examining wave spectra obtained by analyzing records of the waves produced by thirty different storms close to the recording station so that the effect of dispersion could be neglected. In each storm the fetch was over 100 miles and sometimes as much as 1000 miles. The investigation was supplemented for shorter fetches of 1 to 10 miles by using waves recorded at different points on Lough Neagh, and for fetches of 40 to 100 miles by measuring waves with an airborne wave-recorder at different points along a line from Dublin to Liverpool at a time when there was a steady west wind of force 6. The methods used to analyze and extract the information will now be discussed.

(1) *Analysis of wave spectra*

The method of analysis described by Barber & Ursell and others gives a Fourier analysis of a 20 to 30-minute record; the analysis appears in the form of a series of peaks, each corresponding to a harmonic component which is an exact submultiple of the total length of the record. An example of such an analysis containing waves due to a local storm and a band of swell from a distant storm is shown in figure 1. While it cannot be implied that these discrete periodicities are actually present in the sea waves, it is possible, for the duration of the record, to represent the pressure variations at the point of measurement by a combination of independent sine waves with periods which are submultiples of the duration of the record and with amplitudes proportional to the heights of the peaks on the spectrum. The heights of the peaks as they stand refer to the pressure variations at the depth of recording, but they can be adjusted to refer to variations in surface height if it is assumed that the classical hydrodynamical theory relating surface height to wave pressure can be applied to each wave component separately. It is reasonable to assume this for calm conditions because it has been shown in this case that the wave pattern consists of a number of sine waves travelling independently, and it is unlikely that any great error will be introduced by making the same assumption when the waves are acted on by the wind. Accordingly the analyses for the pressure variation at the sea bottom were converted into analyses of surface height.

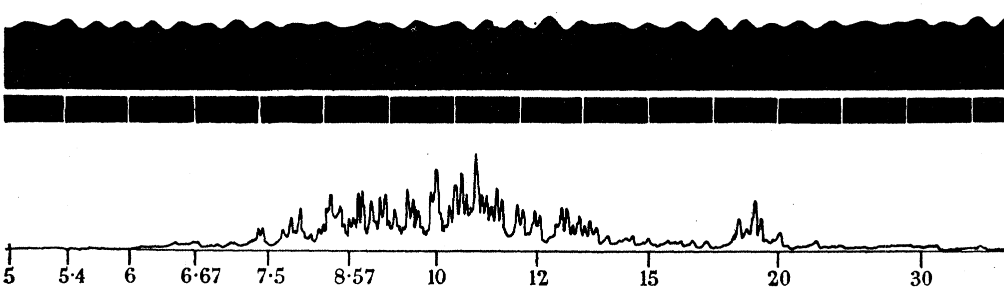


FIGURE 1. Typical wave record and spectrum.

The state of the sea can best be described in terms of the wave energy. Assuming that for the duration of the record the wave pattern consists of a combination of independent sine waves with periods and amplitudes corresponding to those of the peaks on the spectrum, it is possible to evaluate the wave energy. Since the energy per unit area of a single sine wave of height h is $\frac{1}{8}g\rho h^2$, the total energy for all the waves in the spectrum $= \frac{1}{8}g\rho\Sigma h_n^2$ where h_n corresponds to the height of the nth peak in the spectrum. If H is the height of a hypothetical single sine wave train which has the same energy per unit area as the complicated wave pattern, then $\frac{1}{8}g\rho H^2 = \frac{1}{8}g\rho \Sigma h_n^2$ and H can be defined as the equivalent height of the waves. To compare such equivalent height with the maximum heights measured on the wave records, corresponding values of H and maximum height $H_{\text{max.}}$ are plotted in figure 2. The graph shows that $H_{\text{max.}} = 2H$.

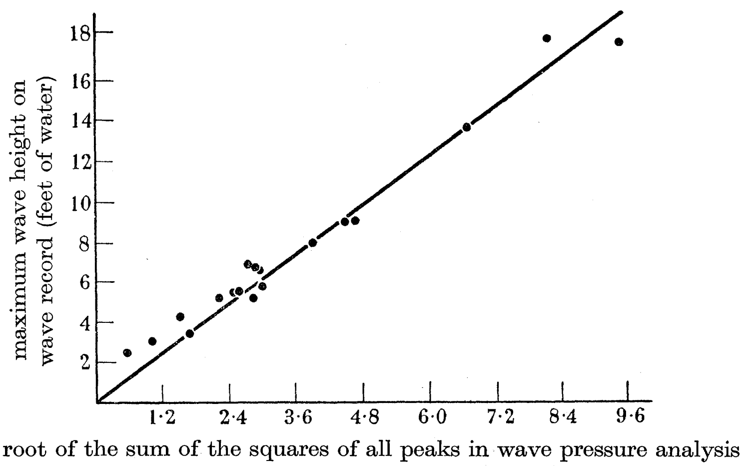


FIGURE 2. Comparison between maximum wave height and equivalent wave height.

The idea of an equivalent height can be extended to parts of a wave spectrum as well as the whole. The energy E_T in a unit wave-period interval $T-\frac{1}{2}$ to $T+\frac{1}{2}$ is given by $E_T = \frac{1}{8}g\rho \sum_{T-\frac{1}{2}}^{T+\frac{1}{2}} h_n^2 = \frac{1}{8}g\rho H_T^2$. H_T can be defined to be the equivalent height for waves of period between $T-\frac{1}{2}$ and $T+\frac{1}{2}$, and it is obtained from the Fourier spectra by measuring and taking the square root of the sum of the squares of the peaks which represent each harmonic within successive one-second intervals of wave period. The values obtained were expressed in feet of water by calibrating the analyzer with a function whose Fourier spectrum is known and the values for each period interval were divided by the classical attenuation factor appropriate to the period and the depth of the recorder to make them refer to surface heights.

The division of the energy into one-second intervals is made solely for reasons of arithmetical convenience. It will be shown later that a more fundamental division would be into unit intervals of the ratio of wave velocity to wind velocity.

(2) *Analysis of wind data*

There are not usually sufficient observations of wind strength in the wave generating area to allow detailed comparisons between the wave and wind characteristics, and it was made a general practice to compute the wind from isobaric charts. Six-hourly charts provided by the Naval Weather Service were found most convenient, and gradient wind speeds were calculated according to the instructions in the Admiralty Weather Manual. The relation between surface wind and gradient wind varies with the atmospheric stability and other factors, but Gordon (1950), using observations over the sea, found that the ratio between the two varied between 0·60 and 0·80, the mean value being 0·66. The best that can be done at present is to assume a constant ratio of 0·66 and to compare the wave characteristics with the gradient wind speeds. There is some advantage in doing this since in any application of this work, it is more likely that weather charts will be available than wind observations.

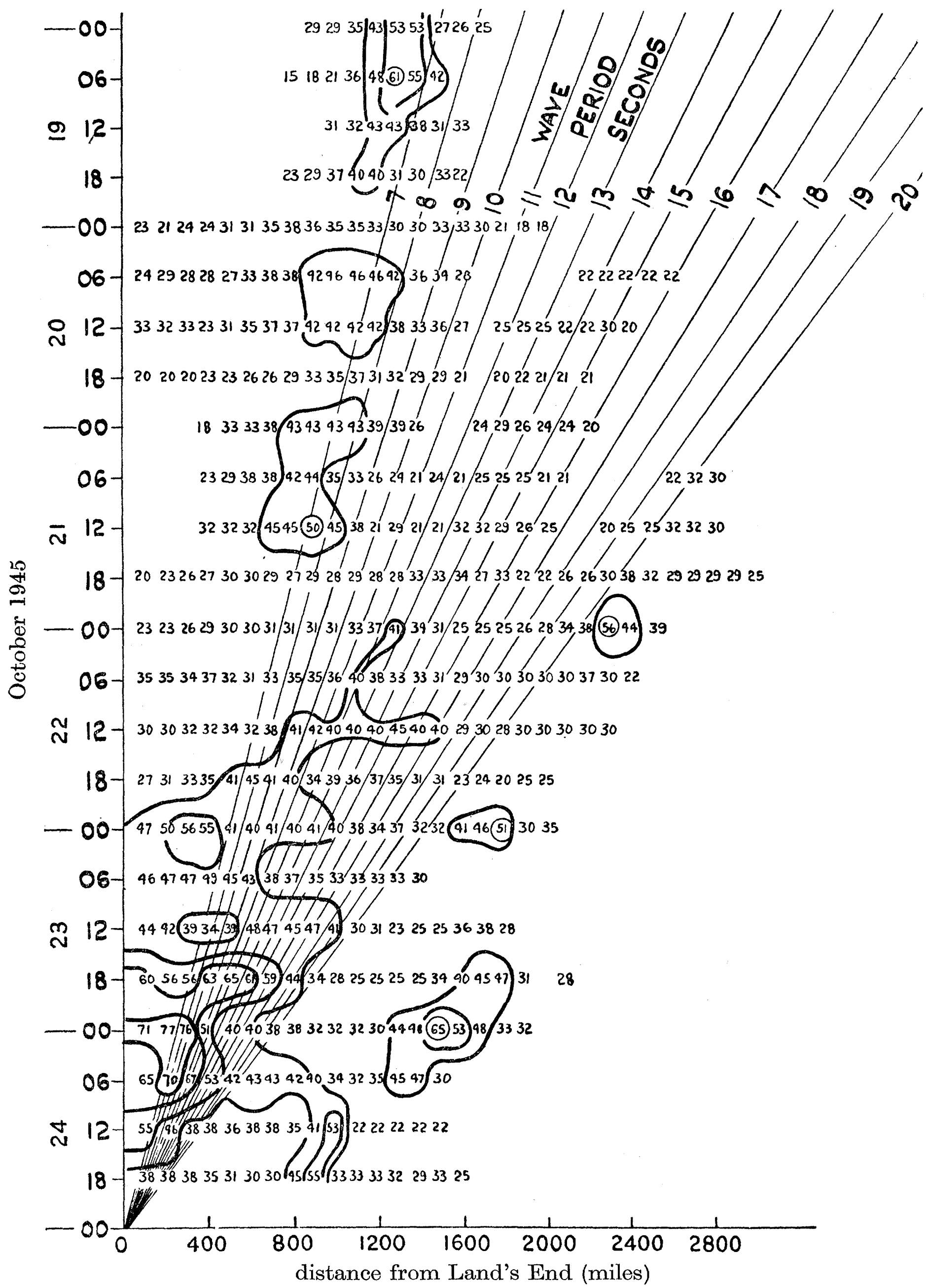


FIGURE 3. Wind data diagram.

The gradient wind in each region traversed by the waves was determined by superimposing on the chart a reference grid drawn on tracing paper with lines radiating at intervals of 10° from the recording station, and concentric circles at intervals of 200 miles. Where the gradient wind had a component in the direction of the wave recording station, the component was determined from the mean value of the isobar spacing (over an area narrow enough to exclude any great variation) and the angle between the wind direction and the direction of the wave recording station. The values of the gradient wind speed derived by this method were plotted at 100-mile intervals for every six hours as in figure 3. This is called a wind data diagram. To find the mean gradient wind speed along the paths taken by the wave components of each period which arrive at the recording station at a particular time, a set of wave propagation lines diverging from the point representing the wave recording station was superimposed on the wind data diagram. The lines correspond to the group velocities of all integral values of wave period between 6 and 24 sec, the time and distance scales being the same as in the wind data diagram. Applying the hypothesis that waves of different periods travel independently, it can be argued that waves of a particular period, arriving at the recording station at a particular time, must have travelled through the areas and times traversed by the line whose slope is proportional to the group velocity of waves of that period, provided that the wind data refer to areas which lie on approximately the same bearing from the recording station. This will generally be so when there is one storm near the recording station and one distant storm because the near storm must subtend a large angle at the recording station. When there are two distant storms on widely different bearings, a separate wind data diagram must be drawn for each storm.

Effect of wind strength on the spectral distribution of energy

In investigating the growth of a wave spectrum, the relationships of the gradient wind strength to the maximum wave period present in the spectrum and to the wave period of greatest amplitude are both important.

The relationship between the maximum wave period and the greatest gradient wind strength was found by taking the upper limits of wave period in the analyses for storms at distances up to 3000 miles from Perranporth and plotting them against the corresponding maximum gradient wind strengths in figure 4. In each example, the fetch was over 100 miles. The graph in figure 4 shows a large scatter but the maximum wave period expressed in seconds is approximately one-third of the maximum gradient wind speed in knots.

To see how the period of the highest waves was related to the wind strength for large fetches, the value of T in seconds corresponding to the maximum value of H_T in the wave spectra from sixty storms was plotted against the mean (instead of the maximum) gradient wind speed. The mean was evaluated over that part of the generating area in which the gradient wind speed in knots exceeds $3T$ (T sec), and is therefore, according to the results of figure 4, capable of generating waves of period T sec. The best evidence (derived from storms near the recording station only) is shown in figure 5a. The second diagram, figure 5b, is derived from the spectra of swell from distant storms, and for these the period for which H_T is a maximum is likely to be displaced slightly to the long period end of the spectrum because of the smaller attenuation of long wave components. The two diagrams are, however, not appreciably different and they give a clear indication that a wave component reaches its maximum amplitude when the ratio of its period, expressed in seconds, to the prevailing mean gradient speed, expressed in knots, is approximately 1 to 4.

The two diagrams indicate that when the fetch is large, the period of the highest waves $T_{\text{max.}}$ depends only on the wind strength U, and that in these equilibrium conditions $T_{\text{max.}}/U = 0{\cdot}25$ (T sec, U knots). Since the wave speed is proportional to the period, the relation suggests that there is an optimum ratio of wave speed to mean wind speed: waves of slower speed and shorter period have heights less than

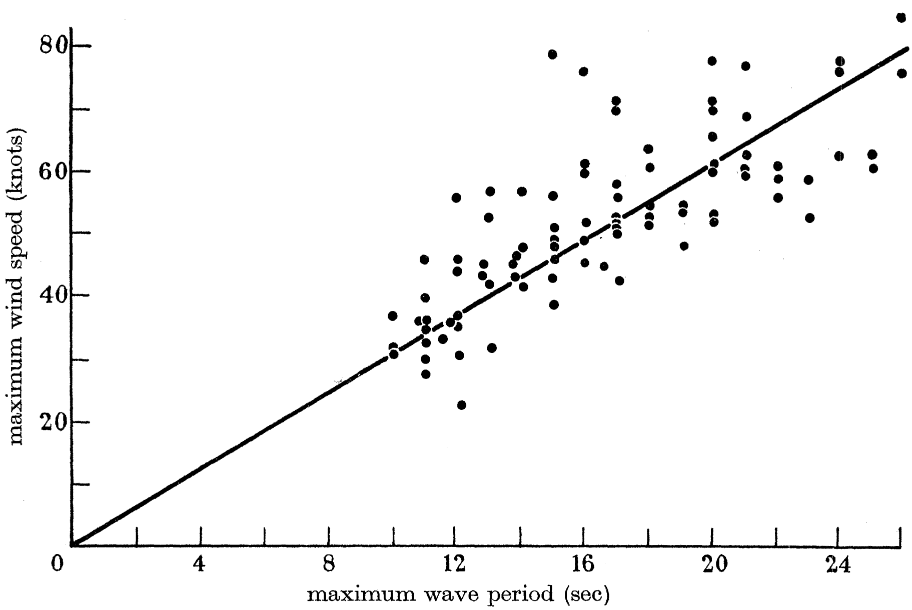


FIGURE 4. Correlation of maximum wave period with maximum wind strength for all storms.

the maximum, presumably because they become unstable before they attain such a height, and waves of greater speed and longer period are also smaller, probably because they are less able to absorb energy as their velocities become nearer that of the wind.

These two relations between the maximum wind speed and the maximum wave period, and between the mean wind speed and the period of the highest waves suggest that if H_T is plotted against T/U to give the envelope of the wave spectrum, the envelope should always have the same shape and an increase in wind speed should change only the vertical scale of the curve. This implies that the envelope can be expressed in the form $H_T = U^n f(T/U)$ where U is the mean wind speed and $f(T/U)$ is a function which is a maximum at T/U (T sec, U knots) $= 0{\cdot}25$, and nearly zero when T/U is $0{\cdot}33$. Assuming $H_T = U^n f(T/U)$, $H_T/T^n = (U/T)^n f(T/U)$ and the values obtained by dividing H_T by some power of the corresponding value of

T, should lie on a curve which is a function of T/U or the ratio of the wave speed to the wind speed. Values of H_T were divided by various powers of T and those obtained by dividing by the first power only were found to lie most closely on a single curve. These values of H_T/T are plotted for every wave component in the spectra of waves

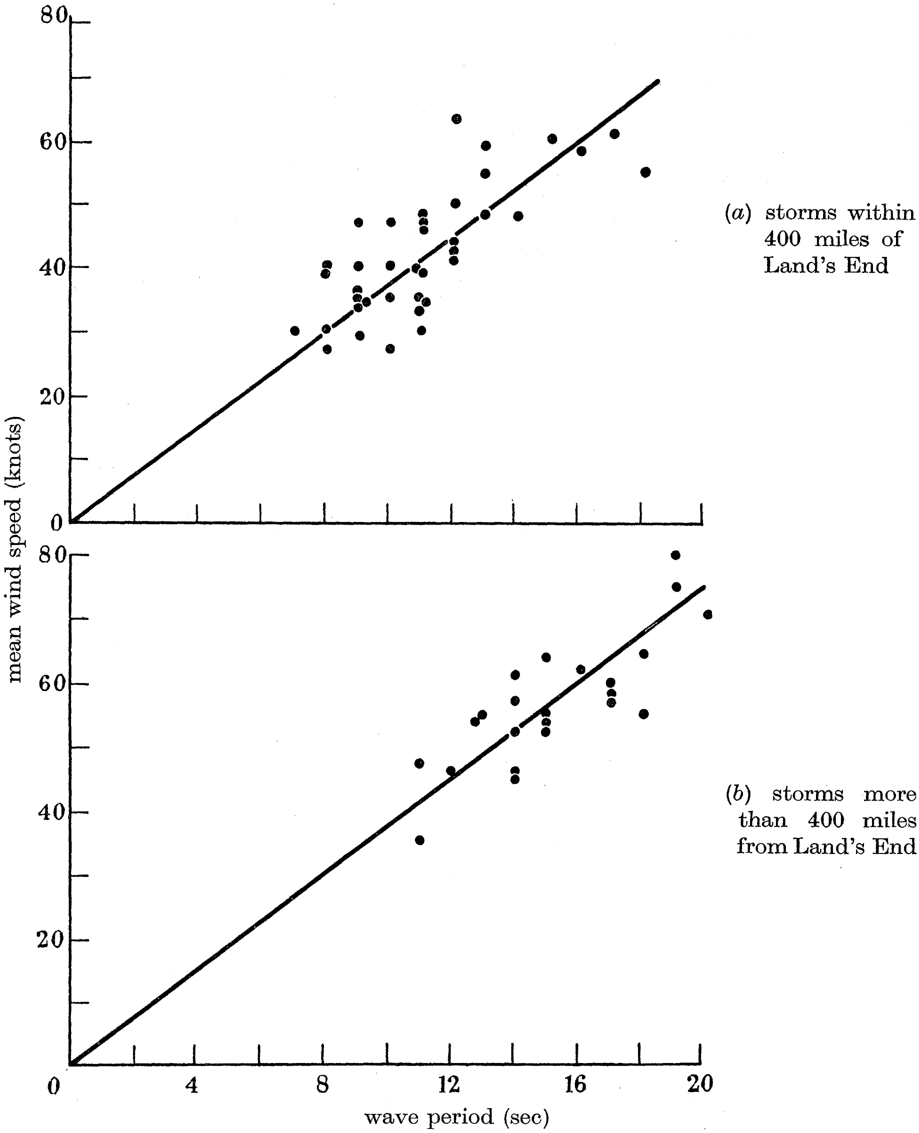
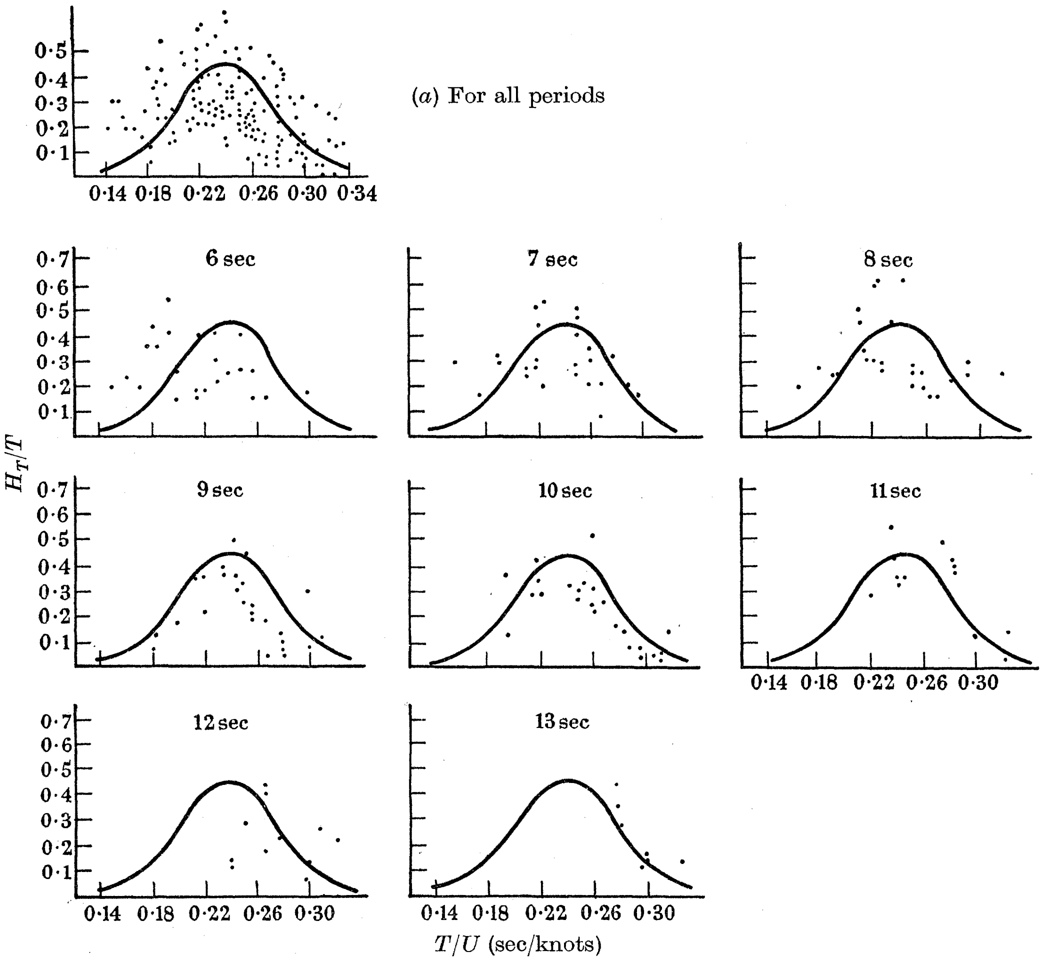


FIGURE 5. Correlation of period of maximum amplitude with mean gradient wind speed in strongest part of storm.

generated near the wave recorder in figure 6*a*. The scatter of the points in each diagram is large but there is some indication that the ratio H_T/T remains more or less the same function of T/U. The scatter must be large because we are dealing with data which are subject to many causes of error, some instrumental and some attributable to the use of isobaric charts which cannot be expected to give a perfect representation of the variable conditions through which the waves have travelled. No significant difference in the distribution curve becomes apparent when the data are sub-divided into classes corresponding to different fetches.



(*b*) For each one-second period interval separately.

FIGURE 6. Graphs of H_T/T against T/U.

Attempts to fit curves to the data suggest that the relation between H_T/T and T/U is represented by a Gaussian function of the form

$$y = K \exp\{-(x-a)^2/2\sigma\}^2.$$

The full line drawn in figure 6*a* corresponds to the function

$$H_T/T = 0{\cdot}44 \exp\{-(T/U - 0{\cdot}24)^2/0{\cdot}0027\}, \qquad (1)$$

where H_T is in feet, T in seconds, U in knots.

This relation gives a maximum value of 0·44 for H_T/T, and the value 0·0027 which is equal to $2\sigma^2$ determines the ratio of H_T/T for any other value of T/U to the maximum value. The same curve is fitted to the observations of H_T/T for each period separately in figure 6*b*.

The relation (1) was obtained by considering wave components of the same period under the action of winds of different speeds, but it can be immediately extended to find the variation of H_T with T when the wind speed U is constant. It then agrees with the results of figures 4 and 5, for when $T = U/3$ the value of H_T is very small, only one-fourteenth of the maximum value of H_T.

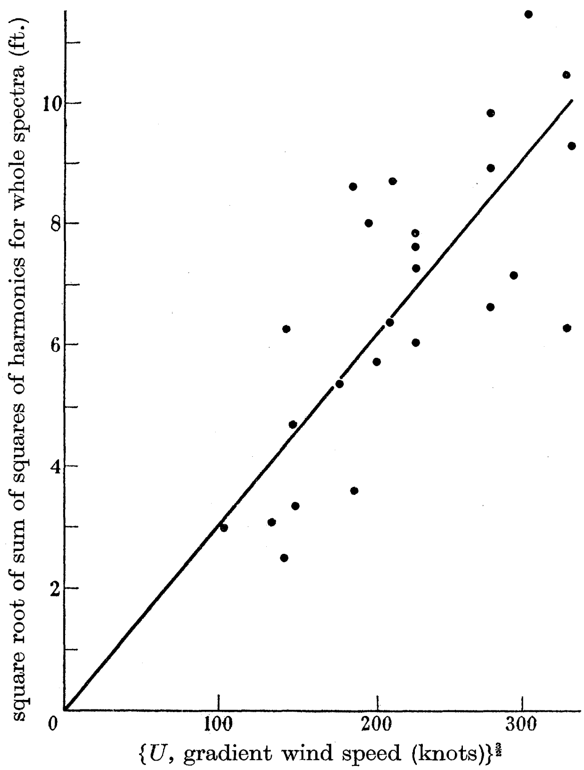


FIGURE 7. Graph of H against $U^{\frac{3}{2}}$.

The equivalent height H can be found by integrating $H_T^2\,\mathrm{d}T$ from 0 to ∞ and taking the square root of the result, which gives

$$H = 0{\cdot}027U^{\frac{3}{2}} \quad (H \text{ feet, } U \text{ knots}). \tag{2}$$

Values of H found directly from the spectra are plotted in figure 7. The straight line which corresponds to the relation (2) appears to give an adequate representation of the data.

The constants in (1) and (2) are not dimensionless, the dimensions of the constant 0·027 in (2) being $L^{-\frac{1}{2}}T^{\frac{3}{2}}$. The spectrum curve can, however, be expressed in terms of non-dimensional constants if the wave energy is sub-divided into unit intervals of the ratio of the wave speed to the surface wind speed instead of unit intervals of wave period. If U is the gradient wind speed, then the surface wind speed is $\frac{2}{3}U$, and the ratio of wave speed to surface wind speed becomes $c/(\frac{2}{3}U) = \frac{3}{2}(c/U)$. If $3U/2c = \mu$, $E_T\,\mathrm{d}T = E_\mu\,\mathrm{d}\mu$ (cf. distribution of energy in spectrum of thermal radiation), and taking c (in knots) as equal to $3{\cdot}03T$ (T in seconds), and $E_\mu = \frac{1}{8}g\rho H_\mu^2$, $H_\mu = H_T\sqrt{\{2U/(3\times 3{\cdot}03)\}}$ and from (1)

$$H_\mu = 0{\cdot}068U^{\frac{3}{2}}(c/U)\exp-\{(c/U-0{\cdot}72)^2/0{\cdot}025\} \quad (H\text{ ft., } U\text{ knots}).$$

If this expression is divided by (2) one obtains

$$H_\mu/H = 2{\cdot}52(c/U)\exp-\{(c/U-0{\cdot}72)^2/0{\cdot}025\}. \tag{1a}$$

The constant 2·52 is now dimensionless and the ratio H_μ/H is a function of c/U only.

Equation (1*a*) might also be applicable to observed wave profiles. The calculations on wave energy discussed above are based on the assumption that waves of periods corresponding to the harmonic components are continuous throughout the record. The observer sees a wave profile which is the sum of these component wave trains and the wave periods which he observes change from wave to wave within the limits of the spectrum. It would be useful, in the further study of wave steepness and wind pressure on waves, to be able to estimate the mean height distribution over all such observed wave periods. It is difficult to formulate such a distribution but it is reasonable to assume that it is similar in form to that given by (1*a*) and an estimate of its scale can be obtained by making the reasonable assumption that the mean height of the dominant waves is not very different from the maximum value of wave height recorded. Equation (1*a*) would thus approximate to this distribution in form and scale for (1*a*) becomes a maximum at $T = 0{\cdot}24$, or $c/U = 0{\cdot}72$ which gives $H_\mu/H = 2{\cdot}52\times 0{\cdot}72 = 1{\cdot}81$, which also agrees fairly closely with the maximum wave height.

The effect of fetch

The data employed so far were obtained under wind conditions in which the fetch, the distance over which the wind acted on the waves, was at least 100 miles and in some instances ten times as much; they indicate that for wave periods up to 13 sec (the limit for which sufficient observations are available), the increase of fetch above 100 miles appears to have no important effect on the relation between H_T/T and T/U.

For shorter fetches, the relation might well be different and the question has been examined with the help of analyses of wave records with fetches up to 16 miles on Lough Neagh. Values of H_T were calculated as before but the wind speeds recorded at the neighbouring R.A.F. station at Aldergrove were used instead of calculations based on isobaric charts; there is little doubt that these would be more representative of the variable conditions over the lake and in dealing with such small waves and short fetches the wind and wave observations had to be identified as closely as possible with each other. The winds were measured at a height of 15 metres above lake level and at a distance of 2 to 3 miles from the lake. The wind values were multiplied by $\frac{3}{2}$ to make them comparable with the calculations based on isobaric charts over the sea and are referred to as gradient winds.

The fetch varied according to the position of the wave recorder which was moored at various distances from the weather shore. The values of H_T for $\frac{3}{4}$ to 2 miles fetch and 7 to 16 miles fetch are plotted in figure 8*a* and sub-divided into diagrams for each one-second period interval in figure 8*b*.

A curve corresponding to the expression

$$H_T/T = 0{\cdot}44 \exp - \{(T/U - \epsilon)^2/0{\cdot}0027\} \quad (H_T\,\text{ft.}, T\,\text{sec}, U\,\text{knots}), \tag{3}$$

is drawn through the points. This expression is the same as equation (1) used for long fetches except that the constant ϵ is no longer 0·24 but varies with the fetch, being about 0·05 for $\frac{3}{4}$ to 2 miles and 0·11 for 7 to 16 miles. The fit, although not so good as that for long fetches, is not unsatisfactory since the waves were caused by winds which had passed over the land and were subject to greater variations than winds blowing over the sea.

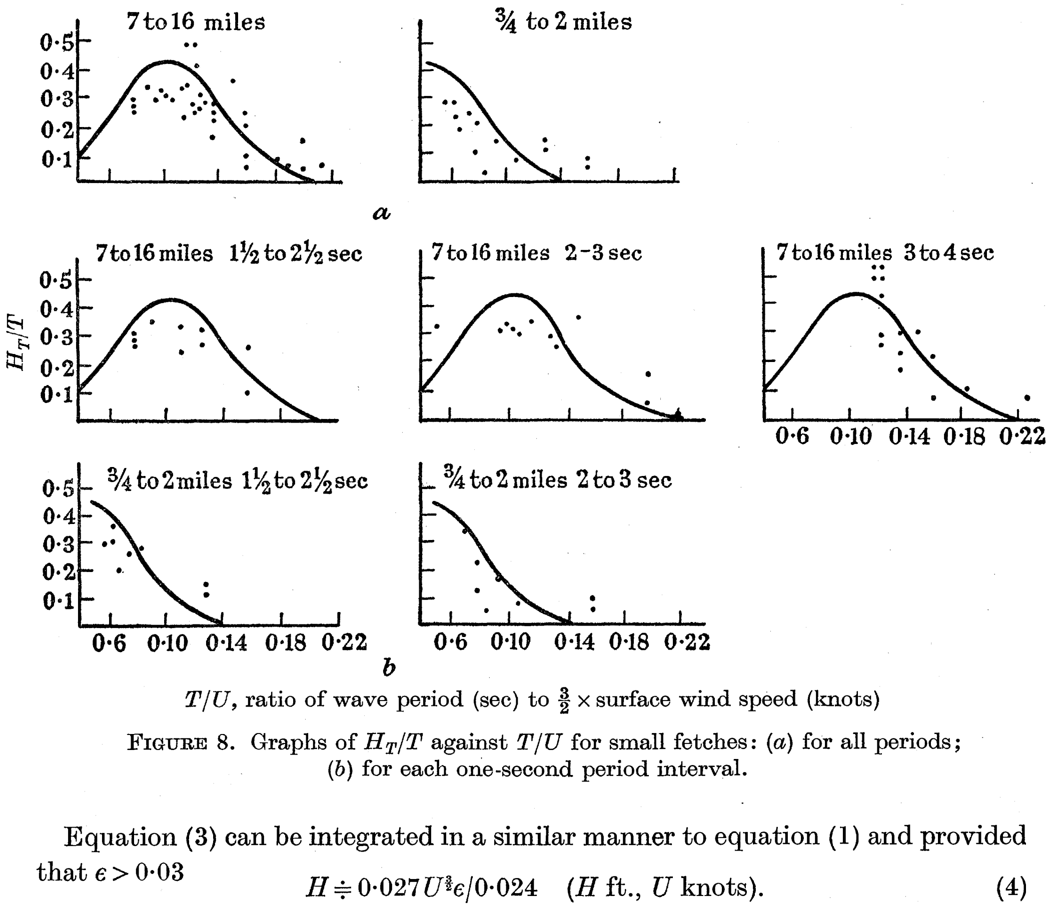


FIGURE 8. Graphs of H_T/T against T/U for small fetches: (*a*) for all periods; (*b*) for each one-second period interval.

Equation (3) can be integrated in a similar manner to equation (1) and provided that $\epsilon > 0{\cdot}03$

$$H \doteqdot 0{\cdot}027\,U^{\frac{3}{2}}\epsilon/0{\cdot}024 \quad (H\,\text{ft.}, U\,\text{knots}). \tag{4}$$

The variation of ϵ for fetches between 40 and 100 miles was found from the line of observations across the Irish Sea. The records obtained by an airborne wave recorder contain some irregular variations due to the weave of the aircraft, which prevent such a precise determination of the upper and lower limits of wave bands as is usually possible in the analyses of wave records from a stationary instrument on the sea bed, but the period of the waves of maximum amplitude could be determined with an accuracy of 1 sec from most of the analyses of the records. In all the analyses, the value of the period corresponding to the maximum value of H_T was also that corresponding to the maximum value of H_T/T, so that the values of

ϵ corresponding to fetches of 40 to 100 miles can be calculated. They are plotted against the fetch in figure 9. The two values 0·03 and 0·11 from Lough Neagh are included in the diagram. The distribution suggests an exponential relationship and the curve drawn is $\epsilon = 0{\cdot}24\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}$, x being the fetch in nautical miles. When the same distribution is plotted for long fetches, as in the inset to figure 9, there is some indication that the relationship becomes linear. The curve drawn is

$$\epsilon = 0{\cdot}24(1+1{\cdot}25\times10^{-4}x)\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}.$$

At 400 miles the difference between this and the simpler expression is 5 %, and since longer fetches are rare, the simpler equation will generally be adequate.

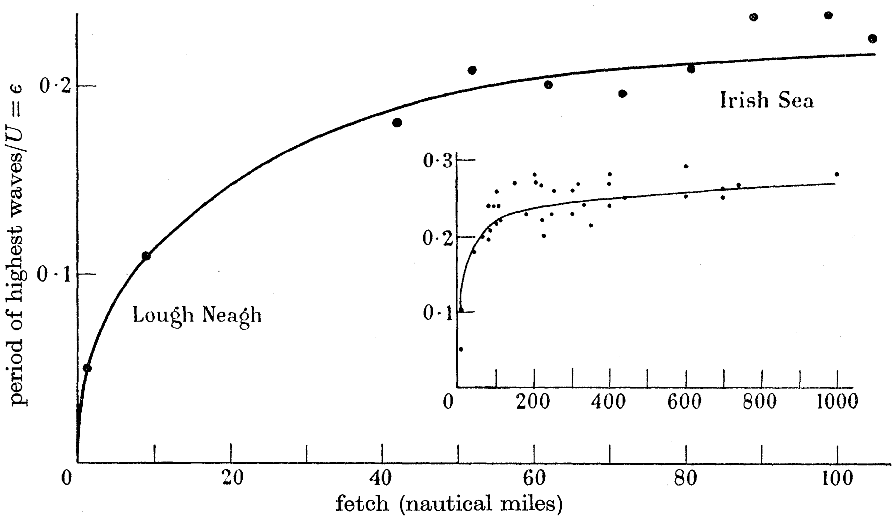


FIGURE 9. Graph of ϵ against fetch.

Using the simpler expression and then the more accurate, one obtains

$$\left.\begin{aligned} H_T/T &= 0{\cdot}44\exp-[T/U-0{\cdot}24\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}]^2/0{\cdot}0027 \\ \text{or} \\ H_T/T &= 0{\cdot}44\exp-[T/U-0{\cdot}24(1+1{\cdot}25\,10^{-4}x)\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}]^2/0{\cdot}0027 \end{aligned}\right\} \quad (3a)$$

(H_T ft., T sec, U knots, x nautical miles).

Similarly from equation (4) one obtains

$$\left.\begin{aligned} H &= 0{\cdot}027U^{\frac{3}{2}}\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}, \\ \text{or}\quad H &= 0{\cdot}027U^{\frac{3}{2}}(1+1{\cdot}25\times10^{-4}x)\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}. \end{aligned}\right\} \quad (4a)$$

These equations hold when $\epsilon > 0{\cdot}03$, as it does with fetches greater than about 1 mile. The expressions incorporating the simpler relationship reduce to equations (1), (2) and (1a) for infinite fetch; the other expressions reduce to

$$H_T/T = 0{\cdot}44\exp-\{T/U-0{\cdot}24(1+1{\cdot}25\times10^{-4}x)\}^2/0{\cdot}0027$$

and

$$H = 0{\cdot}027U^{\frac{3}{2}}(1+1{\cdot}25\times10^{-4}x).$$

These expressions for infinite fetch may represent a correction on the formulae derived earlier in the sense that the wave characteristics continue to increase slightly with fetch but the difference between them lies within the margin of uncertainty of the data, and the simpler expression is used throughout this paper.

Significant wave height and wave period

For an examination of the effect of waves on ships and for other practical purposes, it has been found convenient to use the average height and period of the highest third of the waves observed and this statistical wave model has been called the significant wave (Sverdrup & Munk 1947) or the operational wave (Seiwell 1948). Its value in relation to the mean wave height has been found by measurement of wave records to be 1·57 (Seiwell 1948), and 1·51 for swell and 1·61 for sea by Harney, Saur & Robinson (1949). Barber (1950) has worked out the relative distribution of heights in a wave record and finds that $y = x/\sigma^2 \exp(-x^2/2\sigma^2)$* where $y\,dx$ is the probability of the wave having a height between x and $x+dx$. It can be shown from this that the mean height $H_{\text{mean}} = 1{\cdot}25\sigma$ and that only 1 % of the waves exceed the height given by $3{\cdot}03\sigma = 2{\cdot}4H_{\text{mean}}$ which can be taken as a reasonable estimate of the maximum wave height. It can also be shown that the mean height of the highest third of the waves is 1·61 times the mean wave height. Table 1 summarizes the definitions and shows approximately how they are related. As the equivalent height is one half the maximum wave height, it can be related to the other mean values.

Table 1

H_{mean}	mean wave height (average of all waves)	1·0
H	equivalent wave height (height of simple sine wave having the same energy content as the complicated wave pattern	1·2 H_{mean}
H_s	significant wave height (average of one-third highest waves)	1·6 H_{mean}
$H_{\text{max.}}$	maximum wave height (highest in 100 waves)	2·4 H_{mean} 1·5H_s

According to Barber's distribution formula, approximately 13 % of the waves have heights greater than the significant height. From (4*a*) and table 1 using the exponential relationship for the effect of fetch

$$H_{\text{mean}} = 0{\cdot}023U^{\frac{3}{2}}\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\} \tag{5}$$

$$H_s = 0{\cdot}036U^{\frac{3}{2}}\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\} \tag{6}$$

$$H_{\text{max.}} = 0{\cdot}054U^{\frac{3}{2}}\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}, \tag{7}$$

where H_s is in feet, U in knots and x in nautical miles.

Examination of wave records and their analyses has shown that the significant period corresponds very closely with the period of maximum amplitude on the analysis so that

$$T_s = 0{\cdot}24U\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\} \quad (T\text{ sec, } U\text{ knots, } x\text{ nautical miles}). \tag{8}$$

* The distribution depends on the method used to measure the waves, and on the size of the smallest surface irregularities which are counted as waves. All the statistical examinations made up to the present have been based on wave pressure records made by instruments laid at a depth of 40 to 80 ft. and it can be assumed that waves with lengths less than twice the depth of recording have been ignored.

The value given for the fetch in equations (5) to (8) is usually taken to be the distance over which the wind has acted on the waves, and when the area of water affected is small compared to the total area over which the wind is blowing, it can be assumed to be the distance from the weather shore. If the wind acts over part of the sea area to windward of the observation point, the definition is not so simple; the main difficulty being to define the outer limit at which the wind becomes sufficiently strong to generate the waves. According to the evidence in figure 4 that waves of period T sec are not generated till the gradient wind speed U knots reaches a value such that $U > 3T$, the fetch will be different for short and long waves. This limit is as easy to use as any other if a wave propagation diagram (p. 304) is used, and it is not necessary for small sea areas limited by land.

To obtain an independent check, the expression for significant height and period have been applied to the wind and wave data listed by Sverdrup & Munk (1947) and the predicted and observed wave heights and periods are compared in figures 10*a* and 10*b*. The agreement between the wave heights and periods calculated from the wind data and the observed values is satisfactory.

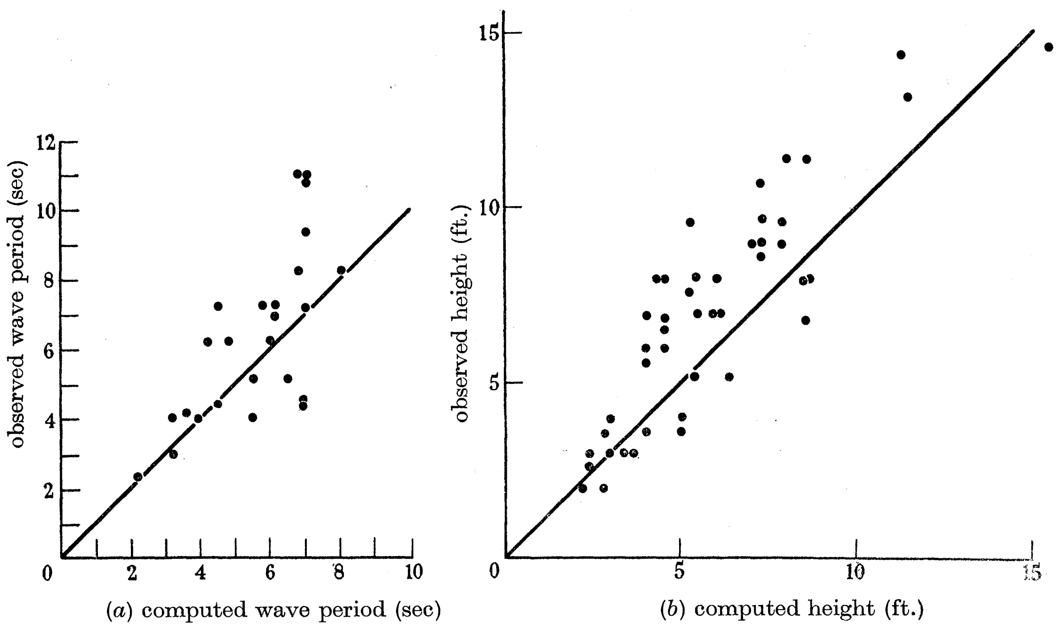


FIGURE 10. (*a*) Comparison of computed and observed wave periods (Sverdrup & Munk's data). (*b*) Comparison of computed and observed wave heights (Sverdrup & Munk's data).

WAVE STEEPNESS

Wave steepness is usually defined as the ratio of height to length. According to equation (7), the maximum height $H_{\max.}$ is given by

$$H_{\max.} = 0{\cdot}054U^{\frac{3}{2}}\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\} \quad (H_{\max.}\text{ ft., } U \text{ knots, } x \text{ miles})$$

and according to equation (8), the period of the highest waves is given by

$$T_s = 0{\cdot}24U\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}.$$

This corresponds to a wave-length of $g \times 0{\cdot}24^2 U^2\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}^2/2\pi$ from which the steepness of the highest waves δ becomes

$$\delta = 0{\cdot}182/[U^{\frac{1}{2}}\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}] \quad (U \text{ knots, } x \text{ nautical miles}) \tag{9}$$

or in terms of surface wind V, assuming $V = 2U/3$

$$\delta = 0{\cdot}149/[V^{\frac{1}{2}}\{1-\exp(-0{\cdot}23x^{\frac{1}{2}})\}]. \tag{10}$$

This empirical relation takes into account losses of energy due to breaking and other causes and agrees with the known fact that waves become less steep with increasing fetch. It implies a steepness of approximately $0{\cdot}149/V^{\frac{1}{2}}$ for fully developed waves which have travelled a long distance under the action of the wind. For small values of wind speed and small fetches, the formula gives a ratio of height to length greater than 1 in 7 but all the measurements that have been made indicate that this is probably beyond the limit at which waves become unstable and till more is known of the factors involved, it must be assumed that the values given by the formula must not exceed this limit.

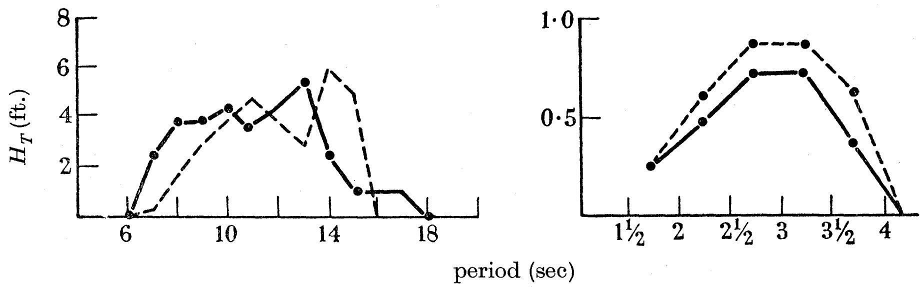
The formula indicates that the steepness is a function of the square root of the wind speed as well as of the stage of development of the waves under particular wind conditions. Sverdrup & Munk (1947) used all the available data, mostly visual observations, to show that wave steepness is a function of the ratio of the wave velocity to the wind velocity c/V for which they used the term 'wave age'. Using equation (10) and (8) and the relation c (knots) = $3{\cdot}03T$ (sec), the steepness

$$\delta = 0{\cdot}163/\{V^{\frac{1}{2}}(c/V)\} \quad \text{and if } \beta = c/V \text{ then } \delta\beta = 0{\cdot}163/V^{\frac{1}{2}}. \tag{11}$$

Over a range of 15 to 50 knots which covers the measurements made, the presence of $V^{\frac{1}{2}}$ in the denominator causes the steepness to vary by a factor of less than two, and in view of the scatter usually present with data of this sort, the effect of V would not be very apparent and there might appear to be a relation between δ and β.

Prediction of wave spectra produced by local winds

Equations (1) and (3a) can be used to estimate the height of each wave component in turn. A transparent sheet on which wave propagation lines are drawn is placed over a plan of the gradient wind speeds as outlined in p. 304, to form a wave propagation diagram as shown in figure 3. The distance over which the wind speed in knots is greater than $3T$, where T is each wave period in turn, is used as the fetch (x in equation (3a)). If the estimate is made for waves which have been generated over a sea area such as the English Channel or the Irish Sea, which is so small that the gradient wind can be assumed to be constant, the width of the sea can be taken as the fetch. If the fetch for any period is longer than 200 miles, the simpler of the two equations is used. The mean gradient wind speed U for each period is taken as the mean of the wind speeds that are greater than $3T$. For short fetches, the values of U are the same for each period. Calculations made for each wave period in turn give results such as those plotted in figure 11 which compares predicted spectra with those observed at Perranporth and Lough Neagh.



(*a*) Perranporth 24.00 G.M.T. 24 October 1945 (*b*) Lough Neagh 11.25 G.M.T. 12 April 1949

FIGURE 11. Comparison of observed (—) and predicted (---) wave spectra.

SWELL FROM DISTANT STORMS

Before the procedure for waves generated near the recording station can be applied to waves generated a large distance away, other factors have to be taken into consideration:

(*a*) the effect of tides on wave period, and possibly wave amplitude, as the waves approach the coast;

(*b*) even when the hypothesis that the wind acts on the wave components independently is accepted, it is necessary to see how closely the group velocity of wave components when under the action of the wind compares with the theoretical value for calm conditions;

(*c*) accepting the same hypothesis it is necessary to see if useful generalizations can be made about the attenuation of wave components with time or distance as they travel from the generating area.

Effect of tidal streams on wave periods and amplitudes

Wave spectra which contain narrow bands of swell reaching the coast of Cornwall from very distant storms show that the period of the swell does not decrease regularly with the time as would be expected from the classical theory but is subject to an oscillation of about 12 hours' period. Barber & Ursell (1948) and Barber (1949) have shown that the swell period is reduced during the time of maximum stream in the north-east direction at Perranporth and increased at the time of maximum stream in the south-west direction and concluded that the oscillation is of tidal origin. If the waves enter an area of slack water with a velocity v and the water begins to move with velocity u, then the velocity of the waves past a stationary observer will be $v+u$ and the wave period will appear reduced by the ratio $v/(v+u)$. The apparent reduction in period will be greatest if the waves enter the area at the time of maximum opposing stream and pass the instrument six hours later when the stream has its greatest velocity in the same direction as the waves. Under these conditions u will be the algebraic difference of the ebb and flood streams.

If a narrow band of swell is travelling from the south-west with a group velocity of 30 knots it would take approximately 6 h to travel from the 100-fathom line, through an area in which the tidal streams set north-east and south-west with a maximum rate of 1 knot. The overall change is then 2 knots and if the swell crosses the 100-fathom line when the south-west stream is strongest, the wave velocity of 60 knots will be changed at Perranporth by two parts in 60 and the wave period is correspondingly reduced by $\frac{2}{3}$ sec. This agrees reasonably well with the variations observed for swell arriving from this direction. Barber (1949) has extended the theory to swells arriving from other directions and in general a variation of 1 sec in wave period is obtained according to the state of the tide and in the prediction of wave spectra due to distant storms; allowance has to be made for this effect.

There is also some evidence of a small variation in wave amplitude with the state of the tide, but not enough is known of this effect at present to make any allowance for it in wave prediction.

Variation of wave-group velocity

In predicting the characteristics of waves caused by distant storms, the wave-group velocity becomes an important factor. Barber & Ursell (1948), selecting examples of small, intense storms at a great distance, which produced waves that, after leaving the generating area, travelled across the ocean in calm or light winds, found that the wave components travelled independently with the group velocities appropriate to their period. Their conclusions agreed with the classical hydrodynamical theory which can be reasonably applied under such conditions. The present paper extends the investigation to conditions where the waves travel under the action of fairly strong winds, and it is assumed that under these conditions the wave components still travel independently. As the classical theory may no longer be applicable, the values of wave-group velocities appropriate to each period may be different and it is necessary to find whether there are significant differences.

It is one of the implicit assumptions of this paper that wave components travel independently without any interchange of energy whether in the generating area or not. With this assumption, one can extend the conclusion drawn from figure 4, that waves of period T seconds are found after long fetches with a gradient wind speed of $3T$ knots, to infer that waves of period T seconds are formed as soon as the gradient wind speed reaches $3T$ for any fetch and to use this criterion to estimate the time of origin of waves of period T in the storm area.

Accordingly, a contour enclosing that part of the generating area where the wind speed in knots was greater than three times the wave period in seconds was drawn on a propagation diagram of the type shown in figure 3, and it is assumed that the first waves of that particular period to reach the coastal station have travelled along a line which joins the nearest part of the wind contour to the point that represents the time of arrival at the coastal station, and that the slope of the line joining them represents the rate of travel. This rate has been compared with the theoretical group velocity in 23 instances, some of which correspond very closely to the fundamental conditions of the Cauchy-Poisson hydrodynamical theory considered by Barber & Ursell, and others in which the waves were subjected to following winds ranging up to 45 knots after they had left the generating area where U (knots) was greater than $3T$ (sec). An appropriate allowance up to ± 1 sec is made for the effect of tidal streams as indicated above, and the corrected period was used as the value T in deciding the place and time at which the wind first exceeded $3T$.

The rate of travel based on the assumption that waves are generated at the hypothetical place and time at which U first exceeded $3T$ is called the estimated velocity or empirical group velocity and the ratio of this estimated velocity to the theoretical group velocity is found. The mean gradient wind speed directed towards Land's End along the line joining the hypothetical origin of the waves to that representing the time of arrival at the coastal station, is also determined. The ratio of the estimated velocity to the theoretical group velocity is plotted against the square of this value of mean gradient wind speed in figure 12 and there appears to be a linear relationship between the two.

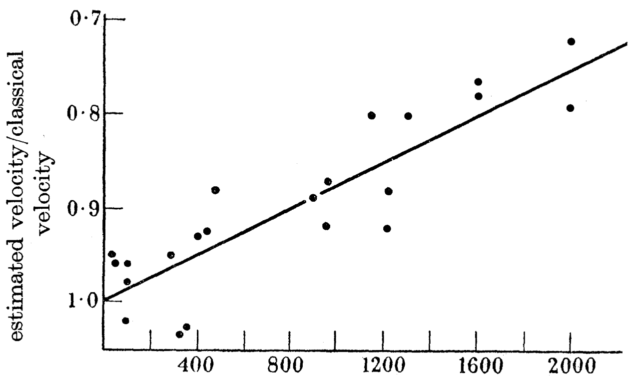


FIGURE 12. Variation of wave-group velocity with wind strength.

As far as can be judged from the comparison of calculated and predicted spectra, the corrected velocities allow a more accurate prediction of the time of arrival of a particular wave period when the waves are acted upon by a following wind. Such an example is illustrated by the propagation lines drawn in figure 13 for a narrow band of swell which arrived at Perranporth with $16\frac{1}{2}$ sec period at 1900 h on 5 May 1945. The rate at which the spectra at Perranporth widened towards shorter periods showed that this narrow band of swell originated in an area 1600 to 2000 miles from the recording station, presumably in the area of the 40-knot winds shown in the propagation diagram. To reach Perranporth at the measured time the $16\frac{1}{2}$ sec waves must have travelled with a velocity which is represented by the full line in the diagram; this velocity is smaller than the theoretical group velocity which is represented by the broken line.

There are as yet not sufficient data to investigate the effect of cross-winds and opposing winds. It appears that successive measurements on the same train of waves as it travels from the generating area will be necessary to further this investigation of the variation of wave-group velocity with wind strength under various conditions.

Attenuation of waves with travel time

Previous studies of the attenuation of waves with distance have shown that there is an increase in mean wave period as the swell travels away from the storm area, but have ignored the dispersion effect. On the present basis, the increase in period may be attributed partly to the effect of dispersion, because of which the longer swell runs ahead of the shorter, and partly to the greater attenuation of the shorter swell components. Previous values of the attenuation coefficients inferred from changes in the dominant waves must therefore depend on the effect of dispersion over a wider area as well as loss of energy.

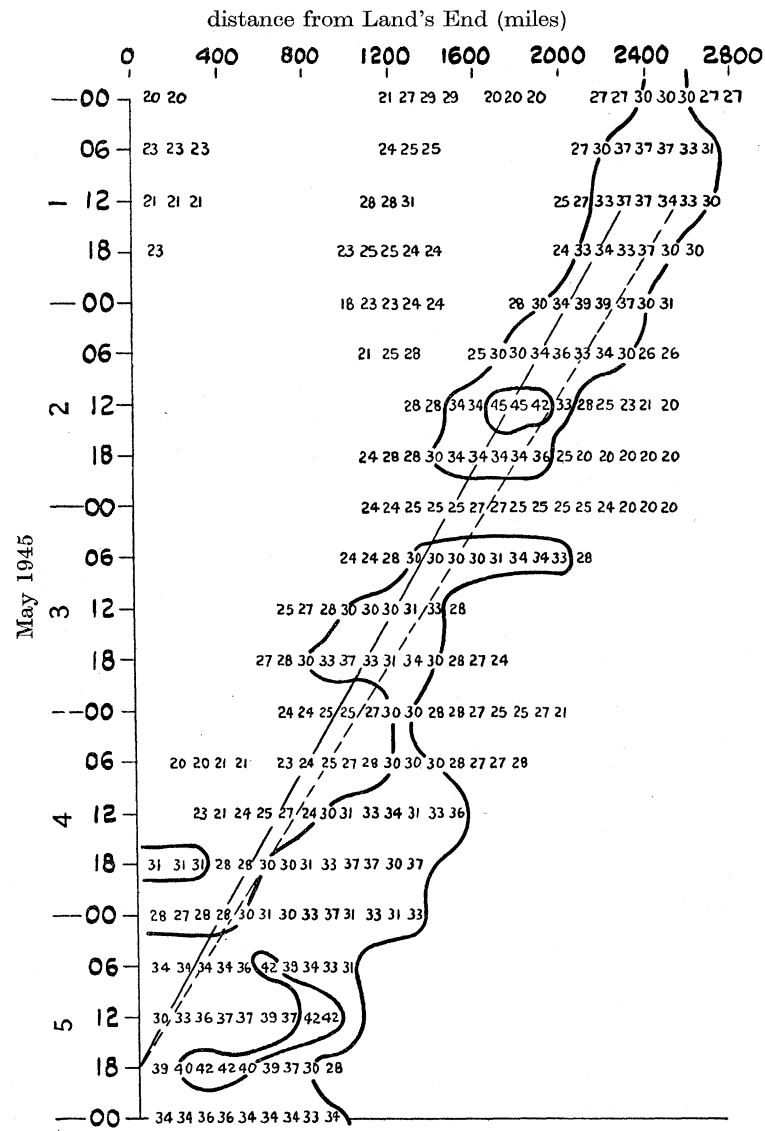


FIGURE 13. Wind data diagram showing path of waves of $16\frac{1}{2}$ sec period: - - -, theoretical group velocity; —, empirical group velocity.

On the assumption that wave components of different periods travel independently, the attenuation due to loss of energy can be isolated, by considering them separately as they leave the storm area, regardless of the composite pattern which they produce at any point.

To make such a study, values of H_T for the spectra of waves arriving at the recording station at particular times were calculated from equations (1) and ($3a$), the wind strengths and fetch being obtained from wind data diagrams, using wave propagation lines based on the empirical group velocities. Comparison of the calculated values with those observed should give a reliable indication of any systematic effect of attenuation. The results showed a good deal of variation, probably attributable to the uncertain character of the wind data, but some general conclusions could be reached:

(1) The attenuation of waves of all periods is much less when there is a following wind.

(2) Short waves are attenuated more than long waves.

(3) Waves of large amplitude are attenuated proportionately more than low waves of the same period.

(4) For distances considered, up to 2000 miles, it was found unnecessary to take any account for the divergence of the waves as they leave the storm area, i.e. it can be assumed that they continue to travel in the great circle along which they were originally generated.

The first and second conclusions are in accordance with Sverdrup's wind resistance theory based on the principle used by Jeffreys (1925, 1926) in his theory of wave generation. This theory is discussed more fully in the next section and it can be shown that when the wind of speed V at the sea surface is less than the wave speed c, the attenuation with the time is given by:

$$a = a_0 \exp\{-gs\rho'(V-c)^2 t/\rho c^3\},$$

where a is the amplitude after decay time t, a_0 the original amplitude, ρ' density of air, ρ density of sea water and s is a non-dimensional constant which Jeffreys took to be 0·27.

The third conclusion is not in accordance with this theory, but it can be made to agree by assuming the sheltering coefficient to be dependent on the wave steepness or a/λ; the modulus of decay will thus be assumed to be proportional to the amplitude at a particular moment and low waves will be attenuated relatively less than high waves. With this assumption, the attenuation formula becomes

$$1/a - 1/a_0 = \{S\rho'(V-c)^2 g^2 t\}/\rho c^5, \tag{12}$$

where Sa/λ replaces the original s, i.e. $s = Sa/\lambda$. Expressed in terms of wave period T, and assuming the speed of the surface wind V is two-thirds of the gradient wind speed U, the formula becomes

$$1/h - 1/h_0 = \text{const.}\,(2U/3 - 3T)^2 t/9T^5, \tag{13}$$

h and h_0 in feet, U gradient wind speed in knots, T wave period in seconds, t time in hours.

Jeffreys's theory was derived for a simple sinusoidal wave and it can be applied to each harmonic component of a complicated wave pattern only if it is assumed that the wind acts on each wave component independently. This assumption, though somewhat arbitrary, is made throughout this paper and it will be assumed therefore that the attenuation of each wave component is governed by its own particular characteristics. Equation (13) was therefore applied to separate values of H_T.

When $U = 0$, the right-hand side of the equation reduces to a constant multiplied by t/T^3, and the constant has been evaluated by investigation of values of H_T measured from spectra of swell which had travelled through areas of calm or light winds.

The observed and predicted values of H_T according to equation (13), are compared in figure 14. There is a good deal of scatter as would be expected because of the many uncertain factors involved, but there is sufficient agreement to suggest that the assumptions are reasonable. The mean values of the constant in equation (13), found for each unit period interval from 10 to 15 sec, are

interval (sec)	10	11	12	13	14	15
constant	29	32	39	32	27·5	33·5

Using the average of these, 32·2, the attenuation formula becomes

$$1/H_T - 1/H_{T_0} = 32{\cdot}2(2U/3 - 3T)^2/9T^5. \tag{13a}$$

Because of the form of equations (13) and (13*a*) the value of the constant must depend on the way in which H_T is defined; if H refers to unit wave velocity, or unit c/U instead of unit period, the value of the constant will be different.

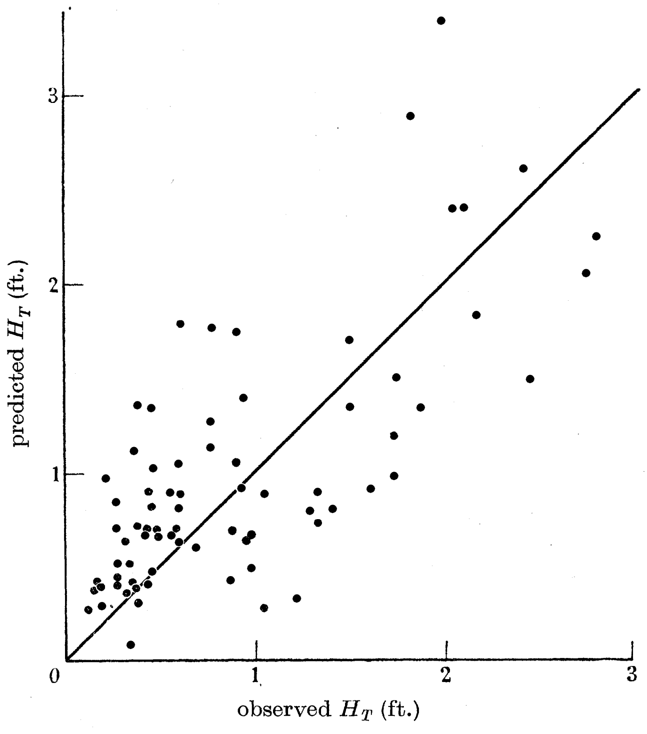


FIGURE 14. Comparison of predicted and observed values of H_T of swell from distant storms.

The formula applies in cases where U is appreciably less than $3T$. When the terms $2U/3$ and $3T$ approach equality, the effect of the tangential stress discussed in the next section becomes comparable to the effect of wind resistance; this would explain why waves of period $3T$ can be generated by a gradient wind of strength U.

There is no direct evidence for the effect of opposing winds but as far as can be judged at present, the attenuation is approximately according to equation (13a) when the negative sign is changed into a positive one.

A comparison of the changes in observed and predicted swell over a period of 3 days in March 1945, is shown in figure 15. The agreement is satisfactory.

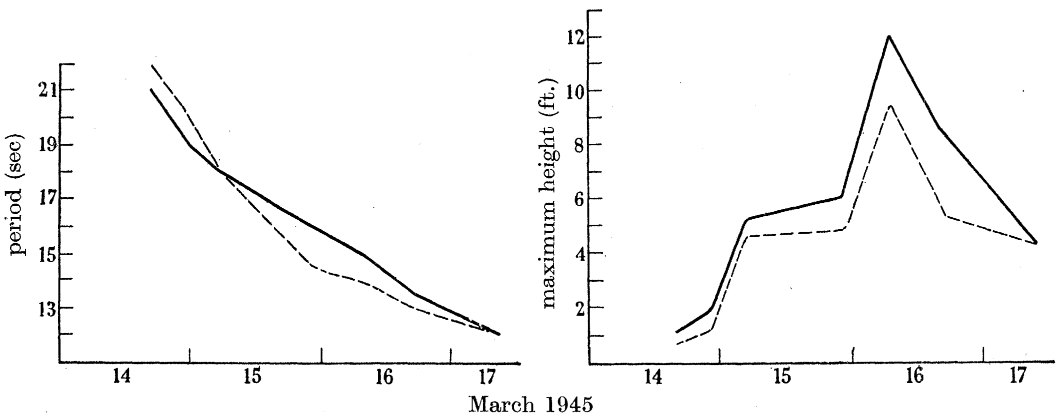


FIGURE 15. Comparison of observed (—) and predicted (- - -) wave period and height for swell at Pendeen, March 1945

PHYSICAL EXPLANATION

The formulation of the factors involved in wave generation can only be approximate and tentative until more is known about the effect of turbulence, but it appears that equilibrium is reached after 200 to 300 miles; after this distance the increase in wave characteristics is so slow as to be almost negligible. An attempt will now be made to fit a physical explanation to the empirical results.

Equilibrium must be reached when each wave component loses as much energy as it gains, and it is therefore necessary to consider ways in which waves can gain and lose energy.

Two factors by which the wind energizes the waves have been suggested and both will be assumed to act together.

(1) The first, discussed by Jeffreys and later by Sverdrup is the tangential stress of the wind on the sea surface. According to Stokes (1847) (and this has been confirmed experimentally), a water particle moves a finite amount after the passage of a wave so that there is a small residual drift velocity. It follows therefore that more work is done on the water particles when they move forward with the wave than is lost when they move back, so that a finite amount of energy is given to the wave particles each wave cycle.

(2) It has been suggested by Jeffreys that waves can be energized by the normal stress of the wind on the sea surface, owing to the formation of an eddy on the

leeward side of the wave so that the normal pressure is greater on the windward side. A finite amount of work is again done on the water particles after the passage of a wave cycle, since more energy is given to the water particles when they move on the windward side with a downward velocity than is taken away from them when they move on the leeward side with an upward velocity. The waves are only energized, however, when they travel slower than the wind; if they travel faster, an eddy tends to be formed on the windward side and the process is reversed, the waves losing their energy to the wind.

The last effect would prevent the formation of waves very much faster than the wind, and fast waves may lose energy to the wind in this manner. Most of the energy must, however, be lost by breaking. Although, theoretically, under calm conditions, all waves of steepness less than 1 in 7 should be stable, waves under the action of the wind are known to break when they are much less steep than this, and equation (10) shows that on the average they only conserve their energy over a distance of 100 miles or more if their steepness is less than $0{\cdot}149/V^{\frac{1}{2}}$, V being surface wind speed in knots.

All these factors may govern the spectral distribution of waves in the storm area and the height of waves of any period is not likely to exceed the value given by the steepness condition. The heights of waves moving faster than the wind would also be controlled by the balance between the energy gained due to the tangential stress and lost due to normal pressure. If it is assumed that the normal-pressure effect depends on a higher power of the wave amplitude value than the tangential effect, then it follows that for each wave speed exceeding the wind speed, there is a limiting value of wave amplitude at which the two effects become equal and cancel out and the wave is in equilibrium with the wind.

Two curves can thus be drawn, showing the maximum wave heights in terms of wave period (or wave speed) allowed by both the breaking effect and normal-pressure-attenuation effect. The curve for the breaking condition will show the height increasing indefinitely with the wave period, while the curve showing the limitation due to normal pressure will start at infinity for wave speeds equal to and less than the wind speed and decrease to zero as the wave speed increases. The optimum wave height and period will be given by the point of intersection of the two curves, waves of this period and height satisfying simultaneously all the required conditions.

These conditions will now be considered in more detail.

(a) *The effect of tangential stress*

At wind velocities exceeding 12 knots, the tangential stress is often quoted as $\tau_{xz} = \gamma^2 \rho' V^2$ where γ^2, the stress coefficient, is a non-dimensional constant of the order of $(2 \text{ to } 3) \times 10^{-3}$, ρ' is the density of air, V surface wind at 10 to 15 metres. This formula is based on the assumption that the aerodynamic roughness, which is a function of γ^2, is constant for wind speeds exceeding 12 knots. Neumann (1948), however, states that γ^2 is not constant for the sea surface and as a result of tilt measurements he derived a law $\tau_{xz} = \rho' \chi V^2$, where $\chi = 0{\cdot}09/V^{\frac{1}{2}}$, c.g.s. units being used.

This equation implies that the aerodynamic roughness of the sea surface decreases with the wind strength. This suggested dependence of γ^2 on the reciprocal of the square root of the wind speed corresponds to the empirical result that the limiting wave steepness is inversely proportional to the square root of the wind speed. The two results would be consistent if the aerodynamic roughness of the sea surface were to be determined by the steepness of the dominant sea waves. It is scarcely justifiable to use Neumann's value, derived from measurements on a different scale and including some terms due to vertical stresses, but providing the constant is changed it leads to results in accordance with those derived empirically in this paper.

Consider a single wave train of length λ. The average work done by tangential stress in unit time per unit area

$$= (1/\lambda)\int_0^\lambda \tau_{xz}(u + 4\pi^2a^2c\lambda^{-2})\,\mathrm{d}x,$$

where u is the horizontal component of the particle velocity and is equal to $ga\cos\theta$ where θ is the slope angle of the wave surface at any point x, c is the wave velocity and a the wave amplitude. The second term in the bracket corresponds to the mass transport velocity which represents the resultant motion of the water particles on the surface after the passage of a wave. Stokes showed that this term was necessary if the waves were to satisfy the irrotational condition and its existence has been verified experimentally by the Beach Erosion Board in America (1941).

On integration, the oscillatory term disappears and the expression becomes $\tau_{xz}4\pi^2a^2c\lambda^{-2}$. Using Neumann's ideas for the relation between the wind and the tangential stress, the average rate of work done per unit area per second is $k\rho' V^{\frac{5}{2}}4\pi^2a^2c\lambda^{-2}$; but $\lambda = 2\pi c^2/g$, where $\tau_{xz} = k\rho V^{\frac{5}{2}}$. Therefore, the average rate of work done by tangential stress is

$$k\rho' V^{\frac{5}{2}}4\pi^2a^2cg^2/4\pi^2c^4 = k\rho' V^{\frac{5}{2}}g^2a^2c^{-3}. \tag{14}$$

(*b*) *The effect of normal stress*

If the normal wind stress on the waves is τ_{zz}, then the rate of work done per unit area per second $= (1/\lambda)\int_0^\lambda \tau_{zz}w_0\,\mathrm{d}x$, where w_0 is the vertical component of the particle velocity at the surface and is given by $2\pi\lambda^{-1}ac\cos 2\pi\lambda^{-1}(x-ct)$. Jeffreys considered τ_{zz} to be equal to $-p_0-\Delta p$, where p_0 is the constant atmospheric pressure and Δp, caused by the formation of an eddy on the leeward side of the wave, is considered to be the sum of a series of harmonic terms of wave-length λ, 2λ, 3λ, etc., of which the one with the same frequency and phase as w_0 does a finite amount of work per wave-length.

Jeffreys thus assumed the variable part of the normal pressure to be effectively $\Delta p = s\rho'(V-c)^2\,\partial\eta/\partial x$, where $\partial\eta/\partial x$ represents the slope of the wave, and s is a non-dimensional constant called the sheltering coefficient. Now

$$\partial\eta/\partial x = 2\pi\lambda^{-1}a\cos 2\pi\lambda^{-1}(x-ct).$$

Therefore, $\Delta p = s\rho'(V-c)^2\,2\pi a\lambda^{-1}\cos 2\pi\lambda^{-1}(x-ct)$ and the rate of work per unit area per second $= (1/\lambda)\int_0^\lambda s\rho'(V-c)^2\,4\pi^2\lambda^{-2}a^2\cos^2 2\pi\lambda^{-1}(x-ct)\,\mathrm{d}x = \frac{1}{2}s\rho'(V-c)^2a^2g^2c^{-3}$.

This holds when the waves are travelling slower than the wind. When the wave velocity exceeds the wind velocity, Δp becomes out of phase with the slope of the wave and energy is lost to the air so that in general the rate of work per unit area per second is $\frac{1}{2}s\rho' \,|\, V-c \,|\, (V-c)\, a^2 g^2 c^{-3}$. In view of the evidence mentioned on p. 319, that high waves are attenuated more than low waves of the same period, it will be assumed that s is not constant but is proportional to a/λ so that $s = Sa/\lambda$, then the rate of work per unit area per second becomes

$$\tfrac{1}{2}S\rho'(V-c)\,|\, V-c \,|\, a^2 g^2 c^{-3} a \lambda^{-1} = \tfrac{1}{2}S\rho'(V-c)\,|\,V-c\,|\, g^3 a^3/2\pi c^5. \qquad (15)$$

(c) *Derivation of energy equation*

The energy per unit area in a wave is $\frac{1}{2}g\rho a^2$, ρ is the density of water. Therefore, from equations (14) and (15), considering the work done by tangential stress and normal stress,

$$\mathrm{d}(\tfrac{1}{2}g\rho a^2)/\mathrm{d}t = k\rho' V^{\frac{3}{2}} g^2 a^2 c^{-3} + \tfrac{1}{2}S\rho'(V-c)\,|\, V-c \,|\, g^3 a^3/2\pi c^5$$

and

$$\mathrm{d}(a^2)/\mathrm{d}t = a^2[2k\rho' V^{\frac{3}{2}} g/\rho c^3 + S\rho' \,|\, V-c \,|\, (V-c)\, g^2 a/2\pi c^5]. \qquad (16)$$

Conditions for a wave to be in equilibrium with the wind

If $c > V$, the second term in equation (16) is negative and $\mathrm{d}(a^2)/\mathrm{d}t$ will be zero when

$$2k\rho' V^{\frac{3}{2}} g/\rho c^3 = S\rho'(V-c)^2 a g^2/2\pi\rho c^5$$

i.e. when

$$a = 4\pi k V^{\frac{3}{2}} c^2/Sg(V-c)^2 = 4\pi k V^{\frac{3}{2}}/Sg(1-V/c)^2 = V^{\frac{3}{2}} f(c/V). \qquad (17)$$

This equation determines the limiting height permitted by the wind resistance conditions when $c > V$.

The second condition is that the waves of all periods must not be too steep for them to travel for a long distance under the action of the wind without losing much energy by breaking. This condition cannot be expressed precisely at present, but it is probably of the same form as the empirical result for the steepness of the highest waves expressed in equation (10) i.e. $a/\lambda = K/V^{\frac{1}{2}}$ where the value of K has to be determined. Therefore $a/2\pi c^2 = K/gV^{\frac{1}{2}}$, so that

$$a = 2\pi K c^2/gV^{\frac{1}{2}} = 2\pi K V^{\frac{3}{2}} c^2/gV^2 = V^{\frac{3}{2}}\phi(c/V). \qquad (18)$$

This gives the limiting height as far as wave steepness is concerned.

It follows that the highest wave to satisfy the conditions for both wind resistance and steepness is one having a wave period corresponding to a value of c that satisfies the equation $f(c/V) = \phi(c/V)$ i.e. when $4\pi k V^{\frac{3}{2}}/Sg(1-V/c)^2 = 2\pi K V^{\frac{3}{2}} c^2/gV^2$ or $2k/SK = (1-V/c)^2\,(c^2/V^2)$.

The value of c/V corresponding to the dominant wave is determined by the values of the constants; it can be shown from equation (8), which gives the value of the period of this wave, that $c/V = 1{\cdot}1$, assuming that $V = 2U/3$ where U is the gradient wind speed.

Application to wave spectrum

The foregoing analysis refers to a single train of waves of a definite wave-length and it may not be strictly applicable to a complex mixture of waves of various periods, although no great error is likely to be introduced when it is applied to determine the conditions for the dominant wave. With regard to its application to waves of other periods, the heights which the waves present to the wind are relevant i.e. the mean height of the waves of each period as seen by a visual observer over a suitable time interval. It has been shown on p. 309 that this height distribution can be inferred from the spectrum formula for H_μ or $H_{c/V}$ which gives a maximum value of H_μ very close to the maximum recorded wave height. Accordingly, it will be assumed that

$$H_\mu = 2a = 4\pi K V^{\frac{3}{2}} c^2/gV^2, \quad \text{when } c/V < 1{\cdot}1,$$

i.e. when wave height is limited by steepness, and

$$H_\mu = 2a = 8\pi k V^{\frac{3}{2}}/Sg(1 - V/c)^2, \quad \text{when } c/V > 1{\cdot}1,$$

i.e. when wave height is limited by wind resistance. The relation between H_μ and H_T can be found as before: $E_T\,\mathrm{d}T = E_\mu\,\mathrm{d}\mu$; therefore,

$$H_T^2\,\mathrm{d}T = H_\mu^2\,\mathrm{d}(c/V), \quad H_T = H_\mu^2(1/V)\,(\mathrm{d}c/\mathrm{d}T),$$

but as $$c = gT/2\pi \quad \text{then} \quad \mathrm{d}c/\mathrm{d}T = g/2\pi$$

and $$H_T = H_\mu\sqrt{(g/2\pi V)}.$$

Substituting this in the expression obtained for H_μ

$$H_T = 2KV(c^2/V^2)\sqrt{(2\pi/g)}, \quad \text{when } c/V < 1{\cdot}1,$$

$$H_T = \{4kV/S(1 - V/c)^2\}\sqrt{(2\pi/g)}, \quad \text{when } c/V > 1{\cdot}1,$$

so that

$$H_T/T = H_T/(2\pi c/g) = (gKVc^2/\pi cV^2)\sqrt{(2\pi/g)} = (gKc/\pi V)\sqrt{(2\pi/g)} \quad \text{for } c/V < 1{\cdot}1$$

and $$H_T/T = [\{2k(V/c)\,g\}/\{\pi S(1 - V/c)^2\}]\sqrt{(2\pi/g)} \quad \text{for } c/V > 1{\cdot}1.$$

Thus, as shown empirically in equation (1), there is some rational basis for regarding H_T/T as a function of c/V only. According to equation (1), H_T/T should equal 0·44 at $c/V = 1{\cdot}1$ and this result enables the values of the constants to be determined: for

$$0{\cdot}44 = Kc2^{\frac{1}{2}}g/\pi^{\frac{1}{2}}V = 2^{\frac{3}{2}}k(V/c)\,g^{\frac{1}{2}}/S(1 - V/c)^2\,\pi^{\frac{1}{2}}$$

solution of these equations gives $K = 0{\cdot}088$ and $k/S = 4{\cdot}3 \times 10^{-4}$. This value of K would give $a/\lambda = 0{\cdot}088/V^{\frac{1}{2}}$ where V is in ft./sec and therefore the steepness $= 2a\lambda = 0{\cdot}176/V^{\frac{1}{2}}$ and expressing V in knots, this becomes $\delta = 0{\cdot}136/V^{\frac{1}{2}}$ which, as might be expected, is slightly less than the value given by (10) which refers to the maximum wave height.

In terms of the wave height H_μ, the spectrum is given by

$$H_\mu = 4\pi \times 0{\cdot}088c^2V^{\frac{3}{2}}/gV^2 = 0{\cdot}034(c^2/V^2)\,V^{\frac{3}{2}} \quad \text{for } c/V < 1{\cdot}1 \tag{19}$$

and $$H_\mu = 8\pi kV^{\frac{3}{2}}/Sg(1 - V/c)^2 = 3{\cdot}37 \times 10^{-4}V^{\frac{3}{2}}/(1 - V/c)^2 \quad \text{for } c/V > 1{\cdot}1. \tag{20}$$

Curves representing (19) and (20) showing variation of H with c/V are shown in figure 16, the two curves intersecting at $c/V = 1{\cdot}1$. When $c/V = 1{\cdot}5$ corresponding to the case where the gradient wind is equal to $3T$ (U knots, T seconds) the height is one-fourteenth of the maximum value; this agrees very closely with the result obtained from equation (1*a*). To obtain a value for the equivalent wave height, the expressions for H_μ are squared and integrated from 0 to infinity and the square root of the result found. This gives $H = 0{\cdot}21U^{\frac{3}{2}}$ where U is the gradient wind speed in knots; this is a significantly lower result than that found from equation (2) which was $0{\cdot}027U^{\frac{3}{2}}$. This is due to the area enclosed by the theoretical curve of H_μ plotted against $c/V = 1{\cdot}1$ and $c/V =$ infinity which is much less than that enclosed by the empirical curve based on (1 a). This discrepancy can be attributed to the assumption that the wind is steady, made in deriving (19) and (20). The wind observations from which the empirical rule was deduced were rarely steady but varied about a mean value. This would modify the expressions obtained for the spectrum, making the result more like the empirical expression.

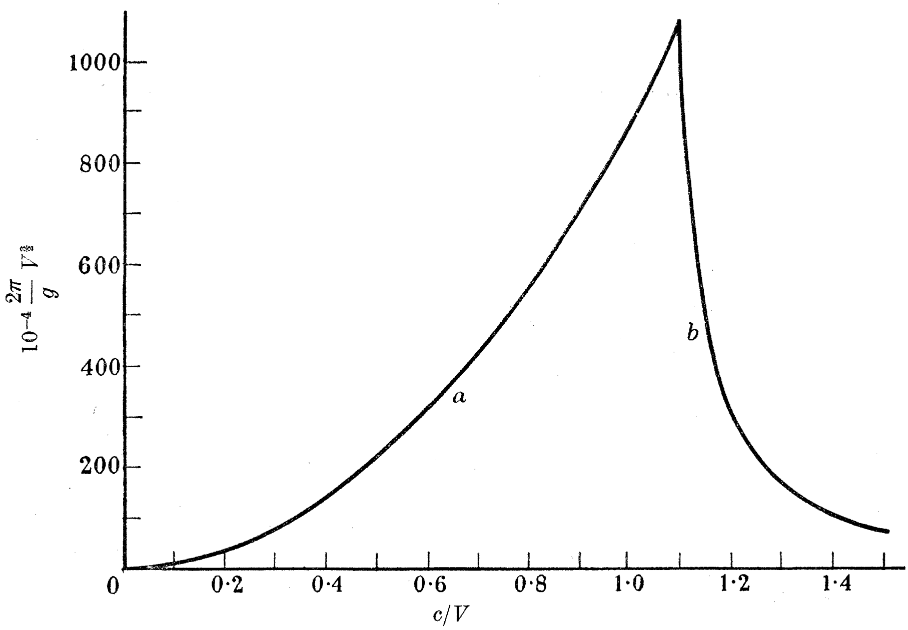


FIGURE 16. Theoretical wave spectrum curve:

$$a, 0{\cdot}088\frac{c^2}{V^2}\left(\frac{2\pi}{g}V^{\frac{3}{2}}\right); \quad b, 8{\cdot}6\times10^{-4}\left(1-\frac{V}{c}\right)^{-2}\left(\frac{2\pi}{g}V^{\frac{3}{2}}\right).$$

The value of the sheltering coefficient

It will be shown later that if in equation (16), $V \ll c$, the numerical value of the first term, which is due to tangential stress, is much less than the second, which is due to normal stress, and then the equation is reduced to

$$\mathrm{d}(a^2)/\mathrm{d}t = -a^3 S\rho'(V-c)^2 g^2/2\pi\rho c^5,$$

and by integration one obtains the relation

$$1/a - 1/a_0 = S\rho'(V-c)^2 g^2 t/4\pi\rho c^5,$$

where a_0 is the initial height, and a the final height after time t. This gives the attenuation of the waves when they travel in regions where there is a light following wind. The equation is identical with equation (13) and the value of S can be determined from the value of the constant in (13) which is 32·2. It is found that $S = 10{\cdot}9$, but since equation (13) refers to H_T values, S must depend on the definition of H_T, and will be denoted by S_T. This is because s the sheltering coefficient is equal to $S_T H_T/\lambda$, and since s must be independent of the way H_T is defined, it follows that there must be a relation between S_T and H_T. As discussed above, H_μ values express more closely than H_T values the mean observed height of waves of given period in the storm area and this will be true for swell as well as waves if the V applies to the speed of the wind that generated the waves originally. Thus

$$2s\lambda = H_\mu S_\mu = H_T S_T \quad \text{and} \quad S_\mu = 10{\cdot}9\sqrt{(g/2\pi V)}.$$

Since only the square root of the wind-speed is involved, no great error is introduced by assuming the mean surface wind to be always 36 ft./sec (gradient wind speed 32 knots); then $S\mu = 4{\cdot}1$. With a value of a/λ between 1/20 and 1/14, corresponding to the steepest waves possible, the value of the sheltering coefficient s would be between 0·20 and 0·29 which agrees closely with the value of 0·27 obtained by Jeffreys. He deduced this value from measurements of the least wind required to maintain waves and was presumably dealing with short steep waves. If so, this theory is in agreement with his observations. Sverdrup obtained a value of 0·013 from the decay of swell and this is again in agreement if the steepness is of the order of 1/150th which is of the right order for swell waves.

Value of k (the stress constant)

Substituting $S = 4{\cdot}1$ in the expression $k/S = 4{\cdot}3\times 10^{-4}$ found above, gives $k = 1{\cdot}76\times 10^{-3}$ in the f.p.s. system of units. k has the dimensions of the square root of a velocity, so in the c.g.s. system of units its value becomes

$$1{\cdot}76\times 10^{-3}\times 30{\cdot}6 = 1\times 10^{-2}.$$

This value is appreciably lower than Neumann's value of 9×10^{-2}, which might be expected since only horizontal stresses are taken into account. The low value of k justifies the assumption made in the last paragraph that the first term in the right-hand side of equation (16) is usually much less than the second.

Conclusions

Assuming that there is no energy exchange between wave components of different period, and that they are acted upon independently by the wind, rules are obtained giving the relative amplitude of the various components generated by a wind of given strength. These agree reasonably well with the results of wave observations in storm areas. With these rules as a basis and making the same basic assumptions, it is possible to estimate the speed of each wave component from a distant storm to the wave recording station and to study the attenuation of each wave component with time of travel. The additional rules so derived are again fairly satisfactory.

Equations which agree more or less with the empirical results can be derived by considering first the tangential and normal stress exerted by the wind on the water; the normal stress is assumed to be due to the mechanism suggested by Jeffreys but the sheltering coefficient is taken to be proportional to the wave steepness. Secondly, it is assumed that at the end of a long fetch there is an equilibrium value of wave steepness in which the gain of energy is balanced by the loss due to breaking and other factors. Attempts to produce a quantitative physical explanation involve using the empirical values of constants such as the sheltering coefficient or stress constant, but this is inevitable in the present state of knowledge.

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