A Further Investigation of Wind Generated Waves

By J. Darbyshire

(Hierzu Tafeln 1 und 2)

Summary. Sixty-four wave records taken by the O. W. S. "Weather Explorer" are investigated and Fourier analysed. The relation between the wind speed and the equivalent height H, and the frequency of maximum amplitude on the spectrum, f_0, do not differ greatly from those previously obtained in 1955. The wave spectrum, however, can now be shown to be reducible to the same form for all wind speeds for large fetch, H_f^2/H^2 being a function of $(f-f_0)$ only. The same function applies for shorter values of fetch, only, then, the relation is between H_f^2/yH^2 and $y\,(f-f_0)$, where y is the ratio of the values of H and $1/f_0$ at the short fetch to those for the same wind speed at infinite fetch.

Agreement can be found with a previous result by R. W. Burling [1955] under certain conditions. For a fetch of 4 miles, agreement to the same order is obtained with the values of mean square slope found by C. S. Cox and W. H. Munk [1954].

It has not been possible to find agreement with the results of Project SWOP (J. Chase, L. J. Cote, W. Marks and others [1957]).

Eine weitere Untersuchung windbedingter Wellen (Zusammenfassung). Vierundsechzig Wellenregistrierungen wurden an Bord des O. W. S. „Weather Explorer" aufgezeichnet und nach der Fourier-Methode analysiert. An dem Verhältnis zwischen der Windgeschwindigkeit und der äquivalenten Höhe H sowie der Frequenz der maximalen Amplitude auf dem Wellenspektrum f_0 wurde im Vergleich zu dem im Jahre 1955 festgestellten Verhältnis nur sehr wenig geändert. Bei großem Fetch kann jedoch das Wellenspektrum als auf die gleiche Form für alle Windgeschwindigkeiten zurückführbar dargestellt werden, da H_f^2/H^2 nur eine Funktion von $(f-f_0)$ ist. Die gleiche Funktion gilt auch für geringere Fetchlängen, nur ist dann die Beziehung zwischen H_f^2/yH^2 und $y(f-f_0)$, wobei y das Verhältnis der Werte H und $1/f_0$ bei kurzem Fetch zu den Werten gleicher Windgeschwindigkeit bei unendlichem Fetch darstellt.

Unter bestimmten Bedingungen kann eine Übereinstimmung mit einem früheren Ergebnis von R. W. Burling [1955] erreicht werden. Bei einem Fetch von 4 sm erhält man mit dem Quadrat des mittleren Gefälles nach C. S. Cox und W. H. Munk [1954] eine Übereinstimmung gleicher Größenordnung.

Eine Übereinstimmung mit den Ergebnissen des Projektes SWOP (J. Chase, L. J. Cote, W. Marks u. a. [1957]) ließ sich nicht erzielen.

Une autre étude des vagues engendrées par le vent (Résumé). Soixante-quatre enregistrements de vagues obtenus par le navire météorologique «Weather Explorer» seront analysés d'après la méthode de Fourier. Le rapport entre la vitesse de vent, la hauteur équivalente H et la fréquence de l'amplitude maximale sur le spectre d'onde n'a pas beaucoup changé lorsqu'on le compare au rapport constaté en 1955. H_f^2/H^2 n'étant qu'une fonction de $(f-f_0)$, on peut représenter le spectre d'ondes comme d'être réductible à la même forme, à laquelle toutes les vitesses de vent peuvent être référées en présence d'un grand «fetch». La même fonction s'applique à une longueur inférieure de fetch, mais dans ce cas la relation se fait entre H_f^2/yH^2 et $y(f-f_0)$; y étant le rapport des valeurs H et $1/f_0$ en présence d'un court fetch à des valeurs d'égale vitesse de vent en présence d'un fetch infini.

Dans certaines conditions, l'accord se laisse atteindre avec des résultats récents de R. W. Burling [1955]. Lorsque le fetch a 4 milles marins de longueur, l'accord se fait dans le même ordre de grandeur à l'aide des valeurs du carré de la pente moyenne d'après C. S. Cox et W. H. Munk [1954].

Il n'était pas possible d'obtenir un accord avec les résultats du projet SWOP (J. Chase, L. J. Cote, W. Marks et d'autres [1957]).

Introduction. Empirical rules concerning the generation of waves by wind have been described in previous papers (Darbyshire 1952, 1955, 1956). The 1952 paper described the results obtained with waves generated near the coast of Cornwall. It became possible later to record waves in the open sea by means of a ship-borne wave recorder (M. J. Tucker [1956]), and the 1955 paper described the results obtained by analysing these wave records. In the paper, 45 analyses of wave spectra were described and it became apparent that the rules governing wave generation in the open ocean differed significantly from those which held near the coast. The 1956 paper extended the open ocean results to cases where the fetch was of the order of 1 to 100 miles.

The wave records considered in the 1955 paper were taken during the period February 1953 to May 1954. These records were of only 7 to 10 minutes duration, and, as discussed previously [1957 b], were not long enough to give an adequate statistical sample of the wave conditions. Wave recording has been continued by the O. W. S. "Weather Explorer" and her successor the O. W. S. "Weather Reporter" and there is now available a large selection of wave records of longer duration. It was decided to use these records to reinvestigate the conditions of wave generation in deep water and to ascertain to what extent the previous results should be modified.

2. **Method of analysis.** About 100 wave records were selected from the period 1954–57, corresponding to wind conditions ranging from force 3 to force 10. These records were converted into a form suitable for analysis on the N. I. O. wave analyser (N. F. Barber, F. Ursell, J. Darbyshire, and M. J. Tucker [1946]). Most of the records were just enlarged optically, traced and blacked in by hand as described in the 1955 paper, but the records with the largest heights could not be dealt with satisfactorily by this means and in those cases the original record was put in a photographic enlarger and the elevations read at 2 sec. intervals and replotted on a suitable scale and the graph blacked in as before. Sixty-four of the wave analyses so obtained were found to be substantially free of the effect of extraneous swell and were investigated further. The duration of the wave record was in each case 1000 secs and spectrum obtained was divided up into equal frequency intervals of 0,007 secs^{-1} corresponding to seven successive peaks on the analysis. This is a departure from the procedure used in the previous papers where the spectrum was divided into equal intervals of 1 sec wave-period. The division into equal frequency intervals was done in this paper to bring the work more into line with that of other workers such as W. J. Pierson, Jr., G. Neumann, and R. W. James [1955].

The sum of the squares of the peaks within the frequency intervals were found and corrected to allow for the frequency response of the ship-borne wave recorder, given by the formula:

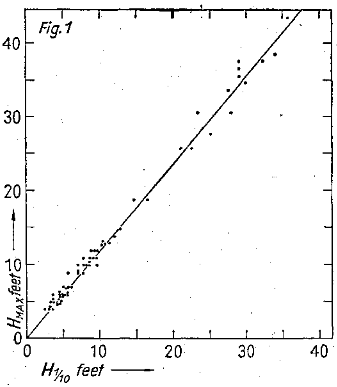
$$\text{(wave amplitude)} / \text{(recorded amplitude)} = 0.83\,[1 + (8.8 \cdot 2\,\pi\,f)^{-2}]^{3/2} \exp\,(4\,\pi^2 f^2\, d/g) \qquad (1)$$

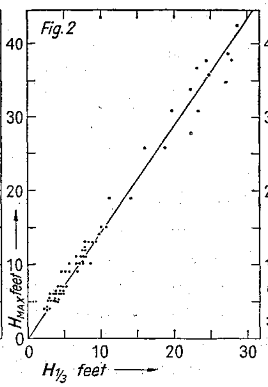
where f is the wave frequency and d is the mean depth below the water-line. Finally the values are smoothed by using binomial coefficients so that:

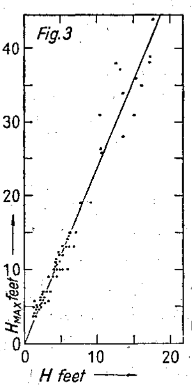
$$H'^2_{f2} = \tfrac{1}{4}\,(H^2_{f1} + 2H^2_{f2} + H^2_{f3})$$

where f_1, f_2, and f_3 refer to any three consecutive frequency intervals. The complete sum of squares, $H^2 = \Sigma\, H_f^2$ was also found for each record.

The original records were also measured to find the maximum wave height $H_{\max}$, the mean height of the highest tenth of the waves, $H_{1/10}$, and the mean height of the highest third, $H_{1/3}$. The values obtained were corrected using equation (1) and assuming a value of







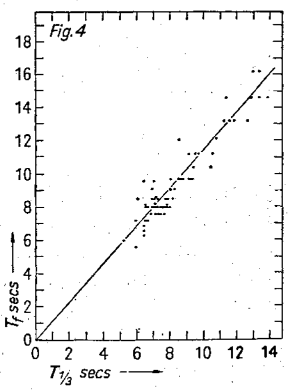


Fig. 1. Plot of $H_{\max}$ against $H_{1/10}$ Fig. 2. Plot of $H_{\max}$ against $H_{1/3}$
Fig. 3. Plot of $H_{\max}$ against H Fig. 4. Plot of T_f against $T_{1/3}$

f equal to f_0 the frequency of the class having the largest value of H_f^2 on the spectrum. The significant period $T_{1/3}$, that is the mean period of the highest third of the waves on the record was also found. $H_{1/10}$, $H_{1/3}$, and H are plotted against $H_{\max}$ in figs. 1, 2, and 3, and $T_{1/3}$ against T_f $(= 1/f_0)$ in fig. 4. The following relations were obtained.

$$H_{\max} = 1.20\ H_{1/10} \tag{2}$$

$$H_{\max} = 1.45\ H_{1/3} \tag{3}$$

$$H_{\max} = 2.40\ H \tag{4}$$

$$T_f = 1/f_0 \quad = 1.14\ T_{1/3}. \tag{5}$$

In fig. 5, T_f is plotted against T_T, the period of the class having the maximum $H^2{}_T$ value when the spectrum is split up into classes of 1 sec. period. It was found that

$$T_f = 1.06\ T_T. \tag{6}$$

Relations between $H_{\max}$, $H_{1/10}$, $H_{1/3}$, and H were worked out theoretically by M. S. Longuet-Higgins in [1952], who obtained lower values of the ratios, $H_{\max}$ to $H_{1/10}$, $H_{\max}$ to $H_{1/3}$, and $H_{\max}$ to H, than those given by equations (2), (3), and (4). He was, however, dealing with a narrow spectrum and better agreement is obtained with the results of a later paper by D. E. Cartwright and M. S. Longuet-Higgins [1956], dealing with broader spectra. Equation (4) is also different from the relation $H_{\max} = 2.0\ H$ originally obtained by the author in 1952. The previous result was, however, based on a collection of swell and wind wave records taken at Perranporth, Cornwall, and so the spectra were narrower than those of the weather ship records because of the attenuation of the shorter period waves at the depth of recording, 50 ft. equations (2) to (5) are based entirely on locally generated waves.

3. Relations with wind speed. In the previous papers, the gradient wind speed was calculated from the synoptic charts but in view of criticisms by G. Neumann and W. J. Pierson, Jr. [1957], it was decided in this investigation

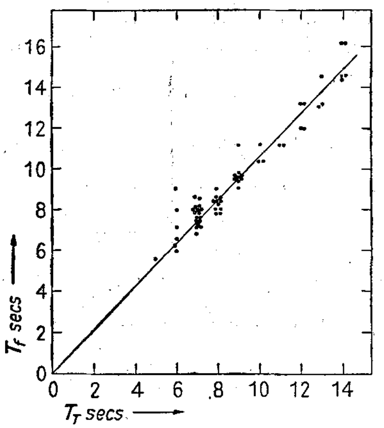
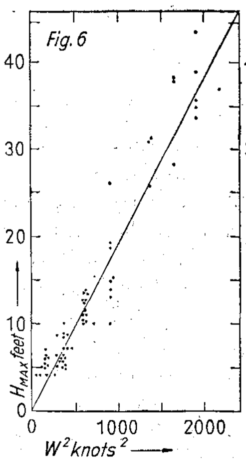
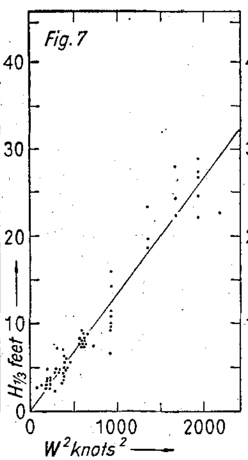
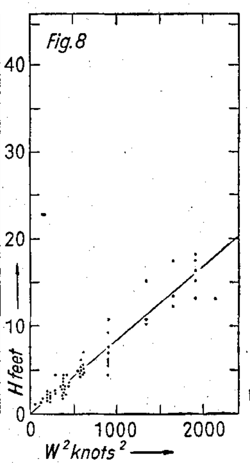


Fig. 5. Plot of T_f against T_T







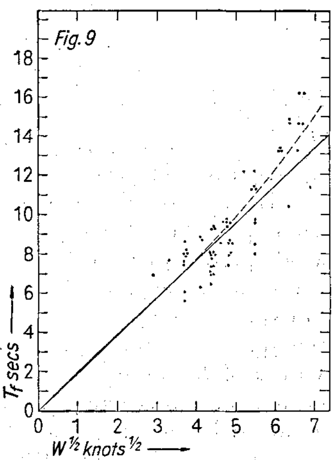


Fig. 6. Plot of $H_{\max}$ against W^2 Fig. 7. Plot of $H_{1/3}$ against W^2

Fig. 8. Plot of H against W^2 Fig. 9. Plot of T_f against $W^{\frac{1}{2}}$

to use the surface wind speeds as measured by the weather ship. There was a large variation in wind speed, fetch, duration, and atmospheric stability, as shown in tables 1, 2, and 3, section 5, but the results are all grouped together in figs. 6, 7, 8, and 9 where $H_{\max}$, $H_{1/3}$, and H are plotted against the square, and T_f against the square root of the wind speed. In view of the diversity of the conditions, the scatter, particularly with $H_{\max}$ and $H_{1/3}$, is very reasonable. The following relations were found.

$$H_{\max} = 0.0193\ W^2 \qquad H \text{ ft, } W \text{ knots} \qquad (7)$$

$$H_{1/3} = 0.0133\ W^2 \qquad H \text{ ft, } W \text{ knots} \qquad (8)$$

$$H = 0.0081\ W^2 \qquad H \text{ ft, } W \text{ knots} \qquad (9)$$

$$T_f = 1.94\ W^{\frac{1}{2}} \qquad T \text{ secs, } W \text{ knots.} \qquad (10a)$$

Equation (10a) is not very satisfactory, however, for high wind speeds, and better agreement is obtained by using

$$T_f = 1.94\ W^{\frac{1}{2}} + 2.5 + 10^{-7}\ W^4. \qquad (10b)$$

Equation (10b) is shown by the broken line in fig. 9. Equations (9) and (10a) are in reasonable agreement with the corresponding equations found in (J. Darbyshire [1955]). If the gradient wind U is assumed to be (3/2) W, equation (9) becomes

$$H = 0.0036\ U^2$$

as against $H = 0.0038\ U^2$ (equation 3a, 1955),

and assuming equations (6) and (10a) and $U = 3/2\ W$,

$$T_T = 1.83\ W^{\frac{1}{2}} = 1.50\ U^{\frac{1}{2}}$$

as against $T_T = 1.64\ U^{\frac{1}{2}}$ (equation 2a, 1955).

4. Derivation of wave spectrum. In the [1955] paper, a spectrum formula was obtained:

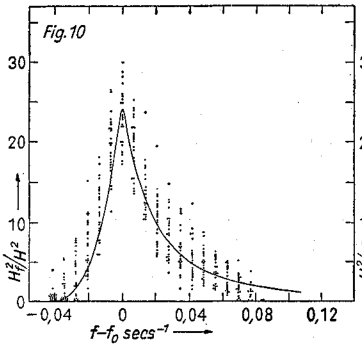
$$H^2{}_T\, dT = 0.0036^2\ (U^{\frac{1}{2}} - 0.43\ T)^2 T^5 dT \qquad (H \text{ ft, } T \text{ secs, } U \text{ knots}), \qquad (11)$$
$$= 0 \text{ for } 0{,}43\ T > U^{\frac{1}{2}}$$

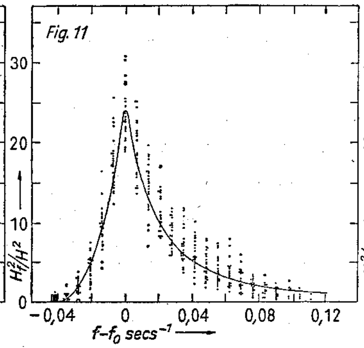
or in terms of frequency f

$$H^2 f df = 0.0036^2\ (U^{\frac{1}{2}} - 0.43/f)^2 f^{-7} d \qquad (f \text{ in secs}^{-1}), \qquad (12)$$
$$= 0 \text{ for } 0.43/f > U^{\frac{1}{2}}.$$

These formulae suffer from the disadvantage that H_T is a maximum at $T = 1.64\ U^{\frac{1}{2}}$ whereas H_f reaches a maximum at $T = 1.79\ U^{\frac{1}{2}}$, indicating a greater variation than that given by equation (6). The formulae also fail in another respect as will be seen in section 5.

In the previous papers, attempts were made to relate the H_T values directly with the wind speed but this method is not very satisfactory as usually minor errors in the values of wind speed lead to major errors in values of H_T and as in the 1955 paper, a large scatter is obtained in the plots of H_T against U. It was, accordingly, decided in the present investigation to express the spectrum entirely in terms of H and f_0. These can be obtained directly from the spectrum and depend directly on the wind speed as shown by equations (9) and (10). Values of H_f were thus normalized by dividing by H^2. It was not possible to obtain distributions of points independent of wind speed by plotting H_f^2/H^2 against f/f_0 but this could be obtained by plotting against the difference $(f - f_0)$. Plots of H_f^2/H^2 against $(f - f_0)$ are shown in figs. 10, 11, and 12, fig. 10 referring to wind forces 3, 4, and 5, fig. 11 to wind forces 6 and 7, and fig. 12 to wind forces 8, 9, and 10. It is clear that the points in all three figures can be represented by the same distribution curve. The curve represented by the solid line is given by:





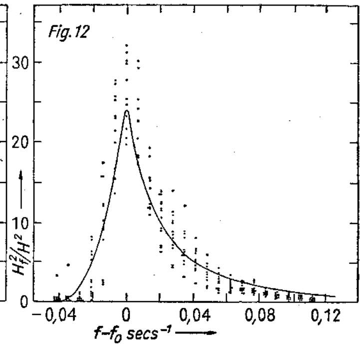


Fig. 10. Plot of H_f^2/H^2 against $(f–f_0)$ for winds of force 3, 4, and 5
Fig. 11. Plot of H_f^2/H^2 against $(f–f_0)$ for winds of force 6 and 7
Fig. 12. Plot of H_f^2/H^2 against $(f–f_0)$ for winds of force 8, 9, and 10

$$H_f^2/H^2 df = 23.9 \exp - \left[\frac{(f-f_0)^2}{0.00847\,\{(f-f_0)+0.042\}}\right]^{\frac{1}{2}} df \tag{13}$$

$$= 0 \text{ when } (f-f_0) < -0.042,$$

$$f,\ f_0, \text{ secs}^{-1}.$$

This is the simplest single expression that can be found to give values near the mean of the points for each value of $(f–f_0)$. A more exact representation is possible by using different functions over different parts of the range but equation (13) is more convenient mathematically and is reasonably accurate. The rather greater fall off at high frequencies in fig. 12 is due to the elevations on the wave records at high wind speeds being read at 2 secs intervals as described in section 2.

5. **Discussion.** The maximum variation of the points for a given value of $(f–f_0)$ is not often more than 3 to 1. This is in agreement with the scatter to be expected statistically (see G. Neumann and W. J. Pierson, Jr. [1957]), as 21 harmonics (3 × 7) were considered, corresponding to 42 degrees of freedom and for this number the χ^2 tables give the 95% confidence limits as 0.675 to 1.48 times the mean value.

As stated above, a wide variety of conditions were represented by the wave records. Table 1 gives an analysis of the wind force and duration, table 2, wind force and fetch, and table 3, wind force and atmospheric stability.

Table 1

Wind force and duration

Wind force	No. of cases	Duration (hours)					
		3–6	6–12	12–18	18–24	24–36	over 36
3	1			1			
4	10	2				8	
5	18	1	1	1		15	
6	14		1	2	1	8	2
7	9	1	4	2	1	1	
8	3			1			2
9	8		2	2	1	2	1
10	1			1			

Table 2

Wind force and fetch

Wind force	No. of cases	Fetch (nautical miles)					
		100–200	200–300	300–500	500–750	750–1000	over 1000
3	1			1			
4	10	1	2	4	2		1
5	18		3	5	6	4	
6	14		3	10			1
7	9			3	5		1
8	3		1	2			
9	8			2	3	1	2
10	1				1		

With force 9, there was one case of 24 hours duration and over 1000 miles fetch and one of 36 hours duration and 800 miles fetch. There were two cases of force 8, 400 miles fetch and 36 hours duration.

Table 3

Wind force and atmospheric stability

Wind force	θ_a–θ_s, Temperature difference (Air-Sea) ^{0}F		
	$\theta_a - \theta_s \leqslant -3^0$	$-3^0 < \theta_a - \theta_s < 3^0$	$\theta_a - \theta_s \geqslant 3^0$
3	1		
4	3	5	2
5	2	6	10
6	4	5	1
7	1	8	1
8	2	1	
9	3	5	
10	1		

According to the curves given by Pierson, Neumann, and James [1955], waves become 'saturated' and cease growing when the fetch and duration exceed certain values at various wind speeds as given in Table 4.

Table 4

Saturation conditions according to Pierson, Neumann and James

Wind force	Duration (hours)	Fetch (nautical miles)
3	2.4	10
4	6.6	40
5	9.0	70
6	14	130
7	23	280
8	34	500
9	52	960
10	69	1420

A glance at the tables will show that almost all the points for force 3, 4, and 5 correspond to saturation conditions and about half the points for force 6 and 7. Almost all the points for force 8, 9, and 10 correspond to unsaturated conditions. The variations observed between the points in figs. 10, 11, and 12 are, however, no more than could be expected statistically and there is certainly no significant difference between the distribution of points on figs. 10 and 12. The only possible discrepancy could be the sharper cut-off at high frequencies in figs. 12 which has already been explained and also the somewhat sharper cut-off at low frequencies. This could be explained by a certain amount of swell of longer period than the wind generated waves being often present at low wind speeds and masking the cut-off. Table 3 shows that the proportion of points corresponding to stable conditions decreases as the wind speed increases but no effect corresponding to this is apparent in figs. 10, 11, and 12.

The fact that H_f^2/H^2 can be expressed entirely as a function of $(f-f_0)$ must be very significant. This relation cannot be derived from the earlier spectrum formula, equation (12), for when it is combined with equations (9) and (10a), one obtains:

$$\begin{aligned} H_f^2/H^2 &= A \cdot (U^{\frac{1}{2}} - 0.43/f)^2 f^{-7} U^{-4} df \\ &= A \cdot (B/f_0 - 0.43/f)^2 f_0^{8} f^{-7}\, df \end{aligned} \quad \text{where } A \text{ and } B \text{ are constants.}$$

This cannot be reduced to a function of $(f-f_0)$ only.

Similarly the spectrum formula proposed by G. Neumann [1954] is of the form:

$$H_f^2\, df = (C/f^6) \exp - k/f^2 W^2 \cdot df$$

from which it can be shown that H^2 varies as W^1 and f_1 as W^{-1} and so

$$\begin{aligned} H_f^2/H^2 &= (D/f^6 W^5) \exp - K_1/f^2 W^2 \cdot df \\ &= (D_1 f_0^{5}/f^6) \exp - K_2/(f^2 f^2_0) \cdot df \end{aligned}$$

where C, D, D_1, k, K_1, and K_2 are constants.

This again cannot be reduced to a function of $(f-f_0)$ only.

Similarly the modification of the Neumann formula due to H. U. Roll and G. Fischer [1956], reduces to:

$$H_f^2/H^2 = (D_3 f_0^{5}/f^5) \exp -K_2/(f^2/f_0^2) \cdot df, \; D_3' \text{ and } K_2 \text{ constants}$$

which is a function of f/f_0 but not $(f-f_0)$.

As the results have been obtained with a ship-borne wave recorder, there may be some uncertainty about the effect of the ship on the waves and whether waves of shorter period are being attenuated exactly as stated by the theoretical formula (1) under actual conditions. A good check for the form of the spectrum is provided by the use of a buoy containing an accelerometer, the output of which is doubly integrated to give the heave. This instrument is floated quite clear of the ship and measures waves of period as low as 1 sec quite satisfactorily. Some of these wave records have been analysed by D. E. Cartwright with a digital computer, using the methods suggested by J. W. Tukey [1949]. The results of two such spectrum analyses are shown in fig. 13 as plots of H_f^2/H^2 against $(f-f_0)$. The frequency interval used is 0,032 $secs^{-1}$ which is wider than the 0,007 $secs^{-1}$ used in this investigation and the values should be compared with values based on equation (13) meaned over this wider interval. The curve corresponding to (13) is superimposed on the plots and there is reasonable agreement particularly over the high er frequencies where the ship-borne wave recorder would be

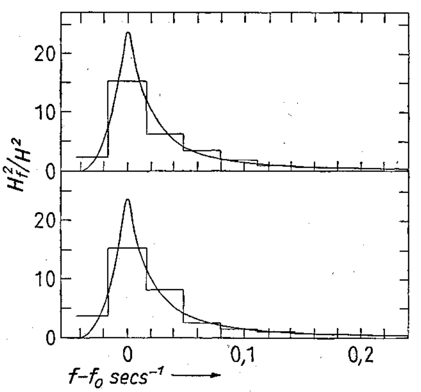


Fig. 13. Comparison of predicted spectrum curve with spectra of waves recorded by the wave measuring buoy

expected to be most inaccurate. The two records were chosen at random and there was a certain amount of swell which rather obscures the fall off at low frequencies. These results provide a check on the use of the N. I. O. analyser as well as the ship-borne wave recorder.

6. **Extension to a shorter fetch.** It is interesting to see how equation (13) becomes modified when the fetch is short and the conditions are definitely 'unsaturated'. No new observations in deep water under these conditions are available and it was decided to re-examine data which have been described before in the [1956] paper, that is the wave measurements at Lough Neagh and Staines Reservoir.

In the [1956] paper, it was stated that in general,

$$T_T = 1.64yU^{\frac{1}{2}} \text{ and } H = 0.0038yU^2,$$

where $y \to 1$ as x the fetch approaches infinity as given by the relation

$$y = \frac{(x^3 + 3x^2 + 65x)}{(x^3 + 12x^2 + 260x + 80)} \qquad x \text{ in nautical miles.} \qquad (14)$$

The original equations for H and T_T for a long fetch found in 1955 have now been modified as shown by equations (9) and (10). It is proposed to modify the short fetch equations similarly so that in general:

$$H = 0.0081yW^2 \qquad H \text{ ft } W \text{ knots,} \qquad (15)$$

$$1/f_0 = T_f = 1.94yW^{\frac{1}{2}} \qquad T \text{ secs } W \text{ knots,} \qquad (16\text{a})$$

$$\text{or } T_f = y(1.94W^{\frac{1}{2}} + 2.5 \cdot 10^{-7}W^4) \qquad T \text{ secs } W \text{ knots.} \qquad (16\text{b})$$

Lines based on these new equations and equations (4) and (5) are still in agreement with the plots of points shown in figs. 2 and 3 of the [1956] paper.

The Lough Neagh measurements were taken with wave pressure recorders at depths of 2 ft and 5 ft. To investigate the spectra, it was thought better to confine the investigation to waves recorded at 2 ft as there is then less correction required for the attenuation of wave pressure with depth. Six such records were analysed, with the fetch varying from $1\frac{1}{2}$ to 10 miles and the wind speed from 8 to 18 knots. The spectra were split into equal frequency intervals of 5 peaks corresponding to a range of 0.015 secs^{-1}. H_f^2/H^2 and $(f-f_0)$ were plotted against each other as before but the points did not fall on the curve given by equation (13). If, however, H_f^2/yH^2 was plotted against $y(f-f_0)$, where y is given by equation (14), the points fall on the curve as shown in fig. 14. In this case 5 harmonics have been taken and smoothed over three successive intervals as before so that there were 30 degrees of freedom for which

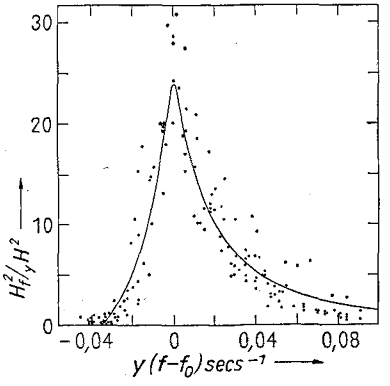


Fig. 14. Plot of H_f^2/yH^2 against $y(f-f_0)$ for Lough Neagh wave data

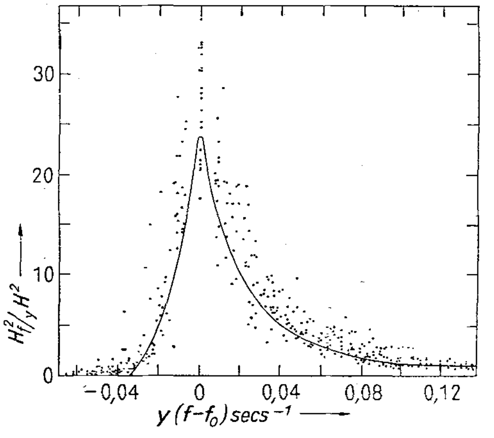


Fig. 15. Plot of H_f^2/yH^2 against $y(f-f_0)$ for Staines Reservoir wave data

the χ^2 tables give for the 95% confidence limits a variation from the mean of 0.56 to 1.56 agreeing with the scatter observed.

In the case of Staines Reservoir, R. W. Burling [1955] has worked on wave records taken with a capacitance surface recorder described by M. J. Tucker and H. Charnock [1954]. Burling gives details about 23 wave spectra where the fetch varied from 400 metres to 1350 metres and the wind speed from 5 to 8.5 metres/sec. The sum of squares of the peaks were taken over frequency intervals of 0.025 secs^{-1} corresponding to five peaks on the analyses. Burling did not take a binomial mean over three successive intervals but analysed from 2 to 16 portions of a long record in each case and took the mean of the H_f^2 values. The least number of degrees of freedom used was thus still about 20 to 30. Burling's data are rearranged and plotted as H_f^2/yH^2 against $y(f-f_0)$ in fig. 15. Equation (14) does not appear to give accurate values of y for x less than 0,5 n. miles and so in this case y was found by dividing the observed T_f by $1.94W^{\frac{1}{2}}$. The points again fall about the curve given by equation (13).

In general then, one can state:

$$(H_f^2/\sigma/yH^2)df = 23.9y \exp - \left[\frac{y^2(f-f_0)^2}{0.00847\ \{y(f-f_0) + 0.042\}}\right]^{\frac{1}{2}} df \qquad (17a)$$

$$= 0 \text{ when } y(f-f_0) < -0.042$$

$$\text{for range } -0.042 < y(f-f_0) < 0.100$$

and combining this with equation (15) gives

$$H_f^2\ df = 1.57 \cdot 10^{-3} y^3 W^4 \exp - \left[\frac{y^2(f-f_0)}{0.00847\ \{y(f-f_0) + 0.042\}}\right]^{\frac{1}{2}} df \qquad (17b)$$

$$= 0 \text{ when } y(f-f_0) < 0.042$$

for range $\quad -0.042 < y(f-f_0) < 0.100.$

$\int H_f^2\ df$ can be evaluated numerically and is equal to:

$$6.6 \cdot 10^{-5} y^2 W^4 \qquad (18)$$

in agreement with equation (15).

7. The spectrum shape at high frequencies. Burling [1955] found that over the range of wind speeds and fetch of his measurements, the energy at high frequencies on the wave spectra was a function only of the frequency and varied as f^{-5}. It is not possible to derive this result analytically from equation (17b) but H_f^2 values at frequencies of 1.00, 1.125, 1.25, 1.50, 1.75, and 2.00 secs^{-1} were estimated from equation (17b) for all the conditions considered by Burling. The results are shown in figs. 16,a and 16,b. In fig. 16,a, the wind speed and the value of f_0 given by Burling are used in each case to evaluate y, by using equation (16,a). In fig. 16,b, only the values of the wind speed and fetch are assumed, y being calculated from equation (14). Burling's measurements for the same frequencies are shown in fig. 16,c. The points in figs. 16,a and 16,b do tend to decrease with the fifth power of the frequency and the apparent scatter is comparable to the scatter on Burling's data. The line drawn on the three diagrams corresponds to:

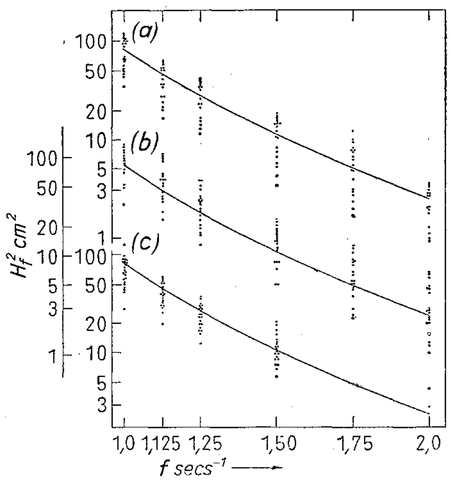


Fig. 16. Variation of H_f^2 with f at high frequencies for Staines Reservoir data. a and b, predicted from formula; c, observed.

$$H_f^2\,(\mathrm{cm}^2) = 83.5 f^{-5}. \tag{19}$$

Equation (17 b) is not therefore inconsistent with Burling's results and it appears that the spectra of short waves should be investigated over a wider variety of conditions to determine whether the f^{-5} rule is really independent of wind speed and fetch.

8. **Mean square slope.** It was shown previously [1956] that with the spectrum formula obtained for a short fetch, agreement could be found with the results of measurements of wave slope taken by C. S. Cox and W. H. Munk [1954] by sun glitter observations at Hawaii, if the fetch was assumed to be 4 or 5 miles.

As shown by Cox and Munk, the mean square slope σ^2 is given by:

$$\sigma^2 = \int_0^\infty (4\pi^2/\lambda^2) \cdot \tfrac{1}{8} H_f^2\, df,$$

or substituting equation (17 b) and assuming $\lambda^2 = g^2/4\pi^2 f^4$

$$\sigma^2 = 1.57 \cdot 10^{-3} y^3 W^4 2\pi^4 g^{-2} \int_0^\infty f^4 \exp - \left[\frac{y^2(f-f_0)^2}{0.00847\,\{y(f-f_0) + 0.042\}}\right]^{\frac{1}{2}} df. \tag{20}$$

The integration can be carried out approximately by splitting the range into eight sections. For purposes of computation, the formula is assumed to hold also for $y(f-f_0) > 0.100$.

(1) $y(f-f_0) > 0.126$
(2) $0.126 > y(f-f_0) > 0.084$
(3) $0.084 > y(f-f_0) > 0.063$
(4) $0.063 > y(f-f_0) > 0.042$
(5) $0.042 > y(f-f_0) > 0.021$
(6) $0.021 > y(f-f_0) > 0$
(7) $0 > y(f-f_0) > -0.021$
(8) $-0.021 > y(f-f_0) > -0.042$

In the range (1), the integral can be reduced approximately to

$$\int_{\frac{0.126}{y}+f_0}^{\infty} f^4 \exp - \left[\frac{y^{\frac{1}{2}}(f-f_0)^{\frac{1}{2}}}{(0.00847)^{\frac{1}{2}}}\right] df;$$

on substituting $u = (f-f_0)^{\frac{1}{2}}$, this reduces to:

$$\int_{(0.126/y)^{\frac{1}{2}}}^{\infty} (u^2 + f_0)^4 2u \exp -u/k_1\, du$$

where $1/k^2_1 = y/0.00847$;

on expanding and integrating term by term by parts, one obtains:

$$(2 \cdot 9!\,k^{10}_1 + 8f_0 7!\,k^8_1 + 12f^2_0 5!\,k^6_1 + 8f^3_0 3!\,k^4_1 + 2f^4_0 k^2_1) \cdot \Big[\exp- u/k_1\Big]_{(0.126/y)^{\frac{1}{2}}}^{\infty}$$

with the values of k_1 and f_0 used in practice, the last term is much greater than the others and the integral for this range becomes:

$$0.00035 f^4_0/y.$$

For other ranges, the integral is approximated by

$$\int_a^b f^4 \exp - y(f-f_0)/C\, df$$

where a, b, and C vary with the range used; substitution of $v = f - f_0$ and $1/k_2 = y/C$ leads to the form:

$$\int_a^b (v + f_0)^4 \exp - v/k_2 \, dv.$$

Splitting up and integrating by parts, one obtains:

$$(k^5_2 4! + k^4_2 4 f_0 3! + k^3_2 6 f^2_0 2! + k^2_2 4 f^3_0 + k_2 f^4_0) \left[\exp - v/k_2\right]_a^b.$$

With values of f_0 and k_2 obtained in practice, only the last three terms are important and integrating over the ranges (2) to (8) and summing up, one obtains:

$$24.31 \cdot 10^{-5} f^2_0/y^3 + 34.82 \cdot 10^{-4} f^3_0/y^2 + 39.70 \cdot 10^{-3} f^4_0/y.$$

The contribution due to range (1) is therefore negligible in comparison with that due to the other ranges. Substituting equation (16a) in equation (20), one obtains:

$$\sigma^2 = 1/y^2(1.90 \cdot 10^{-8} W^3 + 1.42 \cdot 10^{-7} W^{5/2} + 8.21 \cdot 10^{-7} W^2), \tag{21}$$

W in knots, or if W is in cms/sec,

$$\sigma^2 = 1/y^2(1.52 \cdot 10^{-13} W^3 + 0.81 \cdot 10^{-11} W^{5/2} + 3.28 \cdot 10^{-10} W^2). \tag{21a}$$

The W^2 term is the greatest for the normal range of wind speeds and so the formula is not dissimilar in form to the one derived in 1956 in which σ^2 depended on W^2/y^2. Equation (21a) gives somewhat lower values, however, and if a fetch of 4 miles is taken as appropriate to the Cox and Munk measurements, corresponding to $y = 0.26$, a value σ^2 of 0.011 is obtained for a wind speed of 1000 cms/sec. This is lower by a factor of about 2 than the value that was obtained for a sea surface covered with oil slicks. It is probable, however, that the values obtained by Cox and Munk are somewhat high because of the presence of extraneous swell as shown by the large constant which they obtain at zero wind speed in their expression

$$\sigma^2 = 1.56 \cdot 10^{-5} W + 0.008.$$

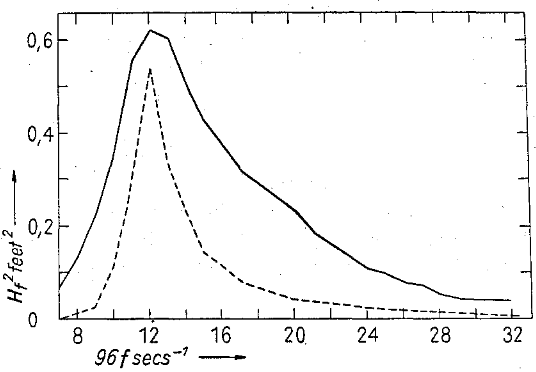


Fig. 17. Comparison of predicted spectrum with spectrum observed in Project SWOP

9. Project SWOP. A determination of a two-dimensional spectrum (Project SWOP) was carried out by J. Chase, L. J. Cote, W. Marks, E. Mehr, W. J. Pierson, Jr., F. C. Ronne, G. Stephenson, R. C. Vetter, and R. G. Walden [1957]. They combined the results of stereo-photography with records taken with the Woods Hole Wave Measuring Pole described by H. G. Farmer, W. Marks, R. G. Walden, and G. G. Whitney [1955]. This work has already been discussed by the author [1957b]. The one-dimensional form of the spectrum is shown in fig. 17. A predicted spectrum using equations (16a) and (17) with a wind speed of 18.7 knots is shown as well. While there is some measure of agreement at the peak, the difference at all other points is much greater than that shown by most of the points in figs. 10, 11, 12, 13, 14, and

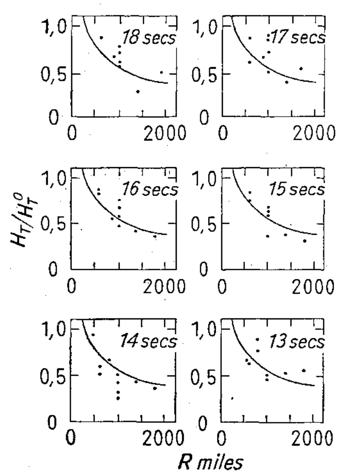


Fig. 18. Variation of H_T/H^0_T with R

15 and must be very significant. The author [1957 b] has pointed out that the conditions in the area where the spectrum was obtained may not be typical.

10. **Attenuation of swell with distance.** It is not the purpose of this investigation to deal at length with the attenuation of waves as they leave the storm area, as this problem has already been studied (Darbyshire [1957 a]). In that investigation, observed H_T values of swell were compared with values computed by using equation (11). As this equation has now been replaced by equations (10) and (17), it was necessary to recalculate the H_T values in the storm area. No significant change was found, however, in the results by using the modified values, and graphs corresponding to those of fig. 4 of the [1957] paper are shown in fig. 18. Equation (4) of the [1957] paper

$$H_T/H^0{}_T = (300/R)^{\frac{1}{2}}, \tag{22}$$

R, attenuation distance in nautical miles, can still be applied and is represented by the solid line in fig. 18.

11. **Conclusions.** The work has shown that expressions for H and $T_f = 1/f_0$ previously found need only slight modification in the light of more observations. The important result is found that the one-dimensional wave spectrum can be expressed as a function of $(f–f_0)$. This function has the same form at all fetch values and the function at any fetch is connected to the one at infinite fetch by the parameter y which is also the ratio of the values of $1/f_0$ and H at this fetch to the value at infinite fetch.

Agreement can be obtained with the rule found by Burling that the energy at high frequencies varies as f^{-5} under the conditions where Burling's data were obtained.

For a fetch of 4 miles, values of mean square slope based on this spectral distribution are of the same order as those obtained by Cox and Munk for a sea surface covered with oil slicks.

The spectral distribution obtained does not agree with the results of Project SWOP.

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List of symbols

f	Wave frequency, (1/Period), in $secs^{-1}$.
f_0	Frequency of class having highest energy when spectrum is split up into equal frequency intervals.
g	Acceleration of gravity.
H	Square root of the sum of the squares of all the peaks in the spectrum. Expressed in feet except when otherwise stated.
H_f	Square root of the sum of the squares of the peaks on the unit frequency interval centred round f $secs^{-1}$.
H_T	When the spectrum is split up into unit period intervals, square root of the sum of the squares of the peaks on the unit period interval centred round T secs.
$H^0{}_T$	Initial H_T value in storm area of swell observed at distance R from storm centre.
H_{max}	Maximum height (crest to trough) observed on a wave record. Expressed in feet except when otherwise stated.
$H_{1/10}$	Mean height of the highest tenth of the waves observed on a wave record.
$H_{1/3}$	Mean height of the highest third of the waves observed on a wave record. Sometimes called the significant wave height.
R	Distance swell has travelled from centre of storm area, in nautical miles.
T	Wave-period in seconds.
$T_{1/3}$	Mean period of highest third of waves observed on record, sometimes called the significant wave-period.
T_f	$= 1/f_0$.

T_T Wave-period of class having the highest energy when the spectrum is split up into equal intervals of 1 sec period.
U Gradient wind speed in knots.
W Surface wind speed, measured at 12 m height, expressed in knots excepted when otherwise stated.
x Fetch in nautical miles.
y Non-dimensional function of x which approaches 1 as x approaches infinity. See equation (14).
θ_a Air temperature in degrees Fahrenheit.
θ_s Sea temperature in degrees Fahrenheit.
λ Wave-length.
σ^2 Mean square slope.

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