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A Finite Amplitude Wave on a Linear Shear Current

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A numerical perturbation procedure is presented that generates water waves propagating over a vertically varying linear shear current. The water surface profile of these waves may be symmetric about the crest, with given height and period, or they may have an irregular water surface profile that has been measured in water of known depth. For waves of the same height the effect of the current is to cause a change in wavelength and hence the kinematics under the wave. Further, the shape of the wave profile is changed significantly.

The mathematical theories for the description of waves propagating on the surface of water have been largely concerned with irrotational motions in the fluid. Realistically, the assumption is not generally valid, since waves seldom propagate in quiescent fluids but rather in water that is acted upon by winds and other current-generating forces.

To better represent waves, a model is developed for waves on linear shear currents that flow in a direction concurrent with the wave direction. The need for better models can be illustrated in the oil industry, where for the design of offshore platforms the effect of ambient currents must be evaluated. Further, some present techniques for evaluating forces on a structure due to waves on a shear current involve simply superimposing the current on an irrotational wave without regard to the nonlinear coupling that occurs at the free surface [*Hall*, 1972].

Early work in regard to waves on shear currents has been done by *Dubreil-Jacotin* [1934], *Daubert* [1961], and *Gouyon* [1961] for arbitrary shear currents and by *Biesel* [1950] and *Tsao* [1959] for linear shear currents. Tsao has obtained a third-order analytic solution of the problem. The present work allows the computer generation of nonlinear waves of any order propagating on a linear shear current, that is, for a fluid with constant vorticity. In all of the above works the shear current is assumed to have been established, and the effect of viscosity is then negligible, at least for a short time.

BOUNDARY VALUE PROBLEM

The mathematical boundary value problem is readily formulated in two dimensions if it is assumed that the wave propagates without change in form with celerity C in the x direction. By moving the reference coordinate system with this celerity the problem is rendered time independent (see Figure 1 for notation). The governing differential equation is determined from the Euler equations:

$$(U + u - C)(U + u - C)_x + (V + v) \cdot (U + u - C)_y = -\frac{1}{\rho} p_x \qquad (1)$$

$$(U + u - C)(V + v)_x + (V + v) \cdot (V + v)_y = -\frac{1}{\rho} p_y - g \qquad (2)$$

where U and V are the horizontal and vertical velocity components of an ambient current; u and v are the horizontal and vertical velocity components of the wave-induced motion; p is the pressure; g is the acceleration of gravity; $C = L/T$, where L is the wavelength and T is the wave period; and the subscripts denote differentiation.

If an incompressible fluid is assumed, the continuity equation is expressed as

$$(U + u - C)_x + (V + v)_y = 0 \qquad (3)$$

A stream function $\psi(x, y)$ can now be defined as

$$(U + u - C) = -\psi_y \qquad (4)$$

$$(V + v) = \psi_x \qquad (5)$$

Substituting the stream function into (1) and (2) and eliminating the pressure term by cross differentiation yield the following nonlinear equation:

$$\partial(\psi, \nabla^2\psi)/\partial(x, y) = 0 \qquad (6)$$

in Jacobian notation. This equation, which states that the vorticity of the fluid $\nabla^2\psi$ is constant along a streamline, may be integrated along a streamline to give the final governing equation for the fluid [*Lamb*, 1945, p. 244]

$$\nabla^2\psi = f(\psi) \qquad (7)$$

If the vorticity distribution $f(\psi)$ is identically equal to zero, the Laplace equation results, giving rise to a class of water wave problems investigated by *Stokes* [1847] and others. Recently, computer procedures to extend this case to nonlinear waves of any order have been given by *Chappelear* [1961] in terms of the velocity potential and by *Dean* [1965] for the stream function representation. For a linear shear current, $f(\psi)$ is equal to a constant, say, $-\omega_0$, giving rise to the Poisson equation

$$\nabla^2\psi = -\omega_0 \qquad (8)$$

It is this case that is to be examined.

There are four boundary conditions that must be imposed on the fluid to guarantee a unique solution. First, there is no flow through the bottom, or

$$\psi_x = 0 \quad \text{on} \quad y = -h \qquad (9)$$

For a wavelike solution, periodicity is required such that

$$\psi(x, y) = \psi(x + L, y) \qquad (10)$$

where L is the length of the wave.

At the free surface $\eta(x)$ the pressure must be a constant, and

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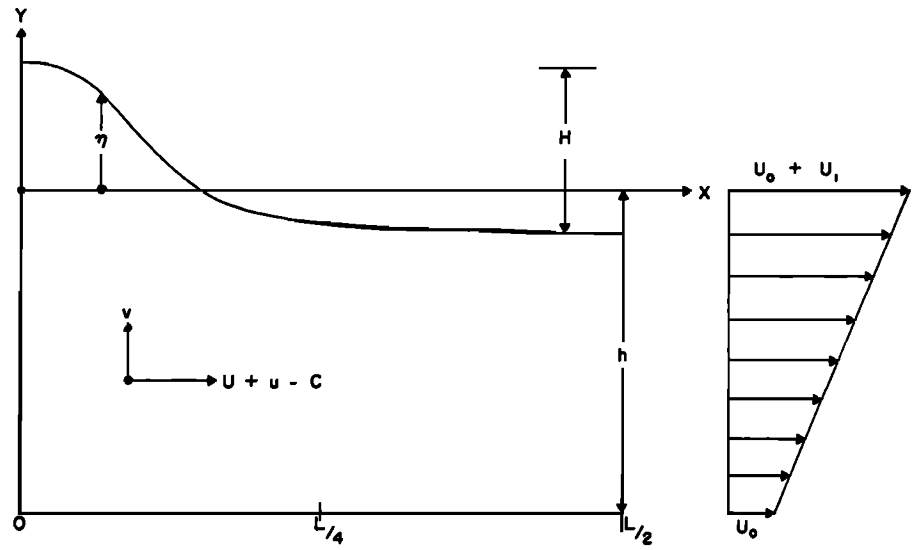

Fig. 1. Definition sketch for a wave on a current U as seen by an observer moving with the wave.

the Bernoulli equation may be used there, since it is a streamline. The dynamic free surface boundary condition is then specified as

$$\eta + (1/2g)[\psi_x^2 + \psi_y^2] = Q = \text{const} \qquad \text{on } y = \eta \tag{11}$$

One further condition at the free surface must also be prescribed, however, not in the usual kinematic sense of requiring the free surface to be a streamline, as is already true by definition, but rather for the purpose of determining value of the free surface streamline. This condition will be discussed further in the following sections.

SYMMETRIC WAVE OF GIVEN HEIGHT AND PERIOD

As a result of the linearity of the governing equation (8) a linearly varying current profile may be superimposed on an irrotational wave field, the following assumed series solution for the stream function being the result.

$$\psi(x, y) = -\left(U_0 - \frac{L}{T}\right)y - \frac{\omega_0(h+y)^2}{2} + \sum_{n=2}^{NN} X(n) \sinh \frac{2\pi(n-1)(h+y)}{L} \cos \frac{2\pi(n-1)x}{L} \tag{12}$$

which is similar to the solution of *Dean* [1965] with the exception of the second term. Note that for a positive value of ω_0 a positive current results. In this expression, U_0 is a uniform (over depth) current component; h is water depth; $X(n)$ is the stream function coefficient; NN is the number of stream function coefficients; and L is the wavelength, defined later as $X(1)$ for convenience, since it is a priori unknown. Here all of the variables with the exception of the $X(n)$ are assumed known. This form of the solution satisfies the boundary condition at the bottom, (9), and also that of periodicity, (10).

The final form of the stream function follows by determining the best values of the $X(n)$ such that the nonlinear free surface boundary conditions are best satisfied. The dynamic free surface boundary condition (11) may be written in a least squares form as

$$E_1 = \frac{2}{L}\int_0^{L/2} [Q(x) - \bar{Q}]^2 \, dx \tag{13}$$

where E_1 would be zero for an exact solution of the problem. The $\bar{Q}$ is the Bernoulli constant for the wave, determined by integration.

$$\bar{Q} = \frac{2}{L}\int_0^{L/2} Q(x) \, dx \tag{14}$$

Note that here the symmetry of the wave profile is used to limit the range of integration to only half of the wavelength. Finally, a second free surface boundary condition is specified for the determination of $\psi(x, \eta)$ (defined for computer convenience as $X(NN + 1)$), which states that the free surface profile η must have a zero mean; i.e., there is no change in the mean water level due to the presence of the wave.

$$\frac{2}{L}\int_0^{L/2} \eta(x) \, dx = 0 \tag{15}$$

The water surface profile η is determined by substituting $y = \eta$ into (12) and using the quadratic formula. The η must be determined iteratively owing to the presence of the transcendental function, $\sinh [2\pi(n - 1)(h + \eta)/L]$.

Owing to the nonlinearities present in the free surface boundary conditions, the procedure for determining the stream function coefficients is an iterative one. First, a trial set of coefficients ($X(1)$, $X(2)$, $X(NN + 1)$) are obtained by the linear theory developed by Biesel and Tsao.

$$\psi(x, y) = -(U_0 - C)y - \frac{\omega_0(h+y)^2}{2} + \frac{H}{2}\left[(U_0 + \omega_0 h - C) \sinh \frac{2\pi(h+y)}{L} \cdot \cos \frac{2\pi x}{L} \Big/ \sinh \frac{2\pi h}{L}\right] \tag{16}$$

with the dispersion relationship

$$(U_0 + \omega_0 h - C)^2 = \frac{gL}{2\pi}\left[1 + \frac{\omega_0}{g}(U_0 + \omega_0 h - C)\right] \tanh \frac{2\pi h}{L} \tag{17}$$

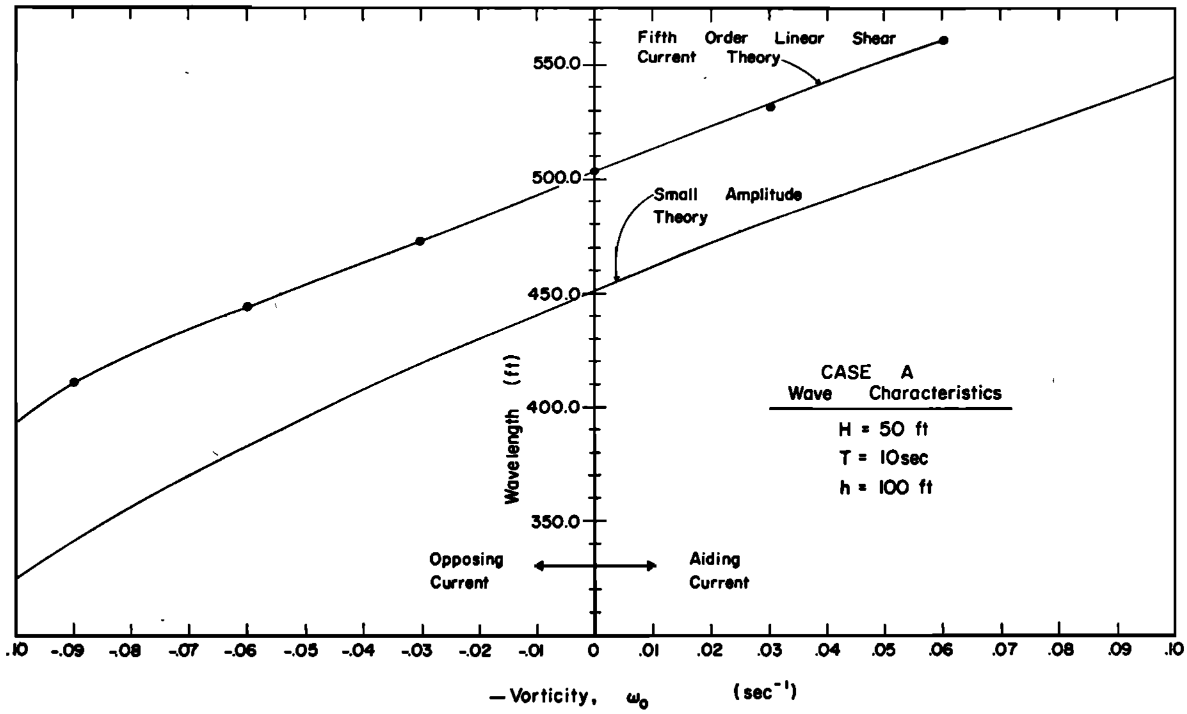

Fig. 2. Wavelength as a function of vorticity $-\omega_0$ for a wave in deep water, case A.

Therefore $X(1) = L$ must satisfy the relationship (17),

$$X(2) = \frac{H}{2} \frac{(U_0 + \omega_0 h - C)}{\sinh (2\pi h/L)}$$

by inspection of (12), and $\psi(x, \eta)$ $(=X(NN + 1))$ is estimated from (16) with $x = 0$ and $y = H/2$. The remaining coefficients ($X(n)$, $3 \leq n \leq NN$) are set to zero. To obtain a better estimate of the $X(n)$, a nonlinear perturbation technique is used to minimize the fit to the dynamic free surface boundary condition. By including the mean sea level condition (15) and a wave height condition, specified as $\eta(0) - \eta(L/2) = H$, as constraints in the minimization by means of a Lagrange multiplier approach [*Hildebrand,* 1965] a drawback of previous numerical wave theories is eliminated; that is, this procedure will converge directly on wave height. Therefore an objective function O may be defined as

$$O = \frac{2}{L} \int_0^{L/2} (Q(x) - \bar{Q})^2 \, dx + \frac{2\lambda_1}{L} \int_0^{L/2} \eta(x) \, dx + \lambda_2 \left[\eta(0) - \eta\left(\frac{L}{2}\right) - H \right] \quad (18)$$

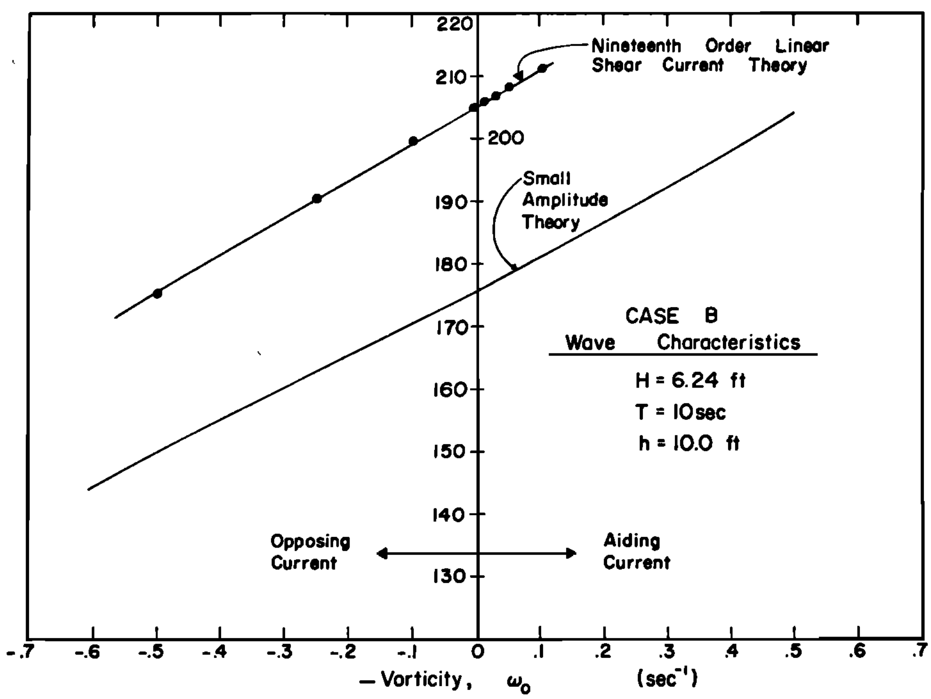

Fig. 3. Wavelength as a function of vorticity $-\omega_0$ for a wave in shallow water, case B.

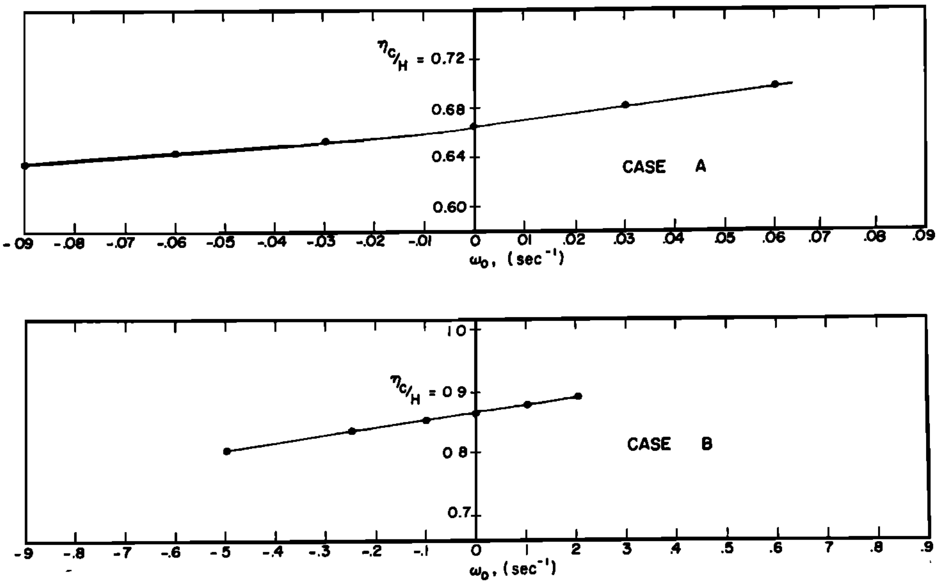

Fig. 4. Dimensionless crest elevation η_c/H as a function of vorticity $-\omega_0$ for cases A and B.

The objective function, which is composed of nonlinear terms, is quasi-linearized by expanding it at iteration $j + 1$ in a first-order Taylor series

$$O^{j+1} = O^j + \sum_{n=1}^{NN+1} (O^j)_{X(n)} X'(n) \tag{19}$$

for each unknown $X(n)$ where the $X'(n)$ are small changes in the $X(n)$. The expanded objective function is then minimized with respect to all the $X(n)$, λ_1, and λ_2.

$$\begin{aligned} (O^{j+1})_{X(m)} &= 0 \qquad m = 1,\ NN + 1 \\ (O^{j+1})_{\lambda_l} &= 0 \qquad l = 1,\ 2 \end{aligned} \tag{20}$$

where only the first-order derivatives need be retained. These $(NN + 3)$ simultaneous equations can now be solved by matrix inversion for the $(NN + 1)X'(n)$, λ_1, and λ_2. The $X'(n)$ are then added to the previous values of the $X^j(n)$ to yield the new $X^{j+1}(n)$. This direct addition becomes unstable for breaking waves, and for those cases only a fraction of the $X'(n)$ is added. This procedure is then iterated numerous times until the desired accuracy of O^j (and consequently E_1) results.

The effect of the linear shear current on a wave of a given height is manifested in a change in wavelength, which in turn results in a change of the kinematics within the wave. (A wave propagating from a zone of no current into a region of a shear current will undergo a change in wave height; in this study, these effects are not considered, since the wave height is held constant in the comparisons). Two different waves have been chosen to illustrate these points. They both have a 10-s period, but the case A wave is in 100 feet (30.5 m) of water and has a wave height of 50 feet (15.2 m), whereas the case B wave is in

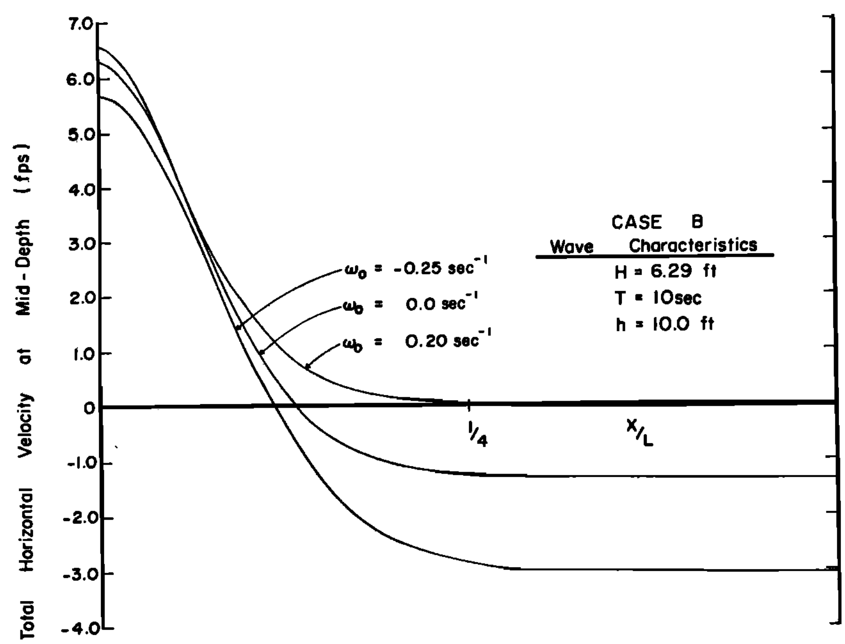

Fig. 5. Change in total horizontal velocity with dimensionless horizontal distance and vorticity $-\omega_0$ evaluated at middepth for a case B wave.

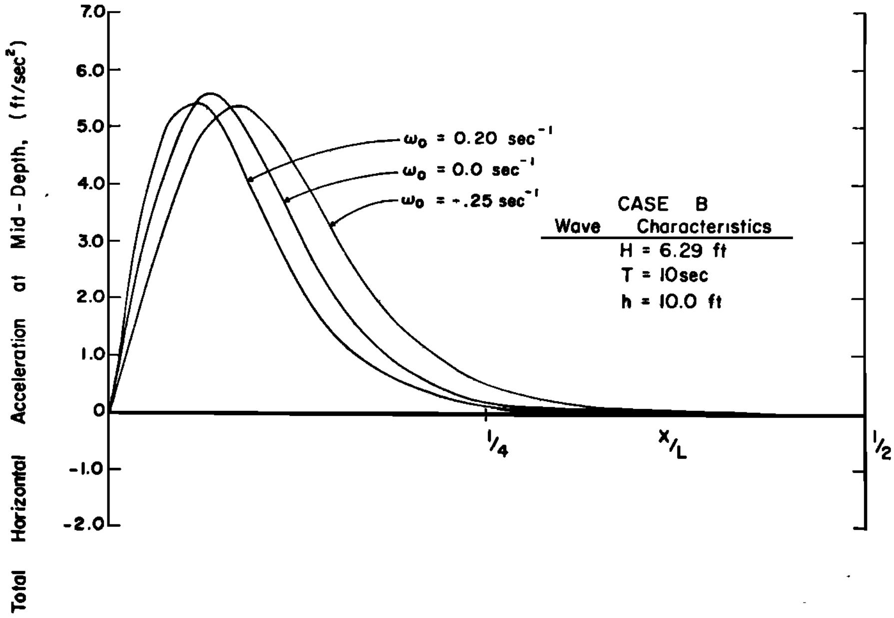

Fig. 6. Change in total horizontal acceleration with dimensionless horizontal distance and vorticity $-\omega_0$ evaluated at middepth for a case B wave.

shallow water, 10 feet (3 m) in depth, and has a height of 6.29 feet (1.92 m).

When only a linear shear current is present ($U_0 = 0$), the effect of the vorticity on the wavelengths of the waves may be seen in Figures 2 and 3. The difference between the small amplitude theory wavelength, (17), and the higher-order linear shear current model is almost due entirely to finite amplitude effects, as may be clearly seen by comparing the two theories at $\omega_0 = 0$, the irrotational case. From the figure the effect of the vorticity is to increase the wavelength of a wave on an aiding current and decrease it on an opposing current. *Sarpkaya* [1955] proposed a superposition technique for approximating the celerity of waves on a current. This procedure assumed that the wave celerity was equal to the celerity of the wave in still water C_{sw} plus the mean value of the current $\bar{U}$, or

$$C = C_{sw} + m\bar{U} \tag{21}$$

where $m = 1.0$. To examine the validity of this superposition procedure, the value of m was obtained for the wavelengths given in Figures 2 and 3, where $\bar{U} = \omega_0 h/2$. These values were 2.0 and 1.2, respectively, indicating that superposition is not generally valid for wave celerity or wavelength, as was also concluded by Sarpkaya on the basis of his experiments.

For the same two waves the change in the maximum crest elevation of the waves due to the vorticity is shown in Figure 4. The vorticity due to an aiding current increases the crest elevation. Note that for the shallow water wave the crest height can approach 90% of the wave height, as compared with the 50% predicted by small amplitude theory.

As an example of the effect of the vorticity on the water particle kinematics within the wave the horizontal velocities and accelerations at middepth for the case B wave are shown in Figures 5 and 6.

The errors in the objective function O for these example waves were quite small, the indication being that the assumed series solution for the stream function and the perturbation procedure is valid. As an example, for the case A and case B waves for a $\omega_0 = 0.03\ \mathrm{s}^{-1}$ the errors after 20 iterations are shown in Table 1. The largest errors occur in the convergence in the specified wave height, but in both cases this is less than 0.01 foot (0.003 m). More iterations and more terms in the series solution would reduce these errors even more.

IRREGULAR WAVE FOR MEASURED WAVE DATA

For an analytic approximation of measured (and digitized) water surface elevation data where a linear shear current was present an irregular form of the linear shear current model is used. The boundary value problem is the same as specified for the symmetric wave model, consisting of (8)–(11) with two exceptions. First, owing to the digitized nature of the measured data, the integral form of the mean value used in obtaining the mean squared error E_1, (13), and the mean Bernoulli constant,

TABLE 1. Errors in Linear Shear Current Model Representation of the Case A and B Waves for $\omega_0 = 0.03\ \mathrm{s}^{-1}$

Case	Dynamic Free Surface Boundary Condition Error, E_1, ft^2 (m^2)	Mean Sea Level Constraint Error, ft (m)	Wave Height Error, ft (m)	Wave Theory Order
A	1.08×10^{-3}	-1.77×10^{-4}	-1.46×10^{-3}	
	(1.00×10^{-4})	(-5.39×10^{-5})	(-4.45×10^{-4})	5
B	2.06×10^{-7}	1.28×10^{-3}	8.26×10^{-3}	
	(1.91×10^{-8})	(3.90×10^{-4})	(2.51×10^{-3})	19

(14), is replaced by the arithmetic means

$$E_1 = \frac{1}{I} \sum_{i=1}^{I} (Q_i - \bar{Q})^2$$

and

$$\bar{Q} = \frac{1}{I} \sum_{i=1}^{I} Q_i$$

where I is the number of data points within the wave. The terms E_1 and $\bar{Q}$ now correspond to *Dean*'s [1965] definitions. Second, the free surface constraint, (15), is altered to require that the predicted water surface elevations η_{pi} at time i coincide in a least squares sense to the measured values η_{mi}. This requirement then is expressed as

$$E_2 = \frac{1}{I} \sum_{i=1}^{I} (\eta_{mi} - \eta_{pi})^2 \qquad (22)$$

Further, the series solution to the boundary value problem is then assumed to be of the following form.

$$\psi(x, y) = -(U_0 - C)y - \frac{\omega_0(h + y)^2}{2} + \sum_{n=4,6\cdots}^{NN-1} \sinh \frac{(n-2)\pi(h+y)}{L} \left[X(n) \cos \frac{(n-2)\pi x}{L} + X(n+1) \sin \frac{(n-2)\pi x}{L} \right] \qquad (23)$$

Note that this states that the wave form is periodic in space and travels without change in form, an assumption that would best be satisfied by ocean swell. The form of $\psi(x, y)$ satisfies the governing differential equation exactly and all of the boundary conditions but those at the free surface η_{mi}. The predicted free surface η_{pi} is found by substituting η_{pi} into (23) for y and solving the quadratic equation iteratively. As the wavelength L, the wave period T, and the free surface streamline value $\psi(x, \eta)$ are unknown, it is convenient to define them as $X(1)$, $X(2)$, and $X(3)$, respectively. The final solution of the problem follows by finding values of the $X(n)$ coefficients that minimize the total error E_T $(=E_1 + E_2)$.

The initial step in the perturbation procedure for the $X(n)$ is to obtain a trial set of the coefficients. One procedure is to obtain them by minimizing the error E_2 with the assumption of zero vorticity. This procedure, used by Dean, results in a linear set of equations that may be solved for the initial $X(n)$. These initial values, which were determined without considering the dynamic free surface boundary condition and the vorticity, are then improved by determining the incremental changes, $X'(n)$, found by minimizing the total error E_T. The E_T at a particular iteration $j + 1$ may be approximated by a first-order Taylor series in the $X(n)$ at iteration j.

$$E_T^{j+1} = E_T^{j} + \sum_{n=1}^{NN} (E_T^{j})_{X(n)} X'(n)$$

By minimizing E_T with respect to all the $X(n)$, NN equations result for the NN unknowns, which are solved for the perturba-

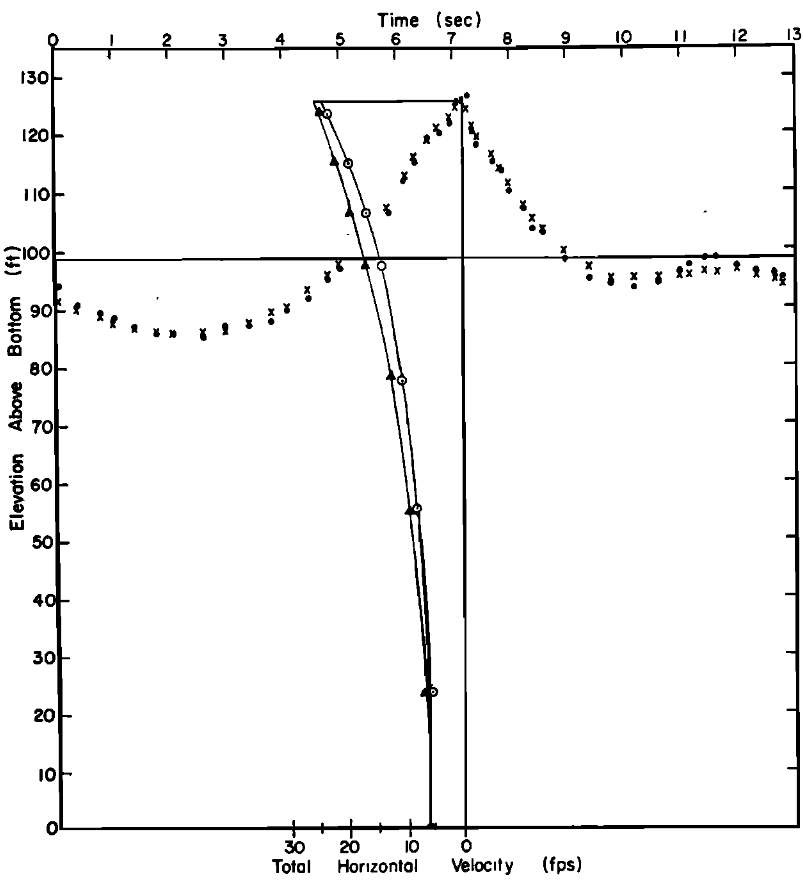

Fig. 7. Measured wave fit by stream function wave theory and linear shear current model with total horizontal velocity profile predicted under wave crest. For surface profile the solid circles are measured wave data and the crosses are irregular forms of the stream function wave theory and linear shear current model. For the velocity profile the open circles are stream function wave theory and triangles are numerical shear current model.

tion corrections $X'(n)$, which are then added to the previous $X(n)$. This procedure then iterated until E_T became suitably small.

In this development it was assumed that the wave profile was measured spatially. More often, the wave profile is measured as a function of time, and thus a change of variables must be made in this formulation. The change is accomplished by replacing the argument $[(n - 2)\pi x]/L$ with $[-(n - 2)\pi t]/T$.

As an example of the usage of this model a wave with a height of 38.8 feet (11.8 m) was measured in the Gulf of Mexico during hurricane Carla in 1961. The water depth was approximately 98 feet (29.9 m). The measured water surface profile is shown in Figure 7 [corresponds to *Dean*, 1965, Figure 5] along with the predicted profile obtained with the stream function wave theory and the linear shear current model. The predicted profiles of both wave theories are the same despite the assumption of $\omega_0 = 0.025\ s^{-1}$ for the linear shear current theory. The error in the dynamic free surface boundary condition was also the same. The horizontal velocity profiles calculated under the wave crest for each theory are also shown in the figure.

CONCLUSIONS

A finite amplitude wave model has been presented that represents, to any order, water waves propagating on a linear shear current, that is, on a fluid that has a constant vorticity. The model can represent either symmetric waves, when the wave height, water depth, wave period, and vorticity are specified, or an irregular wave with a measured profile and an assumed linear shear current.

The validity of the model in representing the actual natural phenomena is assured in the analytical sense in that the errors in the boundary conditions O^j can be made as small as desired. However, this only guarantees that the wave model is a solution to the prescribed boundary value problem. Experimental validity must still be sought.

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