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Solution of the propagation of the waves in open channels by the transfer matrix method

R. Daneshfaraz [a], B. Kaya [b,*]

[a] Engineering Faculty, Department of Civil Engineering, Ege University, Bornova, Izmir, Turkey
[b] Engineering Faculty, Department of Civil Engineering, Dokuz Eylül University, Kaynaklar Campus, Buca, Izmir, Turkey

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ABSTRACT

Many problems in mechanics can be solved by the use of the transfer matrix method. The use of this method in hydraulics engineering is not widespread and only limited studies are available. In this study, linearized St. Venant equations were used and the use of transfer matrix in ocean engineering was investigated for long waves in open channels, and numerical application was carried out. The results obtained through the transfer matrix method, which is quite easy to use, program and comprehend, showed similar results obtained from the characteristics method and finite differences method.

1. Introduction

St. Venant equations can be used in solving equations for wave movements in shallow waters. Since St. Venant equations are non-linear and second-degree hyperbolic, they cannot be integrated directly. By linearizing St. Venant equations can be integrated. In this study, non-linear term is usually neglected and the linearized version of St. Venant equations was used and the transfer matrix method (TMM) was considered for the purpose of obtaining a solution for the long waves in open channels.

In the TMM, which is known as the method of initial values where the aim is to convert the problem of boundary values into the problem of initial value to prevent new constant values and to express equations of the problem by initial constant values (Inan, 1968). With the help of this method, many problems in mathematics can be solved (Dimarogonas, 1996). The method can be used essentially for the solution of 1D linear differential equations; however, after a proper linearization process it can also be used to solve nonlinear problems.

The application of transfer matrix for the solution of hydraulic problems is very limited. According to the finite elements method, matrix dimensions are small, constant and independent of the number of elements. Computer programing of the method is easy and practical (Daneshfaraz and Kaya, 2007). The TMM can be used in determining movements of waves in shallow waters.

When the studies on long wave in literature are viewed, it can be seen that a series of studies were carried out. However no study on the solution of long waves using transfer matrix is available. Studies using other methods and approaches are given below:

Tsai (2002) conducted theoretical investigations on the propagation of long waves of one-dimensional, unstable, viscous and turbulent open canal currents, and discussed the effect of the Froude number on the formation of channel flows in shallow waters depending on the location and time. Shi et al. (2005) investigated the fundamental behavior of long water waves propagating through branching channels of uniform depth and width. They carried out numerical simulations based on the Boussinesq long wave model to verify the effects of width of channel branches on wave transmission and reflection. Koutitas (1983) solved the linear long wave equation by using the finite elements method. In the study, it was accepted that the flow generated a sinusoidal vibration.

Onzikua and Odai (1998) proposed the Burgers equation model for unsteady flow in open channels. In this model to simulate slow transients in wide rectangular open channels of finite length, the St. Venant equations are approximated by a single Burgers equation for flow depth. Flow velocity is expressed as a function of flow depth and its gradient to satisfy the continuity of the flow. Tsai and Yen (2001) suggested a method for linear analysis of shallow water wave propagation in open channels. In this study, the Laplace transform method is adopted to obtain first-order analytical spatiotemporal expressions of upstream and down-stream channel response function.

The methods in the discussed studies are numerical methods. However, the TMM is based on analytical solution. The TMM is used in studies of Baume et al. (1998) and Litrico and Fromion

* Corresponding author.
E-mail address: birol.kaya@deu.edu.tr (B. Kaya).

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(2006). Baume et al. (1998) expressed a need for using linear control theory while pointing out the difficulty of complex hydraulic systems to control. They obtained a reach transfer matrix by liberation of St. Venant equations near a steady flow regime. In this study, St. Venant equations were used in a hydraulic application for the first time. However, the equations were used between the two given points only. Litrico and Fromion (2006) investigated the control of oscillating modes occurring in open channels due to the reflection of propagating waves on the boundaries. They characterized the effect of a proportional boundary control on the poles of the transfer matrix by a root locus which derived to an asymptotic result for high-frequency closed-loop poles. Baume et al., (1998) and Litrico and Fromion (2006) have solved linearized equations via the Laplace method. In these studies, separation of variables method is used.

The proposed method in this study is also extendable and applicable to study of wave interaction generated by vessels moving in either parallel or opposite directions (Wu et al., 2001).

2. Equations for long linear waves in open channels

The dynamic behavior can be described by a set of equations known as the St. Venant equations (Chaudhry, 1993):

$$\frac{\partial y}{\partial t}+D\frac{\partial u}{\partial x}+u\frac{\partial y}{\partial x}-\frac{q}{B}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-g(S_0-Sf)+g\frac{\partial y}{\partial x}=0 \tag{2}$$

where B is the top water surface width (m), D is the hydraulic depth (m), y the water depth (m), g the gravity acceleration (m^2/s), x the longitudinal abscissa in the direction of the flow, t the time, S_0 the bottom slope, S_{f} the energy gradient slope, u the average velocity (m/s) and q the lateral inflow or outflow per unit length.

If there is not lateral inflow or outflow $q=0$, the St. Venant equation for very wide rectangular cross-section channels. Eq. (1) can be written as

$$\frac{\partial \xi}{\partial t}+\frac{\partial (hu)}{\partial x}=0 \tag{3}$$

For $S_{\mathrm{f}}=\tau_{\mathrm{b}}/\rho gh$, Eq. (2) can be written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial \xi}{\partial x}-\frac{\tau_{\mathrm{b}}}{\rho h}+gS_0 \tag{4}$$

where ξ is the amplitude, h is the undisturbed flow depth, ρ is the mass density of water and τ_{b} is shear stress at the base and is a function of the velocity, and is described with the following equation:

$$\frac{\tau_{\mathrm{b}}}{\rho}=ku \tag{5}$$

where k is equivalent friction coefficient. If Eq. (5) is substituted in Eq. (4), Eq. (6) is obtained

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial \xi}{\partial x}-\frac{ku}{h}+gS_0 \tag{6}$$

For shallow water wave propagation, Eqs. (3) and (6) can be written. For gradual variations in $\xi(x,t)$ (propagation of long waves) and small variations in $h(x)$ the non-linear term $u(\partial u)/(\partial x)$ is usually neglected and the linearized version of Eqs. (3) and (6) is

$$\frac{\partial u}{\partial t}=-g\frac{\partial \xi}{\partial x}-\frac{ku}{h}+gS_0 \tag{7}$$

$$\frac{\partial \xi}{\partial t}+\frac{\partial}{\partial x}(hu)=0 \tag{8}$$

If the two equations above are combined and reorganized, a second-degree, linear, and hyperbolic equation will be obtained

$$\frac{\partial^2 \xi}{\partial t^2}=g\frac{\partial}{\partial x}\left(h\frac{\partial \xi}{\partial x}\right)-\frac{k}{h}\frac{\partial \xi}{\partial t} \tag{9}$$

If channel bottom slope is invariable, derivate of 'gS_0' is zero. The flow domain is discretized into equal elements of length Δx. The water depth is assumed constant along each element (i), $h_i=$ const. (Koutitas, 1983).

3. Solution by the TMM

In various engineering problems, as the number of constants to be determined by the use of boundary condition increases, the calculation becomes more tedious and the possibility of making errors increases. Therefore, in the formulation of such problems, ways of reducing the number of constants to a minimum are sought. The method of transfer matrix makes this possible. The main principle of this theory, which is applied to problems with one variable, is to convert all the boundary value problems into problems of initials values, and thus new constants that may result from the use of intermediate condition are eliminated. Therefore, it is a method of expressing the equations in terms of the initials constants. This method thus makes no distinction between the so-called determinate and indeterminate problems of elastomechanics (Inan, 1968).

There are a number of methods for solving the differential equations, one of which is the TMM. The TMM is ideally suited to solve mechanical systems, because only successive matrix multiplication is necessary to fit the elements together. One of the used solutions of differential equation is separation of variables (Riley et al., 1998). The method of separation of variables can be used to obtain the solution of Eq. (9). Assuming that

$$\xi(x,t)=\xi(x)\xi(t) \tag{10}$$

and substituting for ξ in Eq. (9), we obtain

$$\ddot{\xi}(t)\cdot\xi(x)-gh_i\xi''(x)\xi(t)+\frac{k}{h_i}\dot{\xi}(t)\xi(x)=0 \tag{11}$$

If simplifications are made, the following equation will be obtained:

$$\frac{\ddot{\xi}(t)+(k/h_i)\dot{\xi}(t)}{\xi(t)}=gh_i\frac{\xi''(x)}{\xi(x)}=-\alpha^2 \tag{12}$$

where α is a constant and equals to $2\pi/7$. Thus, we find that $\xi(x)$ and $\xi(t)$ satisfy the ordinary differential equations

$$gh_i\frac{\xi''(x)}{\xi(x)}=-\alpha^2 \tag{13}$$

and

$$\frac{\ddot{\xi}(t)+(k/h_i)\dot{\xi}(t)}{\xi(t)}=-\alpha^2 \tag{14}$$

then Eq. (13) can be written as

$$\xi''(x)+\frac{\alpha^2}{gh_i}\xi(x)=0 \tag{15}$$

The solution of Eq. (15) may be written as follows:

$$\xi(x)=C_1\cos\left(\frac{\alpha}{\sqrt{gh_i}}x\right)+C_2\sin\left(\frac{\alpha}{\sqrt{gh_i}}x\right) \tag{16}$$

The first derivative of $\xi(x)$ can be expressed as

$$\frac{\mathrm{d}\xi(x)}{\mathrm{d}x}=-C_1\frac{\alpha}{\sqrt{gh_i}}\sin\left(\frac{\alpha}{\sqrt{gh_i}}x\right)+C_2\frac{\alpha}{\sqrt{gh_i}}\cos\left(\frac{\alpha}{\sqrt{gh_i}}x\right) \tag{17}$$

If Eqs. (16) and (17) are written in matrix form

$$\begin{bmatrix} \xi(x) \\ \dfrac{d\xi(x)}{dx} \end{bmatrix} = [A]\cdot\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{18}$$

where $[A]$ is

$$[A] = \begin{bmatrix} \cos\left(\dfrac{\alpha}{\sqrt{gh_i}}x\right) & \sin\left(\dfrac{\alpha}{\sqrt{gh_i}}x\right) \\ -\dfrac{\alpha}{\sqrt{gh_i}}\sin\left(\dfrac{\alpha}{\sqrt{gh_i}}x\right) & \dfrac{\alpha}{\sqrt{gh_i}}\cos\left(\dfrac{\alpha}{\sqrt{gh_i}}x\right) \end{bmatrix} \tag{19}$$

In the initial points of the ith element Eq. (18) can be written as

$$\begin{bmatrix} \xi(0) \\ \dfrac{d\xi(0)}{dx} \end{bmatrix} = [A_0]\cdot\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{20}$$

where A_0 is

$$[A_0] = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\alpha}{\sqrt{gh_i}} \end{bmatrix} \tag{21}$$

The matrix for coefficient can be obtained by the help of Eq. (20)

$$[C] = [A_0]^{-1}\cdot\begin{bmatrix} \xi(0) \\ \dfrac{d\xi(0)}{dx} \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} \xi(x) \\ \dfrac{d\xi(x)}{dx} \end{bmatrix} = [A][A_0]^{-1}\cdot\begin{bmatrix} \xi(0) \\ \dfrac{d\xi(0)}{dx} \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} \xi_{i+1}(x) \\ \dfrac{d\xi_{i+1}(x)}{dx} \end{bmatrix} = [A_{i+1}][A_i]^{-1}\cdot\begin{bmatrix} \xi_i(x) \\ \dfrac{d\xi_i(x)}{dx} \end{bmatrix} \tag{24}$$

is obtained. Where $[T]$ is element transfer matrix and $[T] = [A_{i+1}]\cdot[A_i]^{-1}$. An element of transfer matrix can be written as

$$[T] = \begin{bmatrix} \cos\left(\dfrac{\alpha}{\sqrt{gh}}x\right) & \dfrac{\sin(\alpha/\sqrt{gh}(x))}{\alpha/\sqrt{gh}} \\ -\dfrac{\alpha}{\sqrt{gh}}\sin\left(\dfrac{\alpha}{\sqrt{gh}}x\right) & \cos\left(\dfrac{\alpha}{\sqrt{gh}}x\right) \end{bmatrix} \tag{25}$$

and the following equation can be written and values of $\xi(x)$ can be determined as follows:

$$[T]\cdot\begin{bmatrix} \xi_i(x) \\ \dfrac{d\xi_i(x)}{dx} \end{bmatrix} = \begin{bmatrix} \xi_{i+1}(x) \\ \dfrac{d\xi_{i+1}(x)}{dx} \end{bmatrix} \tag{26}$$

Eq. (14) can be written as

$$\ddot{\xi}(t) + \frac{k}{h}\dot{\xi}(t) + \xi(t)\alpha^2 = 0 \tag{27}$$

The solution of Eq. (27) may be written as follows:

$$\beta = \sqrt{\left(\frac{k}{h}\right)^2 - 4\alpha^2}\Big/2 \tag{28}$$

$$\xi(t) = e^{-(k/2h)t}[c_1\cos(\beta t) + c_2\sin(\beta t)] \tag{29}$$

Assuming the liquid friction free, and taking $k = 0$, the following equation can be written:

$$\ddot{\xi}(t) + \xi(t)\alpha^2 = 0 \tag{30}$$

The solution of this equation is

$$\xi(t) = [c_1\cos(\alpha t) + c_2\sin(\alpha t)] \tag{31}$$

where c_1 and c_2 are the integral constants and can be determined by using the boundary conditions. Consequently, the final $\xi(x,t)$ equation, which is the input of the TMM, can be obtained from Eqs. (10), (26) and (29 or 31).

4. Numerical examples

Wave propagation problem in a friction free flow (Koutitas, 1983) is handled as the first application example of the method. In the example, results of TMM and other methods are given and compared with each other. The second example, on the other hand, is due to Nas (2006), who reported data of an experiment on a frictional bed. The results of the TMM solution are compared with those of the characteristics method (CM), finite differences method (FDM) and experimental data by means of this example.

4.1. Example 1

In the open channel given in Fig. 1, wave propagation is solved by the TMM and the result is given in Fig. 2. The example solved was taken from Koutitas (1983) and the results were compared with those obtained by CM and FDM. Results of CM and FDM are given in Figs. 3 and 4.

In this example, the fluid was assumed to be friction free ($k = 0$), the period (T) was taken as 60 s, $\xi_0 = 1$ m, and channel bed slope 1%.

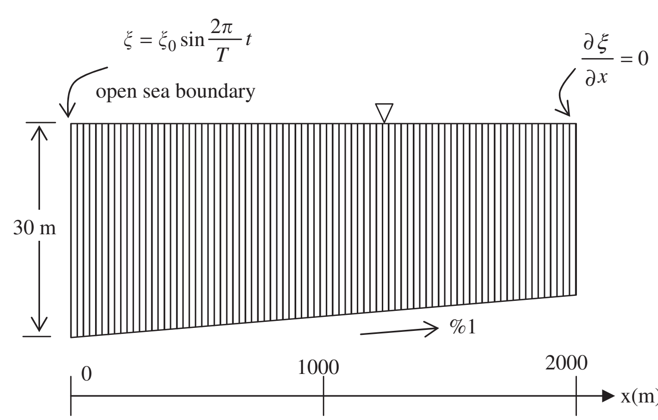

Fig. 1. Wave propagation in the channel.

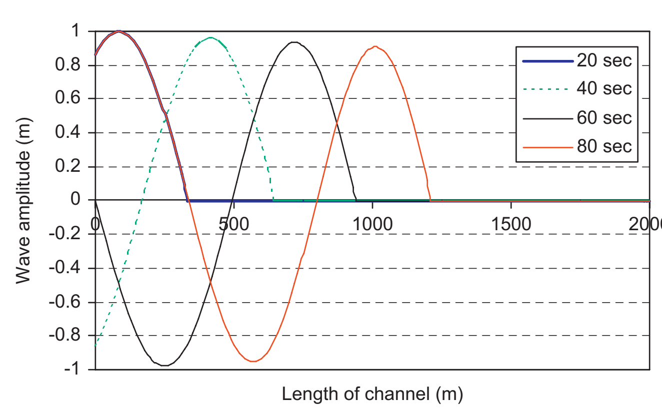

Fig. 2. Results obtained from the solution by using TMM.

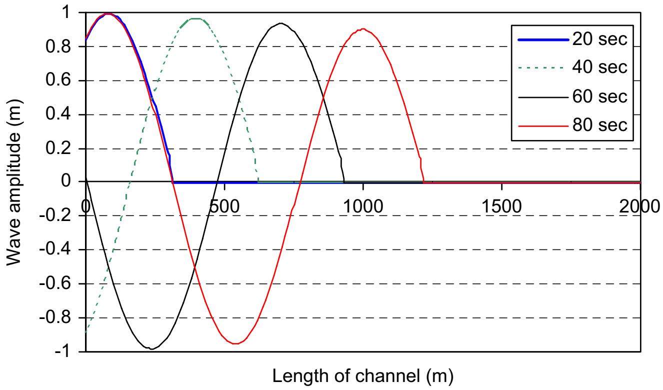

Fig. 3. Results obtained by the characteristics method.

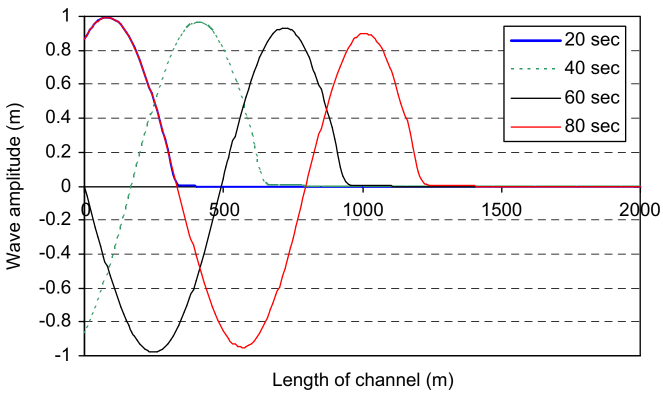

Fig. 4. Results obtained by the FDM.

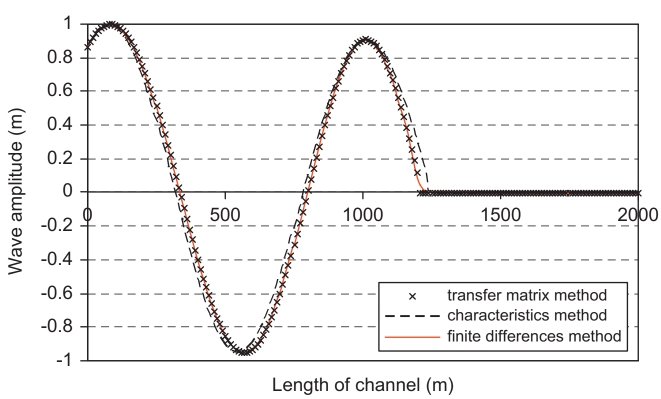

Fig. 5. Comparison of the results obtained by TMM, CM and FDM for $t = 80$ s.

Initially, the surface of water was horizontal. For the entrance cross-section ($x = 0$), ξ is given as initial boundary condition (open sea boundary) using the following equation:

$$\xi(0, t) = \xi_0 \sin(2\pi t/T) \tag{32}$$

where t is the time and T is the period. Using Eq. (31), Eq. (32) is rewritten as follows:

$$\xi(t) = \xi_0 \sin 2\pi t/T \tag{33}$$

For the last cross-section, $\partial\xi/\partial x = 0$ is given as boundary condition.

The results obtained for $t = 80$ s by using the TMM, CM and FDM are compared in Fig. 5. As it can be seen from Fig. 5 the results are very close to each other. Differences between TMM and other methods are given in Table 1.

When TMM is used, the matrix to be solved is in 2×2 size. The size of the matrix is independent from the number of solution points. In finite differences methods, the coefficient matrix is in big size. The matrix size is dependent on numbers of calculation points for a suitable solution. The big size of the matrix lengthens the process time and increases the mistakes as a result. In TMM, the size of the matrix being constant makes the method preferable. Computational elapsed times for TMM, FDM and CM are found as 0.0056, 0.0151 and 0.0266 s, respectively. The TMM has been found as a much faster solution scheme as demonstrated by these results. Besides, to achieve a stable solution, courant condition must be followed in CM and FDM. There is no obligation in TMM.

4.2. Example 2

The experimental setup of Nas (2006) is used as a second numerical example. In the problem, Nas (2006) has investigated the variation of the water level at the inlet cross-section of a channel. The experimental results are compared with the outcomes of characteristics, explicit and implicit FDM. The variation of water depth value at the first cross-section, namely the boundary condition at the beginning of the channel, is given in Fig. 6. In this example, the fluid was assumed to be Strickler friction coefficient as 100, the channel width was taken as 15.8 cm, the channel length was taken as 801 cm, the channel slope $S_0 = 0$ and initial water depth $h_0 = 8.6$ cm.

The results of implicit finite difference method (IFDM), explicit finite difference method (EFDM) and CM, in comparison with the TMM outcomes and experimental observations at 6.1 m from the beginning of the channel are given in Fig. 7. It can be clearly observed from Fig. 7 that TMM results are thoroughly convergent, in comparison with the other methods. Besides, it can be seen that experimental results are in accordance with the model outcomes.

Table 1
Differences between TMM and other methods

X (m)	0	100	200	300	400	500	600
TMM–CM	0.0000	0.0137	0.0790	0.1382	0.1356	0.0599	−0.0450
TMM–FDM	0.0000	0.0072	0.0113	0.0114	0.0078	0.0006	−0.0072
X (m)	700	800	900	1000	1100	1200	1300
TMM–CM	−0.1108	−0.0978	−0.0313	0.0104	−0.0411	−0.1696	0.0000
TMM–FDM	−0.0084	−0.0073	0.0005	0.0037	0.0129	0.0339	0.0000

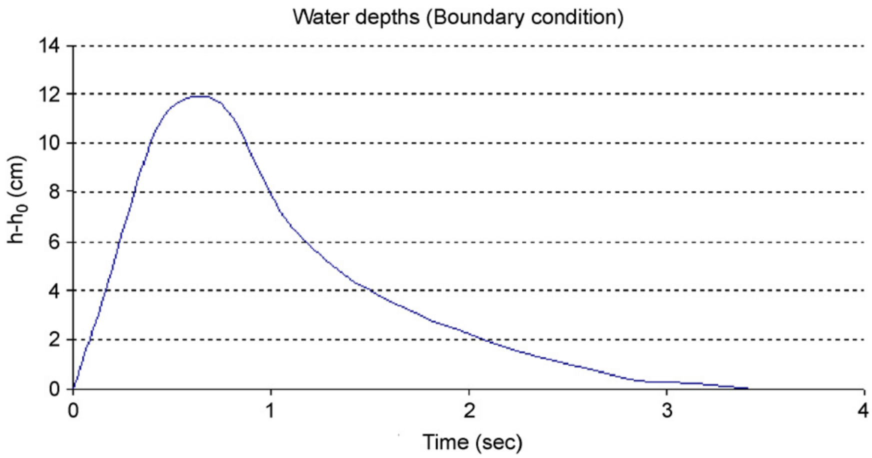

Fig. 6. The variation of water depth value at the first cross-section (boundary condition).

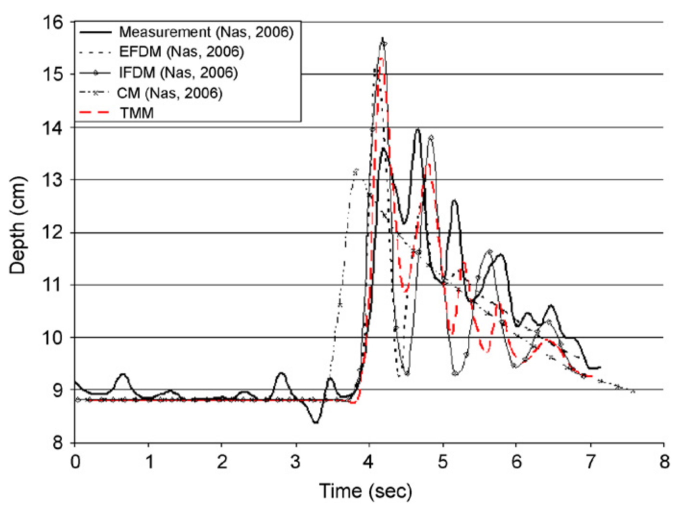

Fig. 7. The results of experimental observations, TMM and other methods.

5. Conclusion

In this study, the application of TMM to solve the equations of long waves that are formed in open channels was investigated. For this purpose, the propagation of waves having a sinusoidal character in entrance cross-section was examined using TMM. This method is an efficient and easily computerized method and also provides a fast and practical solution, because the dimension of the matrix explaining the 1D flow never changes. The TMM solution can be applied to solve of various 1D flow systems possessing different boundary conditions. Besides, it can be seen that experimental results are in accordance with the model outcomes. In the case of the TMM; independent from the number of nodes and the computation time the problem may be always solved by using a matrix of 2×2, which is a noticeable advantage. The results obtained by this method were found to be close to those obtained by characteristic method and finite differences method.

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