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The Normal Modes of a Rotating, Elliptical Earth

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Summary

It is possible to calculate precisely the theoretical eigen-frequencies of any Earth model which is non-rotating, spherically symmetric, and which has an isotropic static stress field and an isotropic dynamic stress–strain relation. In this paper Rayleigh's principle is used to provide a formalism which allows the approximate computation of the normal mode eigen-frequencies of any Earth model which is slowly rotating and slightly aspherical and anisotropic. This formalism is used to compute, correct to second order, the effects of the Earth's angular rotation, and correct to first order, the effects of the Earth's ellipticity of figure on the normal mode eigenfrequencies. It is found that for an arbitrary poloidal or toroidal multiplet, the central ($m = 0$) member of the multiplet is shifted slightly in frequency and that the other members of the multiplet are split apart asymmetrically by the effects of the Earth's rotation and ellipticity. The results may be used to make a preliminary correction for rotation and ellipticity to the Earth's raw normal mode data.

1. Introduction

The elastic-gravitational normal modes of the Earth have been excited by major earthquakes and observed on various low-frequency seismological instruments. Records of these observations can be used to measure the angular frequencies of oscillation of the Earth's normal modes. In recent years it has also become possible, using high-speed computers, to calculate quickly and precisely the theoretical angular frequencies of oscillation of the elastic-gravitational normal modes for a large class of Earth models; namely, for any model having the following characteristics:

(1) the Earth model is spherically symmetric;

(2) the angular velocity of steady rotation is zero;

(3) the dynamic stress–strain relation at every point is perfectly elastic, and furthermore is isotropic;

(4) the static stress field in the equilibrium configuration is at every point isotropic.

Any such model of the Earth will be called a SNREI (spherical, non-rotating, elastic, isotropic) Earth model. For the purpose of computing the theoretical eigen-frequencies, a SNREI Earth model of radius a can be completely characterized by three functions of r, the radial distance from the centre. These three functions are the density $\rho_0(r)$, the bulk modulus $\kappa(r)$, and the shear modulus $\mu(r)$, the latter two being the *in situ* elastic parameters appropriate to the hydrostatically compressed state of the material.

The attention of many geophysicists has now turned toward investigations of the so-called normal mode inverse problem. In the most general sense, this is the problem of enumerating and exploring the collection of all possible, not necessarily SNREI Earth models, whose theoretical eigenfrequencies are in agreement with the measured eigenfrequencies of the real Earth. Before attempting such a problem it is essential to have a knowledge of the precision of the normal mode data.

Probably the major source of error in the data is the presence in the records of various types of noise: instrumental noise, stationary and non-stationary seismic noise, and noise introduced by the methods of data analysis. Another possible source of error arises from the fact that a time series of the Earth's free oscillations is not in general a record of a single impulse response of the Earth, but rather represents the response of the Earth to a main shock as well as to a series of foreshocks and aftershocks. The manner in which the presence of an aftershock sequence can affect the measurement of the eigenfrequency and the Q of a normal mode has been investigated by Press (1966, 1967). Press (1967) found that if a record is analysed as if it were a single impulse response, then this effect could lead to an error in the measurement of the eigenfrequency of a normal mode of about 0·1 per cent. This effect should certainly be recognized in future analyses of the data. Another factor which could greatly affect the precision of the data is the possibility of mode misidentification. Some mode-identifying criteria can be obtained by observing correlations among several records at one station (various components of strain, gravity, and tilt) (Gilbert & Backus 1965), and these have been utilized (Smith 1966; Nowroozi 1966). Further mode-identifying criteria would of course be provided by any future world-wide array of low-frequency seismological instruments (Gilbert & Backus 1965).

The real Earth is of course not a SNREI Earth model; in fact all of the assumptions (1)–(4) are false for the real Earth. However, because of the very large extent of the non-uniqueness in the inverse problem (Backus & Gilbert 1967), and because of the relative mathematical simplicity, it is customary to include only SNREI Earth models in inverse problem calculations. In this case the fact that the real Earth is not a SNREI Earth model may be looked upon as one of the factors affecting the precision of the raw data. If it is desired to compare the raw normal mode data to the theoretical normal mode data of various SNREI Earth models, then the raw data may be looked upon as contaminated by the asphericity, rotation, and anisotropy of the real Earth. It is desirable to try to remove as much of this contamination as is possible. At the present time of course, this contamination cannot be completely removed because very little is known about the Earth's anisotropies and deep inhomogeneities.

The deviations from sphericity and isotropy of the real Earth are probably small enough that they only slightly perturb the theoretical calculations of the normal mode eigenfrequencies. As a first step toward a correction of the raw normal mode data, it is necessary to gain a general idea of the kind of effect that the Earth's slow angular rotation and small asphericities and anisotropies will have on the theoretical eigenfrequencies of an arbitrary SNREI Earth model. It is customary to use the following notation for the normal modes of a SNREI Earth model: ${}_nS_l^m$ denotes the nth overtone of a poloidal mode characterized by a spherical harmonic Y_l^m of degree l and order m, while ${}_nT_l^m$ denotes the nth overtone of a toroidal mode characterized by Y_l^m. In this paper Y_l^m will denote the fully normalized complex surface spherical harmonic

$$Y_l^m = (-1)^m \left[\frac{2l+1}{4\pi}\right]^{\frac{1}{2}} \left[\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}} P_l^m(\cos\theta)\, e^{im\phi}$$

where θ is the colatitude and ϕ is the longitude. This normalization is such that

$$\int_S |Y_l^m|^2 \, d\Omega = 1,$$

where S is the surface of the unit sphere. Here l and n can take on all non-negative integral values, while m takes on all integral values between $-l$ and l. For a fixed l and n, all the $2l+1$ poloidal modes ${}_nS_l^m$ of a SNREI Earth model have the same angular frequency ${}_n\omega_l^S$, while all the $2l+1$ toroidal modes ${}_nT_l^m$ have the same angular frequency ${}_n\omega_l^T$. The angular frequencies ${}_n\omega_l^S$ and ${}_n\omega_l^T$ are said to be $2l+1$-degenerate; i.e. associated with every eigenfrequency ${}_n\omega_l^S$ or ${}_n\omega_l^T$ of a SNREI Earth model, there is a $2l+1$-dimensional eigenspace. The $2l+1$ modes ${}_nS_l^m$ of a SNREI Earth model will be called the poloidal multiplet ${}_nS_l$; the $2l+1$ modes ${}_nT_l^m$ will be called the toroidal multiplet ${}_nT_l$. The effect of slow rotation and small asphericities and anisotropies is to remove the degeneracy of a multiplet ${}_nS_l$ or ${}_nT_l$ by selecting certain elements of the $2l+1$ dimensional eigenspace and shifting their eigenfrequencies by various small amounts. If the deviations from sphericity and isotropy of the real Earth are indeed small, then perturbation techniques can be used to compute their effect. In this paper, Rayleigh's variational principle governing the small oscillations of an arbitrary conservative system about an equilibrium configuration is used for this purpose. If a SNREI Earth model is varied slightly to produce small asphericities and anisotropies, and if it is desired to view the normal mode displacements with respect to a coordinate system rotating with the Earth, then Rayleigh's principle can be used to compute approximately the normal mode eigenfunctions and the eigenfrequencies of the resulting non-SNREI Earth model. As well as allowing corrections to be made to the raw normal mode data, the theory of the normal modes of slightly non-SNREI Earth models reveals other interesting features which should be visible in the data.

Investigations of the effects of rotation and asphericity on the normal modes of rotating bodies have been made by several authors. Love (1889) and Bryan (1889) studied the gravitational modes of a rotating homogeneous, incompressible liquid MacLaurin ellipsoid. Cowling & Newing (1949) used a form of Rayleigh's principle, and Ledoux (1951) used perturbation theory to investigate the oscillations of rotating stars composed of compressible fluid. These authors neglected the aspherical shape of the equilibrium configuration caused by the rotation and studied only the effects of the Coriolis force on the small oscillations. Chandrasekhar & Lebovitz (1962) used a tensor form of the virial theorem to make a study of the small oscillations of a rotating mass of compressible fluid about its aspherical equilibrium configuration. Their method allowed an investigation only of the normal modes of degree $l \leqslant 2$.

Investigations of the effects of the Earth's slow angular rotation on the poloidal and toroidal normal modes of a SNREI Earth model were made independently by several authors soon after the Chilean earthquake of 1960. Backus & Gilbert (1961), MacDonald & Ness (1961), and Pekeris, Alterman & Jarosch (1961) used perturbation theories to show that the first order effect of a slow rotation is to shift slightly the eigenfrequency ${}_n\omega_l^S$ or ${}_n\omega_l^T$ of a normal mode ${}_nS_l^m$ or ${}_nT_l^m$ of a SNREI Earth model. For an Earth model rotating with a steady angular velocity Ω, the modes ${}_nS_l^m$ and ${}_nT_l^m$ have, respectively, the angular frequencies

$$\left.\begin{array}{l} {}_n\omega_l^S + m({}_n\beta_l^S)\,\Omega \\ {}_n\omega_l^T + m({}_n\beta_l^T)\,\Omega \end{array}\right\} \qquad (1)$$

where terms of higher order in $(\Omega/{}_n\omega_l^S)$ and $(\Omega/{}_n\omega_l^T)$ have been neglected. The splitting parameters ${}_n\beta_l^S$ for poloidal modes depend on the SNREI Earth model under consideration, but the parameters ${}_n\beta_l^T$ for toroidal modes are independent of the model (see Appendix A). Note in particular that to first order the rotation of the Earth does not alter the eigenfrequency of the $m = 0$ mode. The second order rotational perturbation to the eigenfrequencies will produce a relative shift of about $(\Omega/{}_n\omega_l^S)^2$ or $(\Omega/{}_n\omega_l^T)^2$, and for the lower order modes this can amount to about 0·1 per cent. It can be shown that the second-order rotational correction does act to shift the eigenfrequency of the $m = 0$ mode. A method for computing the second order rotational correction is given by Backus & Gilbert (1961), and results of the actual computation are presented in the present paper in Tables 1–3 (see Section 3, equations (25) and (26)).

The largest deviation from sphericity of the real Earth is the equatorial bulge; the shape of the Earth is very nearly that of an ellipsoid of revolution with an ellipticity equal to 1/298·3. It may be expected that this deviation from sphericity will produce a relative shift of the eigenfrequencies of a SNREI Earth model by an amount of this order, i.e. approximately 0·3 per cent. Usami & Satô (1962), using ellipsoidal coordinates, computed the first-order effects of small ellipticity on the toroidal normal modes of a homogeneous, elastic, non-gravitating spheroid. Caputo (1963) studied the effects of small ellipticity on the toroidal modes of two simple Earth models: one a homogeneous spheroid, the other consisting of a homogeneous shell, limited inside by a sphere and outside by a spheroid. In the present paper, it is shown how to compute the effects of small ellipticity on the normal modes of an arbitrary SNREI Earth model. Another obvious deviation from sphericity of the real Earth is the differences between continental and oceanic crustal structures. For various reasons, many investigators feel that lateral variations in density and elastic parameters extend to depths of hundreds of kilometres into the upper mantle. Toksöz & Ben-Menahem (1963) and Toksöz & Anderson (1966) have measured phase velocities of mantle Love and Rayleigh waves and have detected small variations (about one to two per cent) between different paths. Their data were partially corrected for the effect of the Earth's ellipticity in that they measured path lengths on an ellipsoidal rather than on a spherical Earth. If the variations in phase velocity for different paths are ascribed solely to regional heterogeneity, then their data reveal that for fundamental normal mode multiplets ${}_0S_l$ or ${}_0T_l$, with l greater than about 25, the perturbing effect of regional heterogeneities is greater than that of the Earth's ellipticity.

The existence of mountains on the surface of the Earth gives positive proof that the static (or secular) stress field in the Earth is not purely hydrostatic, at least not in the upper regions of the Earth. Recent satellite measurements of the low order terms in the expansion of the Earth's gravitational potential (King-Hele 1965; Guier & Newton 1965; Kaula 1966) reveal that there are small deviations from hydrostatic equilibrium in the Earth on an even much larger geographic scale than that of mountain ranges (Jeffreys 1963; MacDonald 1966). There is evidence from seismic refraction surveys at sea that, at least in the upper mantle directly below the crust, the dynamic stress–strain relation is slightly anisotropic (Hess 1964; Backus 1965; Raitt *et al.* 1968; Morris *et al.* 1968). At the present time, no detailed information is available concerning the spatial variation in the Earth of either of these deviations from isotropy. Thus although a theory is provided in this paper which allows the computation of the effects of small anisotropies on the eigenfrequencies of a SNREI Earth model, no detailed calculations are made.

Section 2 of this paper uses Rayleigh's principle to provide an explicit formalism which can be used to compute to first order the small changes in the eigenfrequencies ${}_n\omega_l^S$ or ${}_n\omega_l^T$ of any SNREI Earth model due to a slow angular rotation and small but otherwise arbitrary deviations from sphericity and isotropy. In Section 3 this formalism is used to compute to first order the effects of rotation and ellipticity on the normal modes of three realistic SNREI Earth models. Clairaut's theory is used to compute the ellipticity of figure of the rotating Earth models. The second-order rotational perturbation to the eigenfrequencies is also computed for the same SNREI Earth models. Results presented in Section 3 may be used as a preliminary correction for rotation and ellipticity to the Earth's raw normal mode data. Other interesting features of the normal modes of non-SNREI Earth models are discussed in Sections 4 and 5.

2. Theory

Consider an Earth model which consists of a self-gravitating continuum occupying an arbitrary bounded volume V with surface ∂V, and which has a steady angular velocity of rotation $\mathbf{\Omega}$ about its centre of mass. Assume further that in the equilibrium state of steady angular rotation there is a static, in general non-isotropic stress field $\mathbf{T}_0$. Assume further that the continuum comprising the body is perfectly elastic but that the dynamic stress–strain relation at every point is not necessarily isotropic. Such a body can, of course, undergo small oscillations about the equilibrium configuration. Let ρ_0 denote the density and ϕ_0 the gravitational potential of the body occupying the volume V. Let $e^{i\omega t}\mathbf{S}(\mathbf{r})$ be a possible displacement field for an elastic deformation of the body, and $e^{i\omega t}\phi_1(\mathbf{r})$ the associated disturbance in the gravitational potential, both measured with respect to the reference frame rotating with angular velocity $\mathbf{\Omega}$ about the centre of mass.

Then if the displacement is considered small so that terms of second order in $\mathbf{S}$ may be neglected, the equations of motion for an elastic-gravitational mode of such a body may be written, in the rotating coordinate system, as:

$$-\rho_0\,\omega^2\,\mathbf{S}+2i\rho_0\,\omega\mathbf{\Omega}\times\mathbf{S} = -\rho_0\,\nabla\phi_1-\rho_1\,\nabla(\phi_0+\psi)$$

$$-\nabla[\mathbf{S}.\rho_0\,\nabla(\phi_0+\psi)]+\nabla.\mathbf{E}+\nabla[\mathbf{S}.(\nabla.\boldsymbol{\tau}_0)]-\nabla.(\mathbf{S}.\nabla\boldsymbol{\tau}_0) \quad (2)$$

and $$\nabla^2\phi_1 = 4\pi G\rho_1.$$

In equations (2), $\rho_1 = -\nabla.(\rho_0\,\mathbf{S})$ is the change in density due to the displacement, ψ is the rotational potential due to the centripetal acceleration

$$\psi(\mathbf{r}) = -\tfrac{1}{2}[\Omega^2 r^2-(\mathbf{\Omega}.\mathbf{r})^2], \quad (3)$$

and $\boldsymbol{\tau}_0$ is the static stress deviator

$$\boldsymbol{\tau}_0 = \mathbf{T}_0-\tfrac{1}{3}(tr\,\mathbf{T}_0)\,\mathbf{I}$$

where $\mathbf{I}$ is the second-order identity tensor. The second-order tensor $\mathbf{E}$ is the (Lagrangian) elastic stress tensor. At any point $\mathbf{r}$ in the body and to first order in the displacement field $\mathbf{S}$, $\mathbf{E}$ is related to $\mathbf{S}$ by the linear elastic parameters appropriate to the compressed state of the material at $\mathbf{r}$. The components of $\mathbf{E}$ relative to an arbitrary Cartesian axis system $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$ in the rotating reference frame may be expressed in terms of the components of $\mathbf{S}$ in the following manner (Biot 1965).

$$E_{ij} = \Gamma_{ijkl}\,\sigma_{kl}-\tfrac{1}{2}\tau_{jl}{}^0(\partial_i\,S_l-\partial_l\,S_i)-\tfrac{1}{2}\tau_{il}{}^0(\partial_j\,S_l-\partial_l\,S_j) \quad (4)$$

where $\sigma_{kl} = \frac{1}{2}(\partial_k\,S_l+\partial_l\,S_k)$, and Γ_{ijkl} are the elastic parameters, themselves the components of a fourth-order tensor which will be called the stress–strain tensor. The last two terms on the right arise because the displacement field acts to rotate the static stress field; they only occur if the static stress field is non-hydrostatic.

The equations (2) must be solved relative to certain boundary conditions which are to be applied on the undeformed boundaries of V, both on the external boundary ∂V and on any internal discontinuities (e.g. a mantle–core discontinuity). These boundary conditions are (Alterman, Jarosch & Pekeris 1959; Backus 1967):

$$\left.\begin{array}{l} \mathbf{S} \text{ continuous (note: at a solid–liquid discontinuity, only } \hat{\mathbf{n}}.\mathbf{S} \text{ need be} \\ \quad \text{continuous)} \\ \hat{\mathbf{n}}.\mathbf{E} \text{ continuous, where } \hat{\mathbf{n}} \text{ is the normal to the undeformed boundary} \\ \quad \text{(note: on } \partial V,\ \hat{\mathbf{n}}.\mathbf{E} = \mathbf{0}) \\ \phi_1 \text{ continuous} \\ \hat{\mathbf{n}}.\nabla\phi_1 + \mathbf{S}.\hat{\mathbf{n}}\, 4\pi G\rho_0 \text{ continuous.} \end{array}\right\} \tag{5}$$

It is first convenient to note that the second of equations (2) can be integrated immediately in terms of a Green's function for the volume V.

$$\phi_1(\mathbf{r}) = G\int_V \rho_1(\mathbf{r}')\,\mathscr{G}(\mathbf{r},\mathbf{r}')\,dV - G\int_{\partial V} \rho_0(\mathbf{r}')\,\mathscr{G}(\mathbf{r},\mathbf{r}')\,\mathbf{S}(\mathbf{r}').\hat{\mathbf{n}}\,dS$$

where the second integral is over the surface ∂V of the volume V, and where $\mathscr{G}(\mathbf{r},\mathbf{r}')$ is the Green's function $\mathscr{G}(\mathbf{r},\mathbf{r}') = 1/|\mathbf{r}-\mathbf{r}'|$.

Now let $\mathscr{S}$ be the vector space consisting of all piecewise twice-continuously differentiable vector fields on V. On $\mathscr{S}$, an inner product is defined in the following manner. For any two members $\mathbf{u}$, $\mathbf{v}$ of $\mathscr{S}$, $(\mathbf{u},\mathbf{v})$ is defined as

$$(\mathbf{u},\mathbf{v}) = \int_V \rho_0\,\mathbf{u}^* \,.\, \mathbf{v}\,dV.$$

With this definition, $\mathscr{S}$ becomes an inner product space which can be completed to a Hilbert space. The equations of motion for the elastic-gravitational modes of the body V can thus be written as an eigenvalue equation for a linear operator in a Hilbert space. Written in terms of components in an arbitrary Cartesian axis system, this equation takes the form

$$\rho_0\,\omega^2 S_i - 2i\rho_0\,\omega\,\varepsilon_{ijk}\,\Omega_j\,S_k = HS_i \tag{6}$$

where

$$\begin{aligned} HS_i = {} & -\partial_j(\Gamma_{ijkl}\,\sigma_{kl}) + \partial_i\,[\rho_0\,S_j\,\partial_j(\phi_0+\psi)] + \rho_1\,\partial_i(\phi_0+\psi) + \rho_0\,\partial_i\,\phi_1 \\ & -\partial_i\,[S_k\,\partial_j\,\tau_{kj}{}^0] + \partial_j(S_k\,\partial_k\,\tau_{ij}{}^0) + \partial_j\,[\tfrac{1}{2}\tau_{jl}{}^0(\partial_i\,S_l - \partial_l\,S_i)] + \partial_j\,[\tfrac{1}{2}\tau_{il}{}^0(\partial_j\,S_l - \partial_l\,S_j)] \end{aligned}$$

where $\rho_1 = -\partial_i(\rho_0\,S_i)$ and where E_{ij} is defined in equation (4), and ϕ_1 is defined in equation (6), and where ε_{ijk} is the alternating symbol in three dimensions. Note that the operator H is an integro-differential operator. It can be shown that H as defined is a Hermitian operator, i.e. for any two vector fields $\mathbf{u}$, $\mathbf{v}$ in $\mathscr{S}$ which satisfy the boundary conditions (5), the following relation holds:

$$(\mathbf{u}, H\mathbf{v}) = (H\mathbf{u}, \mathbf{v}).$$

The operator $i\boldsymbol{\Omega}\times$ is also Hermitian, i.e. $(\mathbf{u}, i\boldsymbol{\Omega}\times\mathbf{v}) = (i\boldsymbol{\Omega}\times\mathbf{u}, \mathbf{v})$.

Upon taking the inner product of the displacement vector $\mathbf{S}$ and equation (6) one arrives at the equation

$$\omega^2(\mathbf{S},\mathbf{S}) - 2\omega(\mathbf{S}, i\boldsymbol{\Omega}\times\mathbf{S}) = (\mathbf{S}, \rho_0{}^{-1}\,H\mathbf{S}). \tag{7}$$

Now any linear operator L defined on an inner product space $\mathscr{S}$ may be associated with a unique bilinear functional $\mathscr{L}(\mathbf{u},\mathbf{v})$ on $\mathscr{S}$ by the relation $\mathscr{L}(\mathbf{u},\mathbf{v}) = (\mathbf{u}, L\mathbf{v})$ (Reisz & Nagy 1955). To indicate that the inner product terms in equation (7) are in fact bilinear functionals defined on $\mathscr{S}$, the following notation will be used:

$$\mathscr{H}(\mathbf{S},\mathbf{S}) = (\mathbf{S}, \rho_0{}^{-1}\,H\mathbf{S})$$

$$\mathscr{T}(\mathbf{S},\mathbf{S}) = (\mathbf{S},\mathbf{S})$$

$$\mathscr{W}(\mathbf{S}, \mathbf{S}) = (\mathbf{S}, i\boldsymbol{\Omega}\times\mathbf{S}).$$

It is also convenient to write $\mathscr{H}(\mathbf{S}, \mathbf{S})$ as a sum of three bilinear functionals,

$$\mathscr{H}(\mathbf{S}, \mathbf{S}) = \mathscr{V}(\mathbf{S}, \mathbf{S}) + \mathscr{P}(\mathbf{S}, \mathbf{S}) + \Psi(\mathbf{S}, \mathbf{S})$$

where $\mathscr{P}(\mathbf{S}, \mathbf{S})$ includes all terms linear in $\boldsymbol{\tau}_0(\mathbf{r})$, $\Psi(\mathbf{S}, \mathbf{S})$ includes all terms linear in $\psi(\mathbf{r})$, and $\mathscr{V}(\mathbf{S}, \mathbf{S})$ is independent of $\boldsymbol{\tau}_0(\mathbf{r})$ and $\psi(\mathbf{r})$. Several applications of Gauss' theorem with use of the boundary conditions (4.5) allow $\mathscr{V}(\mathbf{S}, \mathbf{S})$ and $\Psi(\mathbf{S}, \mathbf{S})$ to be written as

$$\mathscr{V}(\mathbf{S}, \mathbf{S}) = \int_V dV\,[\Gamma_{ijkl}\,\sigma_{ij}\,\sigma_{kl}^* + \rho_0\,S_i\,S_j^*\,\partial_i\,\partial_j\,\phi_0$$

$$+\rho_0\,\partial_j\,\phi_0(S_i\,\partial_i\,S_j^* - S_j\,\partial_i\,S_i^*) + \rho_0\,S_i^*\,\partial_i\,\phi_1 + \rho_0\,S_i\,\partial_i\,\phi_1^*] + \int_E dV\,\frac{1}{4\pi G}\,|\nabla\phi_1|^2 \quad (8)$$

and

$$\Psi(\mathbf{S}, \mathbf{S}) = \int_V dV\,[\rho_0\,S_i\,S_j^*\,\partial_i\,\partial_j\,\psi + \rho_0\,\partial_j\,\psi(S_i\,\partial_i\,S_j^* - S_j\,\partial_i\,S_i^*)] \quad (9)$$

where E is all of space. Similarly, several applications of Gauss' theorem with use of the boundary condition that $\hat{\mathbf{n}}.\mathbf{T}_0$ must be continuous everywhere (and thus zero on ∂V) allows $\mathscr{P}(\mathbf{S}, \mathbf{S})$ to be written as

$$\mathscr{P}(\mathbf{S}, \mathbf{S}) = \int_V dV\,\tau_{ij}^0\,[S_k^*\,\partial_k\,\partial_j\,S_i - S_i^*\,\partial_j\,\partial_k\,S_k - \sigma_{jk}\,\partial_k\,S_i^* + \sigma_{ik}\,\partial_j\,S_k^*]. \quad (10)$$

In terms of these bilinear functionals, equation (7) can be written as

$$\omega^2\,\mathscr{T}(\mathbf{S}, \mathbf{S}) - 2\omega\mathscr{W}(\mathbf{S}, \mathbf{S}) - \mathscr{V}(\mathbf{S}, \mathbf{S}) - \mathscr{P}(\mathbf{S}, \mathbf{S}) - \Psi(\mathbf{S}, \mathbf{S}) = 0. \quad (11)$$

If the Earth model is non-rotating, then equation (11) becomes

$$\omega^2\,\mathscr{T}(\mathbf{S}, \mathbf{S}) = \mathscr{V}(\mathbf{S}, \mathbf{S}) + \mathscr{P}(\mathbf{S}, \mathbf{S}). \quad (12)$$

Equation (12) is in fact Rayleigh's principle for a non-rotating mechanical system. The term $\omega^2\,\mathscr{T}(\mathbf{S}, \mathbf{S})$ is twice the kinetic energy of a disturbance $\mathbf{S}$, while $\mathscr{V}(\mathbf{S}, \mathbf{S}) + \mathscr{P}(\mathbf{S}, \mathbf{S})$ is twice the potential energy of the disturbance. The kinetic energy of a disturbance is thus equal to the total potential energy. The potential energy term $\mathscr{V}(\mathbf{S}, \mathbf{S})$ includes elastic energy, gravitational energy, and work done against hydrostatic pressure, while the term $\mathscr{P}(\mathbf{S}, \mathbf{S})$ represents the work done against the deviatoric part of the static stress field. If equation (12) is regarded as defining a bilinear functional ω^2 of $\mathbf{S}$, then Rayleigh's variational principle states that that functional is stationary to first order in an arbitrary small variation in $\mathbf{S}$ if and only if $\mathbf{S}$ is the displacement field of a normal mode of oscillation whose angular frequency is ω. Equation (11) for a rotating Earth model also expresses an equality between kinetic and potential energies, but there are extra kinetic energy terms because of the rotation. Also, since the operator $i\Omega\times$ is Hermitian, as is the operator H, there is a variational principle similar to Rayleigh's principle contained in equation (11). The bilinear functional of $\mathbf{S}$ on the left-hand side of equation (11) is stationary to first order in an arbitrary small variation in $\mathbf{S}$ if and only if $\mathbf{S}$ is the displacement field of a normal mode of oscillation of the rotating Earth model whose angular frequency is ω. This can be seen by considering an arbitrary small variation $\delta\mathbf{S}$ in equation (11). In doing so it is necessary to remember that equation (11) is equivalent to equation (7), and that the operators H and $i\boldsymbol{\Omega}\times$ are Hermitian; it is convenient actually to consider a small variation $\delta\mathbf{S}$ in equation (7)

$$\omega^2(\mathbf{S}, \delta\mathbf{S})+\omega^2(\delta\mathbf{S}, \mathbf{S})-2\omega(\mathbf{S}, i\boldsymbol{\Omega}\times\delta\mathbf{S})-2\omega(\delta\mathbf{S}, i\boldsymbol{\Omega}\times\mathbf{S}) = (\mathbf{S}, \rho_0^{-1}H\delta\mathbf{S})+(\delta\mathbf{S}, \rho_0^{-1}H\mathbf{S}).$$

Since H and $i\boldsymbol{\Omega}\times$ are Hermitian operators, this may be reduced to

$$2\omega^2(\delta\mathbf{S}, \mathbf{S})-4\omega(\delta\mathbf{S}, i\boldsymbol{\Omega}\times\mathbf{S}) = 2(\delta\mathbf{S}, \rho_0^{-1}H\mathbf{S})$$

or

$$2\big(\delta\mathbf{S}, (\omega^2\mathbf{S}-2i\omega\boldsymbol{\Omega}\times\mathbf{S}-\rho_0^{-1}H\mathbf{S})\big) = 0.$$

Since $\delta\mathbf{S}$ is an arbitrary small variation, this implies

$$\omega^2\mathbf{S}-2i\omega\boldsymbol{\Omega}\times\mathbf{S} = \rho_0^{-1}H\mathbf{S}$$

which is equivalent to equation (6). The demonstration of the converse is immediate.

Thus, as Rayleigh (1877) points out, the small shift $\delta\omega$ imparted to any eigenfrequency ω by small perturbations $\delta\rho_0$, $\delta\phi_0$, $\delta\boldsymbol{\Omega}$, $\delta\boldsymbol{\tau}_0$, $\delta\Gamma_{ijkl}$ in the static Earth model can be calculated directly from equation (11) if the eigenfunctions $\mathbf{S}$ for the unperturbed body are known. The first order relation among all these perturbations can be computed by taking first-order variations in equation (11), viewing the left-hand side of equation (11) as a functional not only of $\mathbf{S}$ but also of ρ_0, ϕ_0, $\boldsymbol{\Omega}$, $\boldsymbol{\tau}^0$, Γ_{ijkl}. A term containing $\delta\mathbf{S}$ does not appear because of the stationary character of the functional relative to small changes in the eigenfunctions. The unperturbed body will be taken to be a SNREI Earth model; for such a model eigenfrequencies and eigenfunctions are readily computed. Consider the small variations necessary to produce a slightly non-SNREI Earth model from a nearby SNREI Earth model. The rotation vector $\boldsymbol{\Omega}$ must be varied away from $\boldsymbol{\Omega}=\mathbf{0}$, and the static stress deviator $\boldsymbol{\tau}_0$ must be varied away from $\boldsymbol{\tau}_0=\mathbf{0}$. Small deviations from spherical symmetry will give rise to a $\delta\rho_0$ and a consequent $\delta\phi_0$. The elastic parameters Γ^0_{ijkl} of a SNREI Earth model are related to the bulk modulus κ and the shear modulus μ by

$$\Gamma^0_{ijkl} = (\kappa-\tfrac{2}{3}\mu)\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

where δ_{ij} is the Kronecker delta. The elastic parameters Γ_{ijkl} of a nearby slightly non-SNREI Earth model will, in general, have the form

$$\Gamma_{ijkl} = \Gamma^0_{ijkl}+\gamma_{ijkl} \quad \text{where } \gamma_{ijkl} \ll \kappa, \mu. \tag{13}$$

The coefficients γ_{ijkl} are the Cartesian components of a fourth-order anisotropic tensor; the components of the isotropic part γ^I_{ijkl} of this tensor will be denoted by

$$\gamma^I_{ijkl} = (\delta\kappa-\tfrac{2}{3}\delta\mu)\delta_{ij}\delta_{kl}+\delta\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}).$$

Here $\delta\kappa = \frac{1}{9}\gamma_{iijj}$ and $\delta\mu = \frac{1}{10}(\gamma_{ijij}-\frac{1}{3}\gamma_{iijj})$. The remaining anisotropic part of γ_{ijkl} will be denoted by $\gamma^A_{ijkl} = \gamma_{ijkl}-\gamma^I_{ijkl}$. The terms in $\delta\kappa$ and $\delta\mu$ can represent small variations in the SNREI Earth model from spherical symmetry, while the terms in γ^A_{ijkl} represent deviations of the dynamic stress–strain relation from isotropy. The first-order relation among all these perturbations is, from equation (11)

$$\delta\omega^2\mathscr{T}(\mathbf{S},\mathbf{S}) = \delta\mathscr{V}(\mathbf{S},\mathbf{S})-\omega^2\delta\mathscr{T}(\mathbf{S},\mathbf{S})+2\omega\mathscr{W}(\mathbf{S},\mathbf{S})+\mathscr{P}(\mathbf{S},\mathbf{S})+\Psi(\mathbf{S},\mathbf{S}) \tag{14}$$

where

$$\delta\mathscr{T}(\mathbf{S},\mathbf{S}) = \int_V \delta\rho_0|\mathbf{S}|^2\,dV$$

and where

$$\begin{aligned}\delta\mathscr{V}(\mathbf{S},\mathbf{S}) = \int_V dV\,[&\gamma_{ijkl}\sigma_{ij}\sigma^*_{kl}+\delta\rho_0 S_i S_j^*\partial_i\partial_j\phi_0+\rho_0 S_i S_j^*\partial_i\partial_j\delta\phi_0\\ &+\delta\rho_0\partial_j\phi_0(S_i\partial_i S_j^*-S_j\partial_i S_i^*)+\rho_0\partial_j\delta\phi_0(S_i\partial_i S_j^*-S_j\partial_i S_i^*)\\ &+\delta\rho_0 S_i^*\partial_i\phi_1+\delta\rho_0 S_i\partial_i\phi_1^*].\end{aligned}$$

Equation (14) gives explicitly the first-order change in the angular frequency of any normal mode of a SNREI Earth model due to a non-zero angular rotation $\mathbf{\Omega}$, due to a non-zero static stress deviator $\boldsymbol{\tau}_0$, due to any deviation $\delta\rho_0$, $\delta\phi_0$, $\delta\kappa$, $\delta\mu$ from sphericity (the variations $\delta\rho_0$, $\delta\phi_0$, $\delta\kappa$, $\delta\mu$ can be functions of r alone; they need not be deviations from sphericity), and due to a slightly anisotropic stress–strain tensor Γ_{ijkl}. Note that the small change due to the rotational potential ψ, which is really a term of second order in Ω, has the same form as a perturbation $\delta\phi_0$.

The situation is complicated because of the fact that an angular frequency ${}_n\omega_l^S$ or ${}_n\omega_l^T$ of a SNREI Earth model is $2l+1$ degenerate; i.e. there is a $2l+1$-dimensional subspace ${}_n\mathscr{S}_l^S$ or ${}_n\mathscr{S}_l^T$ of $\mathscr{S}$, any element of which is a possible eigenfunction associated, respectively, with ${}_n\omega_l^S$ or ${}_n\omega_l^T$. In order to carry out computations, it is necessary to select a particular orthonormal basis of this subspace ${}_n\mathscr{S}_l^S$ or ${}_n\mathscr{S}_l^T$. Denote the members of the chosen basis by $\{{}_n\mathbf{S}_l^m, m = -l, \dots, 0, \dots l\}$. Orthonormality means that $({}_n\mathbf{S}_l^m, {}_n\mathbf{S}_l^{m'}) = \mathscr{T}({}_n\mathbf{S}_l^m, {}_n\mathbf{S}_l^{m'}) = \delta_{mm'}$. At this point it becomes cumbersome to retain the subscripts n and l and superscripts S or T to indicate which particular multiplet is being discussed. For the remainder of Section 2, the discussion will be restricted to a single multiplet ${}_nS_l$ or ${}_nT_l$ with associated eigenvalue ${}_n\omega_l^S$ or ${}_n\omega_l^T$. The members of the chosen basis will be relabelled simply $\{\mathbf{S}_m, m = -l, \dots 0, \dots l\}$ and the eigenfrequency ${}_n\omega_l^S$ or ${}_n\omega_l^T$ will be called ω. Any eigenfunction $\mathbf{S}$ of a SNREI Earth model associated with the degenerate eigenfrequency ω may be written in terms of the basis vectors

$$\mathbf{S} = \sum_{m=-l}^{l} a^m \mathbf{S}_m. \tag{15}$$

Now equation (14) was obtained by taking the inner product of equation (6) and the displacement vector $\mathbf{S}$ and then taking first variations. If instead, one takes the inner product of equation (6) with a second *arbitrary* member $\mathbf{S}'$ of the $2l+1$ dimensional subspace associated with the ω in equation (6), then equation (11) would have taken the form

$$\omega^2 \mathscr{T}(\mathbf{S}, \mathbf{S}') - 2\omega\mathscr{W}(\mathbf{S}, \mathbf{S}') - \mathscr{V}(\mathbf{S}, \mathbf{S}') - \mathscr{P}(\mathbf{S}, \mathbf{S}') - \Psi(\mathbf{S}, \mathbf{S}') = 0. \tag{11.1}$$

It is easily shown that, since both $\mathbf{S}$ and $\mathbf{S}'$ are eigenfunctions associated with ω, the bilinear functional in equation (11.1) is stationary to arbitrary small variations in either $\mathbf{S}$ or $\mathbf{S}'$. Therefore taking first variations in equation (11.1) yields

$$\delta\omega^2 \mathscr{T}(\mathbf{S}, \mathbf{S}') = \delta\mathscr{V}(\mathbf{S}, \mathbf{S}') - \omega^2 \delta\mathscr{T}(\mathbf{S}, \mathbf{S}') + 2\omega\mathscr{W}(\mathbf{S}, \mathbf{S}') + \mathscr{P}(\mathbf{S}, \mathbf{S}') + \Psi(\mathbf{S}, \mathbf{S}'). \tag{14.1}$$

Now $\mathbf{S}'$ may be written in terms of the chosen basis vectors

$$\mathbf{S}' = \sum_{m=-l}^{l} b^m \mathbf{S}_m. \tag{15.1}$$

Substituting for $\mathbf{S}$ and $\mathbf{S}'$ in equation (14.1) their expansions (15) and (15.1), equation (14.1) reduces to

$$b^i a^i \delta\omega^2 = b^i [\delta\mathscr{V}(\mathbf{S}_i, \mathbf{S}_j) - \omega^2 \delta\mathscr{T}(\mathbf{S}_i, \mathbf{S}_j) + 2\omega\mathscr{W}(\mathbf{S}_i, \mathbf{S}_j) + \mathscr{P}(\mathbf{S}_i, \mathbf{S}_j) + \Psi(\mathbf{S}, \mathbf{S}_j)] a^j.$$

But the b^i are arbitrary, so the perturbation problem for a degenerate multiplet ${}_nS_l$ or ${}_nT_l$ reduces to an eigenvalue problem in the $2l+1$ dimensional subspace ${}_n\mathscr{S}_l^S$ or ${}_n\mathscr{S}_l^T$; this may be written in terms of a Hermitian $2l+1$ dimensional matrix R.

$$R_{ij} a^j = \delta\omega^2 a^i \tag{16}$$

where

$$R_{ij} = \delta\mathscr{V}(\mathbf{S}_i, \mathbf{S}_j) - \omega^2 \delta\mathscr{T}(\mathbf{S}_i, \mathbf{S}_j) + 2\omega\mathscr{W}(\mathbf{S}_i, \mathbf{S}_j) + \mathscr{P}(\mathbf{S}_i, \mathbf{S}_j) + \Psi(\mathbf{S}_i, \mathbf{S}_j). \tag{17}$$

The bilinear functional $\Psi(\mathbf{S}, \mathbf{S})$ has been retained at this stage even though it is of second order in (Ω/ω) in order to assure that R_{ij} is Hermitian. For any small variations $\mathbf{\Omega}$, $\boldsymbol{\tau}_0$, $\delta\rho_0$, $\delta\phi_0$, $\delta\kappa$, $\delta\mu$, γ^A_{ijkl}, the $(2l+1).(2l+1)$ elements of the Hermitian matrix R may be computed in terms of the eigenfunctions $\{\mathbf{S}_m, m = -l, \dots 0, \dots l\}$ of the SNREI Earth model. Denote the $2l+1$ eigenvalues of the matrix R by

$$\{(\delta\omega^2)_k, k = -l, \dots, 0, \dots l\},$$

and denote the eigenfunctions associated with the eigenvalues by

$$\{(a_k^{-l}, \dots, a_k^0 \dots, a_k^l), \ k = -l, \dots 0, \dots l\}.$$

Then the normal modes of a slightly non-SNREI Earth model can be characterized in the following way. To zeroeth order in the small variations, the normal mode eigenfunctions are

$$\sum_{m=-l}^{l} a_k^m \mathbf{S}_m$$

and to first order in the small variations the associated squared eigenfrequencies are $\omega^2 + (\delta\omega^2)_k$. The above notation assumes that none of the eigenvalues of R is degenerate, which implies that the degeneracy of ${}_n\omega_l^S$ or ${}_n\omega_l^T$ is completely removed by the perturbations. Backus & Gilbert (1961) showed that the effect of rotation alone is sufficient to remove the degeneracy completely. If the degeneracy is not completely removed, then the eigenvalues of R in equation (16) will not all be distinct; any eigenvalue which is not distinct will have an associated eigenspace with dimension greater than one.

Written out in full, equation (14) is

$$\begin{aligned}
\delta\omega^2 \int_V & dV[\rho_0 S^2] \\
= \int_V & dV[\delta\kappa K + \delta\mu M + \delta\rho_0 S_i S_j^* \partial_i \partial_j \phi_0 \\
& + \rho_0 S_i S_j^* \partial_i \partial_j \delta\phi_0 + \delta\rho_0 \partial_j \phi_0 (S_i \partial_i S_j^* - S_j \partial_i S_i^*) \\
& + \rho_0 \partial_j \delta\phi_0 (S_i \partial_i S_j^* - S_j \partial_i S_i^*) + \delta\rho_0 S_i^* \partial_i \phi_1 + \delta\rho_0 S_i \partial_i \phi_1^* - \omega^2 \delta\rho_0 S^2] \\
& + 2i\omega \int_V dV[\rho_0 \varepsilon_{ijk} S_i^* \Omega_j S_k] \\
& + \int_V dV \tau_{ij}^0 [S_k^* \partial_k \partial_j S_i - S_i^* \partial_j \partial_k S_k - \sigma_{jk} \partial_k S_i^* + \sigma_{ik} \partial_j S_k^*] \\
& + \int_V dV[\rho_0 S_i S_j^* \partial_i \partial_j \psi + \rho_0 \partial_j \psi (S_i \partial_i S_j^* - S_j \partial_i S_i^*)] \\
& + \int dV[\gamma^A_{ijkl} \sigma_{ij} \sigma_{kl}^*]
\end{aligned} \tag{18}$$

where $K = |\partial_i S_i|^2$ and $M = \Delta_{ij} \Delta_{ij}^*$ where Δ_{ij} is the strain deviator

$$\Delta_{ij} = \tfrac{1}{2}(\partial_i S_j + \partial_j S_i) - \tfrac{1}{3}(\partial_k S_k)\delta_{ij}.$$

Except for the terms in $\mathbf{\Omega}$, τ_0, and γ^A_{ijkl}, equation (18) is given by Backus & Gilbert (1967). Also the contribution from the term in $\mathbf{\Omega}$ is the same as that derived by Backus & Gilbert (1961) using the perturbation theory of Hermitian operators, and by Pekeris, Alterman & Jarosch (1961) and MacDonald & Ness (1961).

If the change in the SNREI Earth model is such that the location of a discontinuity in ρ_0, κ, or μ is moved, then a term must be added to the right-hand side of equation (18) to take account of this. As in Backus & Gilbert (1967, Appendix C), this term takes the following form. Suppose that ρ_0, κ, μ have jump discontinuities at $r = b$ in the SNREI Earth model. For any function $f(r)$ which has a jump discontinuity at $r = b$, denote by $[f]^{+}_{-}$ the limit of $[f(b+\varepsilon)-f(b-\varepsilon)]$ as ε approaches zero through positive values. Suppose the SNREI Earth model is perturbed so that the location of a discontinuity in ρ_0, κ, or μ is moved from $r = b$ to $r = b+h(\theta, \phi)$. The resulting change $\delta\omega^2$ in any eigenfrequency ω is then given, correct to first order in h, by

$$\delta\omega^2 \int_V dV[\rho_0 S^2] = \int_{S_b} h\,dA\Big[\omega^2 \rho_0 S^2 - \frac{1}{4\pi G}|\nabla\phi_1|^2 - \rho_0 S_i^* \partial_i \phi_i - \rho_0 S_i \partial_i \phi_i^* - \kappa K - \mu M - \rho_0 S_i S_j^* \partial_i \partial_j \phi_0 - \rho_0 \partial_j \phi_0 (S_i \partial_i S_j^* - S_j \partial_i S_i^*)\Big]^{+}_{-} + \int_V dV[\rho_0 \partial_j \delta\phi_0 (S_i \partial_i S_j^* - S_j \partial_i S_i^*) + \rho_0 S_i S_j^* \partial_i \partial_j \delta\phi_0]. \quad (19)$$

The surface integral in equation (19) is over the surface S_b defined by $r = b$, and $\delta\phi_0$ is the change in the gravitational potential due to the fact that the discontinuity in ρ_0 is moved from $r = b$ to $r = b+h$. If the perturbation of the SNREI Earth model includes the variation of the locations of any discontinuities (including the surface ∂V), then terms as in equation (19) must be included in the definition of $\delta\mathscr{V}(\mathbf{S}_i, \mathbf{S}_j)$ and $\delta\mathscr{T}(\mathbf{S}_i, \mathbf{S}_j)$ in equation (17).

3. Results

Equations (16) and (17) define an eigenvalue problem which, when solved, allows one to compute to first order corrections to the squared eigenfrequencies of the normal modes of an arbitrary SNREI Earth model due to a slow angular rotation and small asphericities and anisotropies. The main goal is to use this formulation to correct the raw normal mode data of the real Earth. The first step in this programme will be to compute corrections to the eigenfrequencies of several SNREI Earth models which are thought to be fairly good approximations to the real Earth. In this paper the SNREI Earth models used as the starting points of the computations all have theoretical eigenfrequencies which agree well with the Earth's raw normal mode data.

By far the largest deviation from sphericity in the Earth is the ellipticity of figure. It will be assumed that, at least for low order normal modes, the angular rotation and the ellipticity of the Earth are the dominant perturbations. For fundamental modes of higher angular order, for which the displacement $\mathbf{S}$ is very small in the deep interior, this assumption is probably false. Presumably it is the perturbing effect of lateral inhomogeneities in the crust and upper mantle which will predominate when the displacement $\mathbf{S}$ is confined to a region near the Earth's surface. Various techniques have been used to estimate the maximum or the average value of the static stress deviator within the Earth (Jeffreys 1959; Kaula 1963; MacDonald 1966), and most investigators feel that stress differences rarely exceed 10^8 dyne-cm^{-2}. If the average stress difference in the upper mantle is on the order of 10^8 dyne-cm^{-2} (which is about 10^{-4} times the average mantle rigidity modulus of a typical SNREI

Earth model), then the perturbing effect of a deviatoric stress on the Earth's eigenfrequencies can probably be neglected with respect to the effect of the ellipticity. Almost nothing is known about the presence of an anisotropic stress–strain relation in the Earth, so at the present time it must be assumed that this correction is small as well. It is partly because of a lack of information about the Earth's anisotropies and deep inhomogeneities, and partly because of the relative mathematical simplicity that in this paper only the effects of the Earth's rotation and ellipticity are computed. The effect of the distinctions between continental and oceanic crustal structure could also be treated (though it will not be in this paper), but relatively little is known about the vertical extent of lateral inhomogenitiies in the mantle.

The small variations necessary to produce a rotating, elliptical Earth model starting from an arbitrary SNREI Earth model can be easily enumerated. In the rotating body, define a spherical coordinate system (r, θ, ϕ), θ being the colatitude, with centre at the centre of mass of the body and the $\hat{\mathbf{z}}$ axis along the direction $\boldsymbol{\Omega}$ of steady angular rotation. The equations defining the surfaces of constant density in a slightly elliptical Earth model may be written, to first order in the ellipticity, as

$$r[1-\tfrac{2}{3}\varepsilon(b)\, P_2{}^0(\cos\theta)] = b$$

where $P_2{}^0(\cos\theta)$ is the Legendre polynomial, and b is the mean radius of the surface under consideration. Then $\varepsilon(b)$ is the ellipticity of the surface of constant density with mean radius b. If a is the mean radius of the surface of the Earth model, then $\varepsilon(a)$ is the surface ellipticity. Consider the elliptical Earth model which results from the steady angular rotation $\boldsymbol{\Omega}$ of a SNREI Earth model. The ellipticities of the surfaces of constant density in the interior of the rotating Earth model can be shown to satisfy a certain second-order differential equation (Clairaut's equation), provided it is assumed that the rotating and elliptical Earth model is everywhere in hydrostatic equilibrium (Jeffreys 1959; Chandrasekhar & Roberts 1963). The resulting ellipticity ε (a function of radius) depends only on the value of $|\boldsymbol{\Omega}|$ and on the density ρ_0 of the initial SNREI Earth model. A clever transformation of variables in Clairaut's equation introduced by Radau and summarized by Jeffreys (1959) allows the dependence of ε upon ρ_0 to be cast in a very simple manner, provided certain approximations are valid. A careful comparison of the results of the Clairault and Radau hydrostatic theory of the Earth's figure with recent satellite measurements of the Earth's exterior gravitational potential (Jeffreys 1963) reveals that the theory is not quite valid for the Earth. The real ellipticity of the geoid ε_a is 1/298·3, while the surface ellipticity the geoid would have if the Earth were in hydrostatic equilibrium is 1/299·8. The conclusion is that there must be a small non-hydrostatic stress field (a part of the $\boldsymbol{\tau}_0$ in equation (18)) which acts partially to support the Earth's equatorial bulge. Various hypotheses have been put forward to explain the origin of this small stress field (Jeffreys 1963; Munk & MacDonald 1960; Wang 1966; McKenzie 1966). For the present purpose it is sufficient to know that its magnitude will be small, and that the error committed in using the Radau ellipticities for the computation of the shift of an eigenfrequency due to ellipticity will also be small (about [1/298·3 − 1/299·8], or about 0·0016 per cent).

Radau's approximation gives the ellipticity $\varepsilon(r)$ as a function of depth for an Earth model with density $\rho_0(r)$, mean radius a, and angular rotation $|\boldsymbol{\Omega}| = \Omega$: let $\bar{\rho}_0$ be the mean density

$$\bar{\rho}_0 = \frac{3}{a^3}\int_0^a \rho_0(x)\, x^2\, dx$$

and let

$$m = \frac{\tfrac{3}{4}\Omega^2}{\pi G \bar{\rho}_0}.$$

Let

$$z(r) = \frac{2\int_0^r \rho_0(x)\,x^4\,dx}{3r^2\int_0^r x^2\,\rho_0(x)\,dx}$$

and let

$$\eta(r) = \tfrac{25}{4}[1-\tfrac{3}{2}z(r)]^2-1.$$

Then the surface ellipticity $\varepsilon(a)$ is given by the boundary condition

$$\varepsilon(a) = \frac{5m}{2[\eta(a)+2]}$$

and $\varepsilon(r)$ is given by

$$\varepsilon(r) = \varepsilon(a)\exp\left[-\int_r^a \frac{\eta(x)}{x}\,dx\right]. \tag{20}$$

If the SNREI Earth model used for this computation has the correct moment of inertia about the axis of rotation $C/Ma^2 = 0\cdot3309$, where M is the mass of the Earth (Jeffreys 1963), then the value of $\varepsilon(a)$ which results will be the hydrostatic value $1/299\cdot8$. The ellipticity function actually used in the computations was adjusted to have the measured surface ellipticity by multiplying equation (20) by the factor $299\cdot8/298\cdot3$. In terms of $\varepsilon(r)$ for any rotating, initially SNREI Earth model, the small variations $\delta\rho_0$ and $\delta\phi_0$ in equation (18) can be written (Chandrasekhar & Roberts 1963)

$$\left.\begin{aligned}\delta\rho_0(r,\theta) &= [\tfrac{2}{3}\varepsilon(r)\,r\rho_0'(r)]\,P_2^{\,0}(\cos\theta)\\ \delta\phi_0(r,\theta) &= [\tfrac{2}{3}\varepsilon(r)\,r\phi_0'(r)-\tfrac{1}{3}\Omega^2 r^2]\,P_2^{\,0}(\cos\theta),\end{aligned}\right\} \tag{21}$$

where $\rho_0'(r)$ denotes $\dfrac{d\rho_0(r)}{dr}$, etc.

The expression (21) for $\delta\rho_0$ is not valid wherever the derivative $\rho_0'(r)$ does not exist, in particular at any jump discontinuity in $\rho_0(r)$. Since $\phi_0(r)$ is continuous and differentiable for any SNREI Earth model, the expression (21) for $\delta\phi_0$ is valid everywhere. The $\delta\phi_0$ in equation (21) is only that due to $\delta\rho_0$ and does not include ψ, the gravitational potential. If it is further assumed that the elastic moduli are constant on the same elliptical surfaces as the density and gravitational potential (this is the assumption made in correcting body wave travel time data (Bullen 1963)) then one can also write

$$\left.\begin{aligned}\delta\kappa(r,\theta) &= [\tfrac{2}{3}\varepsilon(r)\,r\kappa'(r)]\,P_2^{\,0}(\cos\theta)\\ \delta\mu(r,\theta) &= [\tfrac{2}{3}\varepsilon(r)\,r\mu'(r)]\,P_2^{\,0}(\cos\theta).\end{aligned}\right\} \tag{22}$$

If the SNREI Earth model under consideration has a surface of discontinuity in κ, μ, or ρ_0 at $r = b$, then the ellipticity is such that this surface of discontinuity is moved to $r = b+h(\theta)$ where

$$h(\theta) = -\tfrac{2}{3}b\varepsilon(b)\,P_2^{\,0}(\cos\theta).$$

The effect of moving the discontinuities on the perturbation computation is given in equation (19), except that in this case the volume integral in equation (19) is not used, as the $\delta\phi_0$ in equation (21) includes the effect of slightly varying the position of any discontinuities in density.

The small variations in ρ_0, ϕ_0, κ, μ and in the locations of discontinuities necessary to produce from an arbitrary SNREI Earth model a rotating, elliptical Earth model are enumerated above, and Rayleigh's principle can be used to find to zeroeth order the eigenfunctions and to first order the eigenfrequencies of the more complicated system. To do this it is necessary to select an orthonormal basis of every $2l+1$-dimensional degenerate eigenspace ${}_n\mathscr{S}_l^S$ or ${}_n\mathscr{S}_l^T$ associated with, respectively, ${}_n\omega_l^S$ or ${}_n\omega_l^T$; it turns out that the natural choice of basis is very convenient for the problem of a rotating, elliptical Earth. The natural basis of a subspace ${}_n\mathscr{S}_l^S$ or ${}_n\mathscr{S}_l^T$ consists of the $2l+1$ vectors ${}_n\mathbf{S}_l^m$ defined so that each is characterized by only one spherical harmonic $Y_l^m(\theta, \phi)$, defined with the $\hat{\mathbf{z}}$ axis along the direction of the angular rotation vector $\boldsymbol{\Omega}$ (this is also the axis of greatest inertia). This is the basis that arises naturally in determining the eigenfunctions of a SNREI Earth model. The basis vectors of a poloidal eigenspace ${}_n\mathscr{S}_l^S$ are (Backus 1967)

$$\{{}_n\mathbf{S}_l^m = \hat{\mathbf{r}}\, {}_nU_l(r)\, Y_l^m(\theta, \phi) + \nabla_1\, [{}_nV_l(r)\, Y_l^m(\theta, \phi)], \quad m = -l, \ldots, 0, \ldots l\}$$

and the basis vectors of a toroidal eigenspace ${}_n\mathscr{S}_l^T$ are

$$\{{}_n\mathbf{S}_l^m = -\hat{\mathbf{r}} \times \nabla_1\, [{}_nW_l(r)\, Y_l^m(\theta, \phi)], \quad m = -l, \ldots, 0, \ldots l\}$$

where

$$\nabla_1 = \hat{\boldsymbol{\theta}} \frac{\partial}{\partial \theta} + \frac{1}{\sin\theta} \hat{\boldsymbol{\Phi}} \frac{\partial}{\partial \phi}.$$

These basis vectors ${}_n\mathbf{S}_l^m$ may also be written in terms of vector spherical harmonics (Morse & Feshbach 1953) by defining $\mathbf{P}_l^m = \hat{\mathbf{r}}\, Y_l^m$, $\mathbf{B}_l^m = \nabla_1\, Y_l^m$, $\mathbf{C}_l^m = -\hat{\mathbf{r}} \times \nabla_1\, Y_l^m$.

With this choice of basis, and with $\delta\rho_0$, $\delta\phi_0$, $\delta\kappa$, $\delta\mu$ given by equations (21) and (22) and ψ given by equation (3), the matrix R_{ij} in equation (17) can be shown to be diagonal. The eigenvectors of R_{ij} are thus the $2l+1$ Cartesian unit vectors $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$ $(0, 0, \ldots, 0, 1)$. For this reason it is possible to compute the correction due to the rotational potential ψ separately from the correction due to the ellipticity. The correction due to ψ will be treated as a part of the second-order rotational correction and will be discussed later. Another consequence of the fact that R_{ij} is diagonal is that, to zeroeth order in ε_a and in $(\Omega/{}_n\omega_l^S)$ or $(\Omega/{}_n\omega_l^T)$, the eigenfunctions of an elliptical rotating Earth model are characterized by a single spherical harmonic Y_l^m. The eigenfunction ${}_n\mathbf{S}_l^m$ associated with a poloidal mode ${}_nS_l^m$ is of the form ${}_n\mathbf{S}_l^m = \hat{\mathbf{r}}\, {}_nU_l(r)\, Y_l^m + \nabla_1\, [{}_nV_l(r)\, Y_l^m]$, and the eigenfunction associated with a toroidal mode ${}_nT_l^m$ is of the form ${}_n\mathbf{S}_l^m = -\hat{\mathbf{r}} \times \nabla_1\, [{}_nW_l(r)\, Y_l^m]$. The new eigenfrequency of any normal mode ${}_nS_l^m$ or ${}_nT_l^m$ can be written, correct to first order in ε_a and in $(\Omega/{}_n\omega_l^S)$ or $(\Omega/{}_n\omega_l^T)$, in the form ${}_n\omega_l^m = {}_n\omega_l + {}_n(\delta\omega)_l^m$, where ${}_n\omega_l$ is used to denote either ${}_n\omega_l^S$ or ${}_n\omega_l^T$ and where

$${}_n(\delta\omega)_l^m = \frac{{}_n(\delta\omega^2)_l^m}{2\, {}_n\omega_l}$$

is calculated from Rayleigh's principle (16) in terms of $\boldsymbol{\Omega}$, $\delta\rho_0$, $\delta\phi_0$, $\delta\kappa$, $\delta\mu$, and the eigenfunctions ${}_n\mathbf{S}_l^m$ of the unperturbed SNREI Earth model.

$${}_n(\delta\omega)_l^m = \frac{1}{2\, {}_n\omega_l} [\delta\mathscr{V}({}_n\mathbf{S}_l^m, {}_n\mathbf{S}_l^m) - {}_n\omega_l^2\, \delta\mathscr{T}({}_n\mathbf{S}_l^m, {}_n\mathbf{S}_l^m)] + i\Omega({}_n\mathbf{S}_l^m, \hat{\mathbf{z}} \times {}_n\mathbf{S}_l^m). \tag{23}$$

In Appendix A, it is shown how to evaluate the volume integrals on the right-hand side of equation (23) in terms of the scalar functions ${}_nU_l(r)$, ${}_nV_l(r)$, and ${}_nW_l(r)$ which characterize the eigenfunctions ${}_n\mathbf{S}_l^m$. It is found that the combined action of rotation and ellipticity completely removes the degeneracy. The term $i\Omega({}_n\mathbf{S}_l^m, \hat{\mathbf{z}} \times {}_n\mathbf{S}_l^m)$ on the right-hand side of equation (23) represents the first-order effect of rotation alone, while the other term represents the first-order effect of the ellipticity. It will be recalled that the first-order effect of rotation alone is to remove the degeneracy completely by a symmetric splitting of Zeeman type, as in expression (1). The ellipticity causes the splitting to be asymmetric; for an Earth model which is ellipsoidal as well as rotating, the spacing between the eigenfrequencies of a multiplet ${}_nS_l$ or ${}_nT_l$ will not be uniform. In fact, it is shown in Appendix A that the eigenfrequency ${}_n\omega_l^m$ of a normal mode ${}_nS_l^m$ or ${}_nT_l^m$ of a rotating ellipsoidal Earth can be written, correct to first order in ε_a and $(\Omega/{}_n\omega_l^S)$ or $(\Omega/{}_n\omega_l^T)$ as

$$ {}_n\omega_l^m/{}_n\omega_l = 1+{}_n\alpha_l^e\,\varepsilon_a+m\,{}_n\beta_l^r\left(\frac{\Omega}{{}_n\omega_l}\right)+m^2\,{}_n\gamma_l^e\,\varepsilon_a. \tag{24} $$

The superscript r indicates a parameter depending only on the rotation $\mathbf{\Omega}$, while the superscript e indicates a parameter depending on the ellipticity. The effect of rotation alone is of the form

$$ {}_n(\delta\omega)_l^m/{}_n\omega_l = m\,{}_n\beta_l^r\left(\frac{\Omega}{{}_n\omega_l}\right) $$

while the effect of ellipticity alone is of the form

$$ {}_n(\delta\omega)_l^m/{}_n\omega_l = ({}_n\alpha_l^e+m^2\,{}_n\gamma_l^e)\,\varepsilon_a. $$

Usami & Satô (1962) pointed out that ellipticity acts to split a $2l+1$-degenerate toroidal eigenspace ${}_nT_l$ of a homogeneous, non-gravitating, Earth model into $l+1$ lines; this is in agreement with the above result.

For the Earth it is found that for a few low-order normal modes, the second-order perturbing effect of the Earth's rotation is as important as the first-order perturbing effect of the ellipticity. One effect on an eigenfrequency ${}_n\omega_l+{}_n(\delta\omega)_l^m$ which is of order $(\Omega/{}_n\omega_l^S)^2$ or $(\Omega/{}_n\omega_l^T)^2$ is the effect of the rotational potential ψ; another term of order $(\Omega/{}_n\omega_l^S)^2$ or $(\Omega/{}_n\omega_l^T)^2$ arises from the second-order rotational perturbation theory of Backus & Gilbert (1961). This theory is valid even in the case when ellipticity as well as rotation is considered, since in either case the degeneracy is completely removed to first order. The net effect of all the second-order rotational theory is to impart to all eigenfrequencies ${}_n\omega_l+{}_n(\delta\omega)_l^m$ a small additional shift which will be denoted by ${}_n(\delta\omega_2)_l^m$. As stated above, the effect of the rotational potential ψ is conveniently treated as a $\delta\phi_0$ term in Rayleigh's principle, equation (18). The small shift ${}_n(\delta\omega_2{}^P)_l^m$ in any eigenfrequency ${}_n\omega_l+{}_n(\delta\omega)_l^m$ due to the effect of a rotational potential is expressed in terms of the scalars ${}_nU_l(r)$, ${}_nV_l(r)$ in Appendix B. The additional small shift ${}_n(\delta\omega_2{}^r)_l^m$ arising from the second-order perturbation theory of Hermitian operators applied to the Coriolis force term is given by Backus & Gilbert (1961).

$$ {}_n(\delta\omega_2{}^r)_l^m = {}_n(\sigma_2{}^r)_l^m\quad(\Omega/{}_n\omega_l)^2 \tag{25} $$

$$ [{}_n(\sigma_1{}^r)_l^m]^2-2{}_n(\sigma_2{}^r)_l^m = -2i({}_n\mathbf{S}_l^m,\hat{\mathbf{z}}\times{}_n(\mathbf{S}_1{}^r)_l^m) \tag{26} $$

where ${}_n(\sigma_1{}^r)_l^m = m({}_n\beta_l^r)$ and where ${}_n(\mathbf{S}_1{}^r)_l^m$ is the first-order rotational correction to the eigenfunction. This correction to the eigenfunction is the unique particular solution of the non-homogeneous equation

$$ (-\rho_0\,{}_n\omega_l^2\,I+H_0)\ {}_n(\mathbf{S}_1{}^r)_l^m = \rho_0\,{}_n\omega_l^2\ [2{}_n(\sigma_1{}^r)_l^m\,{}_n\mathbf{S}_l^m-2i\hat{\mathbf{z}}\times{}_n\mathbf{S}_l^m] \tag{27} $$

such that $({}_n\mathbf{S}_l^m,{}_n(\mathbf{S}_1{}^r)_l^m) = 0$, where I is the identity operator and where H_0 is that Hermitian linear operator such that the equation $H_0\,\mathbf{S} = \rho_0\,\omega^2\,\mathbf{S}$ defines the normal mode eigenvalue problem in the unperturbed SNREI Earth model. In order to compute ${}_n(\delta\omega_2{}^r)_l^m$, equation (27) must first be solved for ${}_n(\mathbf{S}_1{}^r)_l^m$ and then the volume integral in equation (26) evaluated. The details are straight-forward and will not be given. Note that Backus & Gilbert (1961) have two extra terms in their expression for ${}_n(\sigma_2{}^r)_l^m$. One of these terms is identically zero and the other arises because they have used a Lagrangian formulation of the equations of motion.

The net result of all the second-order rotational theory is that the additional frequency shift ${}_n(\delta\omega_2)_l^m = {}_n(\delta\omega_2{}^p)_l^m + {}_n(\delta\omega_2{}^r)_l^m$ is of the form

$$ {}_n(\delta\omega_2)_l^m = ({}_n\alpha_l^r + m^2\,{}_n\gamma_l^r)\quad(\Omega/{}_n\omega_l)^2. \tag{28}$$

Thus correct to first order in ε_a and to second order in $(\Omega/{}_n\omega_l^S)$ or $(\Omega/{}_n\omega_l^T)$, a normal mode ${}_nS_l^m$ or ${}_nT_l^m$ of a rotating, elliptical Earth model has an angular frequency of oscillation given by

$$ {}_n\omega_l^m/{}_n\omega_l = 1 + {}_n\alpha_l + m\,{}_n\beta_l + m^2\,{}_n\gamma_l \tag{29}$$

where

$$ {}_n\alpha_l = {}_n\alpha_l^r\,(\Omega/{}_n\omega_l)^2 + {}_n\alpha_l^e\,\varepsilon_a$$

$$ {}_n\beta_l = {}_n\beta_l^r\,(\Omega/{}_n\omega_l)$$

$$ {}_n\gamma_l = {}_n\gamma_l^r\,(\Omega/{}_n\omega_l)^2 + {}_n\gamma_l^e\,\varepsilon_a.$$

The second-order effect of the rotation contributes to the asymmetry in exactly the same manner as does the ellipticity. Tables 1–3 give values of the splitting parameters ${}_n\alpha_l$, ${}_n\beta_l$ and ${}_n\gamma_l$ for several low-order poloidal and toroidal modes for three SNREI Earth models. The models used are model 1 of Backus & Gilbert (1967), and models G1 and Q1 of Gilbert & Backus (1968). All three of these models fit the observed raw fundamental normal mode data in the period range 1·4 to 53·9 min with an r.m.s. relative error of less than 0·3 per cent. All three of these are continental models, and none has a solid inner core; all have the correct mass M and moment of inertia $C = Ma^2$. Several other SNREI Earth models which also fit the raw normal mode data have been found (Anderson 1965; Landisman, Satô & Nafe 1965; Pekeris 1966; Bullen & Haddon 1967; Press 1968). The splitting parameters ${}_n\alpha_l$ and ${}_n\gamma_l$ are of course dependent upon the properties of the SNREI Earth model used. For example, for the three particular SNREI Earth models used in this paper, the splitting parameters listed in the tables differ in some instances by as much as 20 per cent. Fig. 1 gives a schematic representation of the computed asymmetrical

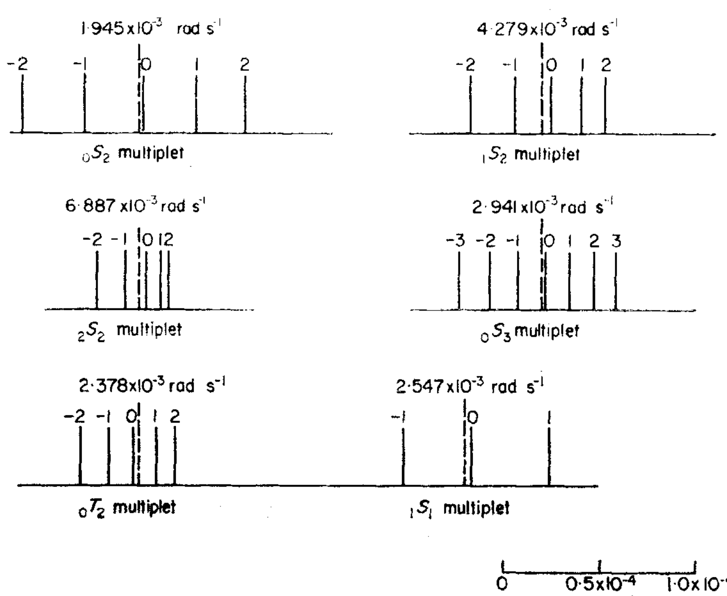

FIG. 1. Schematic indication of line spacing for six low-order multiplets. The splitting parameters are those of model 1. The dashed line in each case represents the angular frequency ${}_n\omega_l{}^S$ or ${}_n\omega_l{}^T$ of the degenerate multiplet before perturbation. Note the relatively close spacing between ${}_0\omega_2{}^T$ and ${}_1\omega_1{}^S$.

line spacing in the spectra of several low-order multiplets for model 1. In the spectrum of the real Earth, all lines appear as broadened peaks because of the dissipation (and because of the finite record length). However, for all modes shown in Fig. 1, the splitting is sufficiently greater than the broadening that it should be possible to completely resolve all the peaks. All but ${}_0T_2$ and ${}_1S_1$ have been partially resolved by Slichter (1967).

A few low-order modes have been partially resolved by several investigators, but discrepancies between individual measurements are about as great as computed asymmetries due to ellipticity and rotation. At the present time, contamination of the data by various types of noise seems to limit severely the quality of the measurements. Fig. 2 shows the results of one of the analyses of data for the ${}_0S_2$ quintet (spectrum from Smith (1961)). The record used was collected after the Chilean earthquake of 1960 on a Benioff quartz strain gauge located at Isabella, California. Smith took two relatively long portions of the record (about 15 days and 30 days), multiplied by a proper fading function and computed the Fourier transforms of these two portions. The shorter record was virtually free of large aftershocks. Fig. 2 is the result. The three highest peaks can with reasonable certainty be identified as, from left to right, $m = -1$, $m = 0$, and $m = 1$. Note that a measurement between the $m = \pm 1$ peaks allows one to determine $2_n\beta_l$, and a measurement of the spacing between $m = 0$ and $m = 1$ or $m = 0$ and $m = -1$ then allows one to determine ${}_n\gamma_l$. The parameter ${}_n\alpha_l$ cannot be measured from the raw data as it represents the amount by which the entire multiplet is shifted; only ${}_n\omega_l\ (1 + {}_n\alpha_l)$ can be measured. If the position of the central $m = 0$ line is taken to be defined by the solid vertical line in Fig. 2, then the computed theoretical positions (using model 1) of the other four lines are indicated by the dashed vertical lines. The agreement between

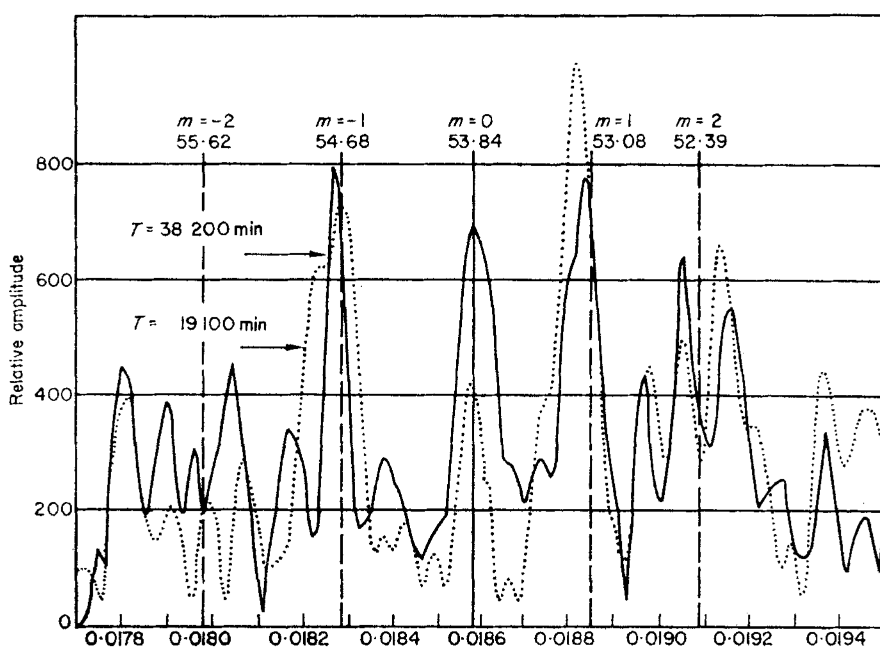

FIG. 2. Fine resolution of poloidal quintet ${}_0S_2$. Recorded after Chilean earthquake of 1960 on a quartz strain gauge located at Isabella, California. Fourier analysis of two lengths of record, 19 100 min and 38 200 min, sample interval 2 minutes (data from Smith (1961)). The dashed vertical lines represent the computed positions (periods designated in minutes) of the $m = \pm 1$ and $m = \pm 2$ peaks for model 1 (see text).

Table 1

Elliptical and rotational splitting parameters for SNREI *Earth model* 1. *The angular frequency $_n\omega_l$ is that of the degenerate multiplet before the application of perturbation theory*

Mode	Angular frequency (rad s^{-1}) $_n\omega_l$	Elliptical splitting parameters 1000 × $_n\alpha_l^e\,\varepsilon_a$	1000 × $_n\gamma_l^e\,\varepsilon_a$	Second order rotational splitting parameters 1000 × $_n\alpha_l^r\left(\frac{\Omega}{_n\omega_l}\right)^2$	1000 × $_n\gamma_l^r\left(\frac{\Omega}{_n\omega_l}\right)^2$	Total splitting parameters 1000 × $_n\beta_l$	1000 × $_n\chi_l$	1000 × $_n\gamma_l$
$_0T_2$	2·3779E–03	1·998	−0·999	−3·037	0·697	5·111	−1·039	−0·302
$_0T_3$	3·6762E–03	1·407	−0·352	−0·840	0·072	1·653	0·567	−0·280
$_0T_4$	4·8037E–03	1·181	−0·177	−0·329	0·015	0·759	0·852	−0·162
$_0T_5$	5·8266E–03	1·063	−0·106	−0·142	0·004	0·417	0·921	−0·102
$_0T_6$	6·7752E–03	0·990	−0·071	−0·059	0·001	0·256	0·931	−0·070
$_0T_7$	7·6698E–03	0·939	−0·050	−0·001	0·000	0·170	0·938	−0·050
$_0T_8$	8·5240E–03	0·903	−0·038	0·074	−0·001	0·119	0·977	−0·039
$_0T_9$	9·3474E–03	0·877	−0·029	0·209	−0·003	0·087	1·086	−0·032
$_0T_{10}$	1·0146E–02	0·857	−0·023	0·494	−0·005	0·065	1·351	−0·028
$_0S_0$	5·1179E–03	0		0·335			0·335	
$_1S_0$	1·0404E–02	0		0·091			0·091	
$_2S_0$	1·5883E–02	0		0·040			0·040	
$_3S_0$	2·0729E–02	0		0·026			0·026	
$_1S_1$	2·5474E–03	−1·191	1·787	2·559	−0·769	14·946	1·368	1·018
$_0S_2$	1·9454E–03	0·753	−0·377	0·266	−0·187	14·929	1·019	−0·564

${}_1S_2$	4·2791E–03	0·689	−0·344	0·532	−0·100	4·098	1·221	−0·444
${}_2S_2$	6·8867E–03	0·490	−0·245	0·040	−0·003	1·348	0·530	−0·248
${}_0S_3$	2·9413E–03	0·496	−0·124	0·309	−0·067	4·619	0·805	−0·191
${}_2S_3$	7·8516E–03	0·860	−0·215	0·006	0·000	0·708	0·866	−0·215
${}_0S_4$	4·0614E–03	0·569	−0·085	0·255	−0·026	1·832	0·824	−0·111
${}_1S_4$	7·3660E–03	0·856	−0·128	0·047	−0·010	1·941	0·903	−0·138
${}_2S_4$	8·6994E–03	1·004	−0·151	0·009	0·000	0·292	1·013	−0·151
${}_0S_5$	5·2747E–03	0·649	−0·065	0·084	−0·007	0·842	0·733	−0·072
${}_1S_5$	8·6068E–03	0·933	−0·093	−0·006	−0·005	1·425	0·927	−0·098
${}_2S_5$	9·5520E–03	0·998	−0·100	0·022	−0·001	0·167	1·020	−0·101
${}_0S_6$	6·5150E–03	0·706	−0·050	−0·033	−0·001	0·410	0·673	−0·051
${}_1S_6$	9·5649E–03	1·086	−0·078	−0·014	−0·002	0·872	1·072	−0·080
${}_2S_6$	1·0592E–02	0·903	−0·065	−0·000	−0·001	0·335	0·903	−0·066
${}_0S_7$	7·7276E–03	0·731	−0·039	−0·120	0·001	0·186	0·611	−0·038
${}_1S_7$	1·0407E–02	1·222	−0·065	−0·013	−0·001	0·560	1·209	−0·066
${}_2S_7$	1·1749E–02	0·852	−0·046	−0·038	−0·000	0·407	0·814	−0·046
${}_0S_8$	8·8646E–03	0·723	−0·030	−0·247	0·003	0·059	0·476	−0·027
${}_1S_8$	1·1314E–02	1·364	−0·057	−0·019	−0·001	0·419	1·345	−0·058
${}_2S_8$	1·2911E–02	0·835	−0·035	−0·078	0·000	0·386	0·757	−0·035
${}_0S_9$	9·8950E–03	0·696	−0·023	−0·520	0·005	−0·009	0·176	−0·018
${}_1S_9$	1·2351E–02	1·523	−0·051	−0·027	0·000	0·339	1·496	−0·051
${}_0S_{10}$	1·0822E–02	0·671	−0·018	−1·297	0·011	−0·037	−0·626	−0·007
${}_1S_{10}$	1·3515E–02	1·683	−0·046	−0·031	0·000	0·278	1·652	−0·046

Table 2

Elliptical and rotational splitting parameters for SNREI *Earth model G1. The angular frequency $_n\omega_l$ is that of the degenerate multiplet before the application of perturbation theory*

Mode	Angular frequency (rad s^{-1}) $_n\omega_l$	Elliptical splitting parameters 1000 × $_n\alpha_l^e\,\varepsilon_a$	1000 × $_n\gamma_l^e\,\varepsilon_a$	Second order rotational splitting parameters 1000 × $_n\alpha_l^r\left(\frac{\Omega}{_n\omega_l}\right)^2$	1000 × $_n\gamma_l^r\left(\frac{\Omega}{_n\omega_l}\right)^2$	Total splitting parameters 1000 × $_n\beta_l$	1000 × $_n\alpha_l$	1000 × $_n\gamma_l$
$_0T_2$	2·3780E–03	2·045	−1·023	−3·037	0·697	5·111	−0·992	−0·326
$_0T_3$	3·6764E–03	1·400	−0·350	−0·840	0·072	1·653	0·560	−0·278
$_0T_4$	4·8040E–03	1·144	−0·172	−0·329	0·015	0·759	0·815	−0·157
$_0T_5$	5·8270E–03	1·001	−0·100	−0·142	0·004	0·417	0·859	−0·096
$_0T_6$	6·7758E–03	0·904	−0·065	−0·060	0·001	0·256	0·844	−0·064
$_0T_7$	7·6706E–03	0·831	−0·045	−0·001	0·000	0·170	0·830	−0·045
$_0T_8$	8·5251E–03	0·773	−0·032	0·074	−0·001	0·119	0·847	−0·033
$_0T_9$	9·3487E–03	0·724	−0·024	0·208	−0·003	0·087	0·932	−0·027
$_0T_{10}$	1·0148E–02	0·681	−0·019	0·490	−0·005	0·065	1·171	−0·024
$_0S_0$	5·1178E–03	0		0·336			0·336	
$_1S_0$	1·0404E–02	0		0·091			0·091	
$_2S_0$	1·5885E–02	0		0·040			0·040	
$_3S_0$	2·0735E–02	0		0·026			0·026	
$_1S_1$	2·5476E–03	−1·213	1·820	2·559	−0·769	14·946	1·346	1·051
$_0S_2$	1·9454E–03	0·740	−0·370	0·268	−0·187	14·931	1·008	−0·557

${}_1S_2$	4·2792E–03	0·680	−0·340	0·532	−0·100	4·099	1·212	−0·440
${}_2S_2$	6·8873E–03	0·474	−0·237	0·040	−0·003	1·349	0·514	−0·240
${}_0S_3$	2·9414E–03	0·466	−0·116	0·310	−0·067	4·620	0·776	−0·183
${}_2S_3$	7·8526E–03	0·831	−0·208	0·005	0·000	0·708	0·836	−0·208
${}_0S_4$	4·0615E–03	0·531	−0·080	0·256	−0·026	1·834	0·787	−0·106
${}_1S_4$	7·3658E–03	0·828	−0·124	0·047	−0·010	1·941	0·875	−0·134
${}_2S_4$	8·7006E–03	0·953	−0·143	0·008	0·000	0·292	0·961	−0·143
${}_0S_5$	5·2749E–03	0·604	−0·060	0·085	−0·007	0·843	0·689	−0·067
${}_1S_5$	8·6063E–03	0·920	−0·092	−0·006	−0·005	1·424	0·914	−0·097
${}_2S_5$	9·5535E–03	0·910	−0·091	0·021	−0·001	0·168	0·931	−0·092
${}_0S_6$	6·5153E–03	0·652	−0·047	−0·032	−0·001	0·411	0·620	−0·048
${}_1S_6$	9·5644E–03	1·081	−0·077	−0·014	−0·002	0·872	1·067	−0·079
${}_2S_6$	1·0593E–02	0·789	−0·056	−0·001	−0·001	0·335	0·788	−0·057
${}_0S_7$	7·7279E–03	0·665	−0·036	−0·120	0·001	0·187	0·545	−0·035
${}_1S_7$	1·0407E–02	1·212	−0·065	−0·013	−0·001	0·560	1·199	−0·066
${}_2S_7$	1·1750E–02	0·731	−0·039	−0·039	0·000	0·407	0·692	−0·039
${}_0S_8$	8·8648E–03	0·640	−0·027	−0·246	0·003	0·059	0·394	−0·024
${}_1S_8$	1·1314E–02	1·353	−0·056	−0·019	−0·001	0·419	1·334	−0·057
${}_2S_8$	1·2912E–02	0·711	−0·030	−0·082	0·000	0·387	0·629	−0·030
${}_0S_9$	9·8949E–03	0·593	−0·020	−0·516	0·005	−0·008	0·077	−0·015
${}_1S_9$	1·2352E–02	1·516	−0·051	−0·027	−0·001	0·338	1·489	−0·052
${}_0S_{10}$	1·0822E–02	0·550	−0·015	−1·272	0·011	−0·036	−0·722	−0·004
${}_1S_{10}$	1·3516E–02	1·680	−0·046	−0·031	0·000	0·278	1·649	−0·046

Table 3

Elliptical and rotational splitting parameters for SNREI *Earth model Q*1. *The angular frequency ${}_n\omega_l$ is that of the degenerate multiplet before the application of perturbation theory*

Mode	Angular frequency (rad s^{-1}) ${}_n\omega_l$	Elliptical splitting parameters 1000 × ${}_n\alpha_l^e\,\varepsilon_a$	1000 × ${}_n\gamma_l^e\,\varepsilon_a$	Second order rotational splitting parameters 1000 × ${}_n\alpha_l^r\left(\frac{\Omega}{{}_n\omega_l}\right)^2$	1000 × ${}_n\gamma_l^r\left(\frac{\Omega}{{}_n\omega_l}\right)^2$	Total splitting parameters 1000 × ${}_n\beta_l$	1000 × ${}_n\alpha_l$	1000 × ${}_n\gamma_l$
${}_0T_2$	2·3775E–03	2·098	−1·049	−3·056	0·702	5·112	−0·958	−0·347
${}_0T_3$	3·6759E–03	1·390	−0·348	−0·847	0·073	1·653	0·543	−0·275
${}_0T_4$	4·8036E–03	1·100	−0·165	−0·329	0·015	0·759	0·771	−0·150
${}_0T_5$	5·8267E–03	0·928	−0·093	−0·142	0·004	0·417	0·786	−0·089
${}_0T_6$	6·7755E–03	0·805	−0·057	−0·095	0·001	0·256	0·746	−0·056
${}_0T_7$	7·6700E–03	0·705	−0·038	0·000	0·000	0·170	0·705	−0·038
${}_0T_8$	8·5241E–03	0·621	−0·026	0·075	−0·001	0·119	0·696	−0·027
${}_0T_9$	9·3473E–03	0·546	−0·018	0·211	−0·003	0·087	0·757	−0·021
${}_0T_{10}$	1·0146E–02	0·478	−0·013	0·498	−0·005	0·065	0·976	−0·018
${}_0S_0$	5·1179E–03	0		0·338			0·338	
${}_1S_0$	1·0404E–02	0		0·092			0·092	
${}_2S_0$	1·5885E–02	0		0·040			0·040	
${}_3S_0$	2·0735E–02	0		0·026			0·026	
${}_1S_1$	2·5489E–03	−1·270	1·906	2·580	−0·772	14·686	1·310	1·134
${}_0S_2$	1·9434E–03	0·655	−0·328	0·271	−0·185	14·888	0·926	−0·513

${}_1S_2$	4·2720E–03	0·660	−0·330	0·544	−0·101	4·038	1·204	−0·431
${}_2S_2$	6·8904E–03	0·406	−0·203	0·034	−0·002	1·341	0·440	−0·205
${}_0S_3$	2·9400E–03	0·405	−0·101	0·312	−0·066	4·580	0·717	−0·167
${}_2S_3$	7·8468E–03	0·749	−0·187	−0·002	0·001	0·685	0·747	−0·186
${}_0S_4$	4·0614E–03	0·472	−0·071	0·255	−0·026	1·805	0·727	−0·097
${}_1S_4$	7·3605E–03	0·782	−0·117	0·048	−0·010	1·932	0·830	−0·127
${}_2S_4$	8·6841E–03	0·849	−0·127	0·001	0·001	0·270	0·850	−0·126
${}_0S_5$	5·2751E–03	0·539	−0·054	0·084	−0·007	0·823	0·623	−0·061
${}_1S_5$	8·6047E–03	0·879	−0·088	−0·006	−0·004	1·420	0·873	−0·092
${}_2S_5$	9·5303E–03	0·775	−0·078	0·018	0·000	0·149	0·793	−0·078
${}_0S_6$	6·5151E–03	0·578	−0·041	−0·033	−0·001	0·395	0·545	−0·042
${}_1S_6$	9·5632E–03	1·033	−0·074	−0·014	−0·002	0·860	1·019	−0·076
${}_2S_6$	1·0571E–02	0·638	−0·046	−0·008	0·000	0·329	0·630	−0·046
${}_0S_7$	7·7274E–03	0·577	−0·031	−0·120	0·002	0·176	0·457	−0·029
${}_1S_7$	1·0406E–02	1·145	−0·061	−0·013	−0·001	0·548	1·132	−0·062
${}_2S_7$	1·1732E–02	0·579	−0·031	−0·056	0·000	0·404	0·523	−0·031
${}_0S_8$	8·8645E–03	0·536	−0·022	−0·248	0·003	0·052	0·288	−0·019
${}_1S_8$	1·1314E–02	1·271	−0·053	−0·019	−0·001	0·408	1·252	−0·054
${}_2S_8$	1·2896E–02	0·557	−0·023	−0·123	0·001	0·384	0·434	−0·022
${}_0S_9$	9·8959E–03	0·472	−0·016	−0·524	0·006	−0·013	−0·052	−0·010
${}_1S_9$	1·2351E–02	1·420	−0·047	−0·027	0·000	0·329	1·393	−0·047
${}_0S_{10}$	1·0824E–02	0·413	−0·011	−1·336	0·011	−0·039	−0·923	0·001
${}_1S_{10}$	1·3516E–02	1·569	−0·043	−0·030	0·000	0·270	1·539	−0·043

the theory for model 1 and the data is certainly well within probable experimental error. The fact that the data is rather severely contaminated by noise may be seen from the fact that the three peaks do not have the symmetrical shapes of typical resonance peaks. The quintet ${}_0S_2$ has also been partially resolved by Slichter (1965, 1967), using data collected on Lacoste gravimeters after the Alaskan earthquake of 1964. Slichter (1967) lists the positions of the centres of the peaks $m = -2$, $m = -1$ and $m = 1$ as recorded by two different gravimeters, both located at UCLA. If the means of the values indicated by the two records are taken, it appears from this data that there is no measurable asymmetry in the splitting of the ${}_0S_2$ multiplet. However, the separate data from the two gravimeters do not agree to better than 0·15 per cent, an amount which is about the same as the theoretical asymmetry. The gravimeter data is apparently contaminated by some type of noise.

The parameter ${}_n\alpha_l$ represents the relative amount by which the position of the eigenfrequency for $m = 0$ is shifted. If a multiplet can be resolved and the position of the $m = 0$ peak measured, then ${}_n\alpha_l$ represents the amount by which this measurement must be corrected before comparison with the theoretical data of SNREI Earth models. It is seen that the corrections to the raw normal mode data due to the Earth's rotation and ellipticity are fairly small, on the order of 0·1 per cent. The values of ${}_n\alpha_l$ in Tables 1–3 may be used as a first step in correcting the raw normal mode data. The SNREI Earth models used to compile Tables 1–3 all fit the raw normal mode data. If it is desired to make a better correction for ellipticity and rotation, it will first be necessary to construct SNREI Earth models which fit the partially corrected raw data, and then to compute new splitting parameters for these SNREI Earth models which fit the corrected (for ellipticity and rotation) raw Earth data. Model 5821 of Backus & Gilbert (1968) was obtained by inverting corrected data. The parameters of Table 1 were used.

The eigenfrequencies of all radial modes ${}_nS_0$ are unaffected to first order by ellipticity and rotation. For fundamental modes ${}_0S_l$ and ${}_0T_l$ of higher angular order, the splitting due to ellipticity will completely dominate that due to rotation, since $(\Omega/{}_0\omega_l^S)$ and $(\Omega/{}_0\omega_l^T)$ decrease as l increases. In this case the splitting looks nothing at all like Zeeman splitting, but instead the $m = 0$ member lies at one end of the multiplet while the $m = \pm l$ members lie very close together and at the other end. Even for extremely high order fundamental modes, there will be a highly asymmetrical splitting of this type due to ellipticity. It has been pointed out by Backus (personal communication) that in the limit of very large l and for $n = 0$ (fundamental modes) the total spacing between the $m = 0$ line and the $m = \pm l$ lines is related to the difference in travel times of surface waves over equatorial and polar paths. The total spacing should in fact be $\Delta\omega_l/{}_0\omega_l = \frac{1}{2}\varepsilon_a = 0{\cdot}17$ per cent, where $\Delta\omega_l = {}_0\omega_l - ({}_{\pm l}\omega_l)$, and ε_a is the surface ellipticity. It is shown in Appendix B that the parameters ${}_n\alpha_l^e$ and ${}_n\gamma_l^e$ describing the ellipticity effect are related by

$$ {}_n\alpha_l^e = -\tfrac{1}{3}l(l+1)\,{}_n\gamma_l^e. $$

Thus, since the total spacing for very large l is very nearly

$$ \Delta\omega_l/{}_0\omega_l = l^2\,|{}_0\gamma_l^e|\,\varepsilon_a, $$

the splitting parameter ${}_0\alpha_l^e$ for very large l will be

$$ {}_0\alpha_l^e\,\varepsilon_a = \frac{\varepsilon_a}{6}\frac{l(l+1)}{l^2} \approx 0{\cdot}55.10^{-3}. $$

One consequence of the fact that high-order fundamental normal modes are split by about 0·2 per cent by the ellipticity is that it will not be possible to measure the Q of these modes by merely measuring the width of the unresolved spectral peak. This is because most of the broadening of an unresolved peak is due not necessarily to dissipation but to splitting. Thus, for these high-order fundamental modes, the lower limit of the spin gap introduced by Gilbert & Backus (1965) must be redefined to include perturbations due to ellipticity. Appendix C of this paper discusses further the manner in which the splitting interferes with measurements of the dissipation.

It has been pointed out that the surface wave data of Toksöz & Ben-Menahem (1963) and Toksöz & Anderson (1966) indicate that for higher order fundamental normal modes, the Earth's rotation and ellipticity is no longer the dominant perturbing effect. It is most likely that the dominant perturbation for these higher order fundamental modes (for which the displacement is concentrated near the Earth's surface) is the effect of lateral inhomogeneities in the crust and upper mantle. A few preliminary studies of the nature of the lateral inhomogeneities in the Earth's upper mantle have been made by utilizing measurements of average phase velocities of surface waves over various great circular paths (Toksöz & Ben-Menahem 1963; Backus 1964; Toksöz & Anderson 1966). At the present time, however, there is probably not sufficient information to write expressions for $\delta\rho_0$, $\delta\phi_0$, $\delta\kappa$ and $\delta\mu$ due to the regional variations in upper mantle structure. Toksöz & Anderson (1966) have shown that the surface wave phase velocity data cannot be explained by postulating just two different types of structure, one under continents and one under oceans. They have suggested that the distinction between continental shield areas and continental tectonic areas is as important as the more obvious continental-oceanic distinction. Even if there were sufficient information about lateral inhomogeneities $\delta\rho_0$, $\delta\phi_0$, $\delta\kappa$, $\delta\mu$, to compute the new matrix elements R_{ij} for any normal mode multiplet ${}_nS_l$ or ${}_nT_l$ the problem would still be more difficult than the ellipticity and rotation problem, because the matrix R_{ij} would not in this case be diagonal. This means that for those normal mode multiplets significantly affected by the lateral inhomogeneities (which includes at least all fundamental modes with l greater than about 25, and probably fundamental modes of considerably lower angular order), a single normal mode cannot be characterized to zeroeth order by a single spherical harmonic Y_l^m. The normal modes of an Earth with asymmetrical lateral inhomogeneities will be to zeroeth order of the general form

$$\sum_{m=-l}^{l} {}_n a_{lk}^m \, {}_n\mathrm{S}_l^m$$

with associated eigenfrequencies ${}_n\omega_l + {}_n(\delta\omega)_{lk}$. (The subscripts n and l dropped from the coefficients ${}_n a_{lk}^m$ in Section 2 have been added here.)

Backus & Gilbert (personal communication) have pointed out an important consequence of the fact that a normal mode can no longer be characterized by a single Y_l^m: their statement (Backus & Gilbert 1961) that a low-frequency geophysical instrument placed at the Earth's north or south pole would observe only a single member of a multiplet (the member with $m = 0$) is erroneous when continentality plays an appreciable role in splitting the multiplet. For fundamental modes of high angular order (at least for $l > 25$) the geographical perturbations appear to be dominant; thus every normal mode is to zeroeth order of the form

$$\sum_{m=-l}^{l} {}_n a_{lk}^m \, {}_n\mathrm{S}_l^m,$$

and in particular every normal mode contains a term of the form ${}_n\mathrm{S}_l^0$. Thus every normal mode can have a non-zero amplitude at the Earth's north or south pole, and an instrument placed there will observe a broad envelope for every multiplet due to the splitting caused by lateral inhomogeneities. It is indeed unfortunate that a single instrument at the Earth's north or south pole will not succeed in producing high-quality normal mode data for modes of higher angular order. The installation of a world-wide array of low-frequency seismological instruments will be necessary if one wishes to resolve the high-order multiplets. For the low-order multiplets for which the rotation and ellipticity perturbations are dominant, polar stations should observe mainly $m = 0$ modes.

4. First-order displacement fields

In order to compute the second-order rotational perturbation of the eigenfrequency of a normal mode ${}_nS_l^m$ or ${}_nT_l^m$, it is necessary to introduce and to compute the first-order correction to the eigenfunctions. Correct to first order in $(\Omega/{}_n\omega_l^S)$ or $(\Omega/{}_n\omega_l^T)$, the displacement field S of a normal mode ${}_nS_l^m$ or ${}_nT_l^m$ of a rotating Earth model may be written $\mathbf{S} = {}_n\mathbf{S}_l^m + (\Omega/{}_n\omega_l)\,{}_n(\mathbf{S}_1{}^r)_l^m$. The vector field ${}_n(\mathbf{S}_1{}^r)_l^m$ is the unique solution of the non-homogeneous equation (27). The perturbation technique used in this case to determine corrections to the eigenfunctions is the perturbation theory of Hermitian operators using an expansion in powers of the perturbation (Kato 1966; Messiah 1966). To deal with an Earth model which is ellipsoidal as well as rotating, it was decided to use Rayleigh's variational principle rather than perturbation theory of Hermitian operators to compute, correct to first order in ellipticity as well as rotation, the change in the eigenfrequencies. If the only desire is to compute the first-order perturbation to the eigenfrequencies, the two methods are equivalent. The only reason that the former is more convenient in the present case is because it is easier to use the bilinear functional $[\delta\mathscr{V}(\mathbf{S},\mathbf{S}) - \omega^2\,\delta\mathscr{T}(\mathbf{S},\mathbf{S})]$ than it is to use the unique Hermitian operator associated with it. Denote this particular Hermitian operator by $[\delta\mathscr{V} - \omega^2\,\delta\mathscr{T}]$. This operator is related to the bilinear form $\delta\mathscr{V}(\mathbf{S},\mathbf{S}) - \omega^2\,\delta\mathscr{T}(\mathbf{S},\mathbf{S})$ by the equation

$$([\delta\mathscr{V} - \omega^2\,\delta\mathscr{T}]\,\mathbf{S},\mathbf{S}) = \delta\mathscr{V}(\mathbf{S},\mathbf{S}) - \omega^2\,\delta\mathscr{T}(\mathbf{S},\mathbf{S}).$$

In terms of this operator it is possible to use perturbation theory to compute the perturbation to the displacement field to first order in ε_a as well as to first order in $(\Omega/{}_n\omega_l^S)$ or $(\Omega/{}_n\omega_l^T)$. If the displacement field S of a normal mode of a rotating elliptical Earth can be expanded in the form $\mathbf{S} = \mathbf{S}_0 + (\Omega/{}_n\omega_l)\,\mathbf{S}_1{}^r + \varepsilon_a\,\mathbf{S}_1{}^e$, then $\mathbf{S}_0$, $\mathbf{S}_1{}^r$, and $\mathbf{S}_1{}^e$ can be computed. It has in fact been shown that rotation and ellipticity act to remove the degeneracy completely and that $\mathbf{S}_0$ is of the form ${}_n\mathbf{S}_l^m$, an eigenfunction of the unperturbed SNREI Earth model characterized by a single spherical harmonic Y_l^m. The associated first-order fields ${}_n(\mathbf{S}_1{}^r)_l^m$ and ${}_n(\mathbf{S}_1{}^e)_l^m$ are then the unique particular solutions of the non-homogeneous equations

$$(\rho_0\,{}_n\omega_l{}^2 I - H_0)\,{}_n(\mathbf{S}_1{}^r)_l^m = -\rho_0\,{}_n\omega_l{}^2\,[2\,{}_n(\sigma_1{}^r)_l^m\,{}_n\mathbf{S}_l^m - 2i\,\hat{\mathbf{z}}\times{}_n\mathbf{S}_l^m], \tag{27}$$

$$(\rho_0\,{}_n\omega_l{}^2 I - H_0)\,{}_n(\mathbf{S}_1{}^e)_l^m = -\{2\rho_0\,{}_n\omega_l{}^2({}_n\alpha_l^e + m^2\,{}_n\gamma_l^e)\,{}_n\mathbf{S}_l^m - \varepsilon_a^{-1}\,[\delta\mathscr{V} - {}_n\omega_l{}^2\,\delta\mathscr{T}]\,{}_n\mathbf{S}_l^m\}. \tag{30}$$

It is clear that if ${}_n(\mathbf{S}_1{}^r)_l^m$ is to be a solution of equation (27), then the inner product $({}_n(\mathbf{S}_1{}^r)_l^m, \hat{\mathbf{z}}\times{}_n\mathbf{S}_l^m)$ must necessarily be non-zero, and similarly if ${}_n(\mathbf{S}_1{}^e)_l^m$ is to be a solution of equation (30), the inner product $({}_n(\mathbf{S}_1{}^e)_l^m, [\delta\mathscr{V} - {}_n\omega_l{}^2\,\delta\mathscr{T}]\,{}_n\mathbf{S}_l^m)$ must be non-zero. Since equations (27) and (30) are solved in the SNREI Earth model by expanding ${}_n(\mathbf{S}_1{}^r)_l^m$ and ${}_n(\mathbf{S}_1{}^e)_l^m$ in series of vector spherical harmonics, this means that the expansion of ${}_n(\mathbf{S}_1{}^r)_l^m$ can contain no vector spherical harmonics not in the expansion of $\hat{\mathbf{z}}\times{}_n\mathbf{S}_l^m$ and that the expansion of ${}_n(\mathbf{S}_1{}^e)_l^m$ can contain no vector spherical harmonics not in the expansion of $[\delta\mathscr{V} - {}_n\omega_l{}^2\,\delta\mathscr{T}]\,{}_n\mathbf{S}_l^m$. Note that the stipulation that the inner product $({}_n(\mathbf{S}_1{}^e)_l^m, [\delta\mathscr{V} - {}_n\omega_l{}^2\,\delta\mathscr{T}]\,{}_n\mathbf{S}_l^m)$ be non-zero is equivalent to the stipulation that the bilinear functional $\delta\mathscr{V}({}_n(\mathbf{S}_1{}^e)_l^m, {}_n\mathbf{S}_l^m) - {}_n\omega_l{}^2\,\delta\mathscr{T}({}_n(\mathbf{S}_1{}^e)_l^m, {}_n\mathbf{S}_l^m)$ be non-zero. It is thus possible to determine a great deal about the nature of the first-order displacement fields ${}_n(\mathbf{S}_1{}^r)_l^m$ and ${}_n(\mathbf{S}_1{}^e)_l^m$ without actually having to solve equations (27) and (30). In order to compute ${}_n(\sigma_2{}^r)_l^m$ from equation (26), it is however necessary to solve completely equation (27) for ${}_n(\mathbf{S}_1{}^r)_l^m$. Let $\boldsymbol{\sigma}_l^m$ denote an arbitrary poloidal vector field of degree l and order m and let $\boldsymbol{\tau}_l^m$ denote an arbitrary toroidal vector field of degree l and order m. Then $\boldsymbol{\sigma}_l^m$ must be of the general form

$$\boldsymbol{\sigma}_l^m = \hat{\mathbf{r}} U_l\, Y_l^m + \nabla_1(V_l\, Y_l^m) = U_l\,\mathbf{P}_l^m + V_l\,\mathbf{B}_l^m,$$

and $\boldsymbol{\tau}_l^m$ must be of the general form $\boldsymbol{\tau}_l^m = -\hat{\mathbf{r}} \times \nabla_1(W_l\, Y_l^m) = W_l\,\mathbf{C}_l^m$ where U_l, V_l, and W_l are arbitrary scalar fields. It is easily shown that if ${}_n\mathbf{S}_l^m$ is poloidal of the form $\boldsymbol{\sigma}_l^m$, then $\hat{\mathbf{z}} \times {}_n\mathbf{S}_l^m$ is of the form $\boldsymbol{\sigma}_l^m + \boldsymbol{\tau}_{l-1} + \boldsymbol{\tau}_{l+1}$, and if ${}_n\mathbf{S}_l^m$ is toroidal of the form $\boldsymbol{\tau}_l^m$, $\hat{\mathbf{z}} \times {}_n\mathbf{S}_l^m$ is of the form $\boldsymbol{\tau}_l^m + \boldsymbol{\sigma}_{l-1}^m + \boldsymbol{\sigma}_{l+1}^m$. Hence if ${}_n\mathbf{S}_l^m$ is poloidal of the form $\boldsymbol{\sigma}_l^m$, then ${}_n(\mathbf{S}_1{}^r)_l^m$ is necessarily of the form ${}_n(\mathbf{S}_1{}^r)_l^m = \boldsymbol{\tau}_{l+1}^m + \boldsymbol{\tau}_{l-1}^m$, and if ${}_n\mathbf{S}_l^m$ is toroidal of the form $\boldsymbol{\tau}_l^m$, then ${}_n(\mathbf{S}_1{}^r)_l^m$ is necessarily of the form ${}_n(\mathbf{S}_1{}^r)_l^m = \boldsymbol{\sigma}_{l+1}^m + \boldsymbol{\sigma}_{l-1}^m$ (Backus & Gilbert, personal communication). One can say that to first order, rotation acts to couple a poloidal field of order l to toroidal fields of order $l \pm 1$ and to couple a toroidal field of order l to poloidal fields of order $l \pm 1$. It may similarly be shown that for an elliptical Earth model, if the bilinear functional $\delta\mathscr{V}({}_n(\mathbf{S}_1{}^e)_l^m, {}_n\mathbf{S}_l^m) - {}_n\omega_l{}^2\, \delta\mathscr{T}({}_n(\mathbf{S}_1{}^e)_l^m, {}_n\mathbf{S}_l^m)$ is to be non-zero, then the nature of ${}_n(\mathbf{S}_1{}^e)_l^m$ is simply related to that of ${}_n\mathbf{S}_l^m$. Namely, if ${}_n\mathbf{S}_l^m$ is poloidal of the form $\boldsymbol{\sigma}_l^m$, then ${}_n(\mathbf{S}_1{}^e)_l^m$ is necessarily of the form

$${}_n(\mathbf{S}_1{}^e)_l^m = \boldsymbol{\tau}_{l+1}^m + \boldsymbol{\tau}_{l-1}^m + \boldsymbol{\sigma}_{l+2}^m + \boldsymbol{\sigma}_{l-2}^m,$$

and if ${}_n\mathbf{S}_l^m$ is toroidal of the form $\boldsymbol{\tau}_l^m$, then ${}_n(\mathbf{S}_1{}^e)_l^m$ is necessarily of the form

$${}_n(\mathbf{S}_1{}^e)_l^m = \boldsymbol{\sigma}_{l+1}^m + \boldsymbol{\sigma}_{l-1}^m + \boldsymbol{\tau}_{l+2}^m + \boldsymbol{\tau}_{l-2}^m.$$

Thus one of the first-order effects of the ellipticity is the same as that of the rotation, namely to couple poloidal modes of order l to toroidal modes of order $l \pm 1$, and to couple toroidal modes of order l to poloidal modes of order $l \pm 1$. Note that if other perturbations such as lateral inhomogeneities are more important than ellipticity, the simple relation given above between the zeroeth order and first-order displacement fields is no longer valid. The perturbation to the displacement field ${}_n\mathbf{S}_l^m$ to first order in arbitrary small lateral inhomogeneities $\delta\rho_0$, $\delta\phi_0$, $\delta\kappa$ and $\delta\mu$ will in general be a much more complicated combination of poloidal and toroidal fields of various degrees l and orders m.

The most important conclusion to be drawn from the above discussion is the following fact. A normal mode of a non-SNREI Earth model will not consist of purely toroidal or of purely poloidal motion. Thus if one observes the normal modes of the real Earth and then somehow effects a separation of the motion into poloidal and toroidal motion (for example a gravimeter can observe only a poloidal displacement field as there is no perturbation of the gravitational potential associated with toroidal motion), then in the spectrum of the poloidal motion there will be peaks at toroidal eigenfrequencies, and in the spectrum of the toroidal motion, there will be peaks at poloidal eigenfrequencies. The amplitudes of the first-order displacement fields will in general of course be small, on the order of $(\Omega/{}_n\omega_l^S)$ or $(\Omega/{}_n\omega_l^T)$ or ε_a times the amplitudes of the zeroeth order displacement fields. In fact the signal to noise ratio has been so low in all existing data that the first-order displacement fields do not seem to have been detected. Nonetheless, their existence should not be overlooked in future analyses of the normal mode data. A qualitative explanation of the fact that the Coriolis force acts to couple poloidal and toroidal motion was given by MacDonald & Ness (1961).

5. Quasi-degeneracy

Whenever two degenerate eigenfrequencies ${}_n\omega_l$ and ${}_{n'}\omega_{l'}$ of the unperturbed SNREI Earth model are so close together that the first-order corrections introduced by the perturbations are larger than $|{}_n\omega_l - {}_{n'}\omega_{l'}|$, the theory in Section 2 becomes invalid. It is however possible to treat this quasi-degenerate case using perturbation techniques (Messiah 1966). Consider a SNREI Earth model which has two eigenspaces ${}_n\mathscr{S}_l$ and ${}_{n'}\mathscr{S}_{l'}$ associated respectively with quasi-degenerate eigenfrequencies ${}_n\omega_l$ and ${}_{n'}\omega_{l'}$, and denote $({}_{n'}\omega_{l'})^2-({}_n\omega_l)^2$ by Δ. Let P' be the orthogonal projection operator on the $2l'+1$-dimensional space ${}_{n'}\mathscr{S}_{l'}$ defined relative to the inner product $(\mathbf{u}, \mathbf{v}) = \int_V \rho_0 \mathbf{u}^* . \mathbf{v}\, dV$. Define the operator $K = H_0-\Delta P'$ where H_0 is the Hermitian operator which defines the normal mode eigenvalue problem in the SNREI Earth model, $H_0\, \mathbf{S} = \rho_0\, \omega^2\, \mathbf{S}$. Then K has eigenspace ${}_n\mathscr{S}_l \oplus {}_{n'}\mathscr{S}_{l'}$, the direct sum of ${}_n\mathscr{S}_l$ and ${}_{n'}\mathscr{S}_{l'}$ with associated eigenfrequency ${}_n\omega_l$. Now the normal mode eigenvalue problem for the SNREI Earth model may be written in terms of the operator K as

$$K\mathbf{S}+\Delta P'\mathbf{S} = \rho_0\, \omega^2\, \mathbf{S}. \tag{31}$$

Taking the inner product of equation (31) and the field $\mathbf{S}$ yields

$$\omega^2 \mathscr{T}(\mathbf{S}, \mathbf{S})-(\mathbf{S}, {\rho_0}^{-1} K\mathbf{S})-(\mathbf{S}, {\rho_0}^{-1}\, \Delta P'\mathbf{S}) = 0. \tag{32}$$

If bilinear functionals $\mathscr{K}(\mathbf{S}, \mathbf{S})$ and $\mathscr{D}(\mathbf{S}, \mathbf{S})$ are defined by $\mathscr{K}(\mathbf{S}, \mathbf{S}) = (\mathbf{S}, {\rho_0}^{-1} K\mathbf{S})$ and $\mathscr{D}(\mathbf{S}, \mathbf{S}) = (\mathbf{S}, {\rho_0}^{-1}\, P'\, \mathbf{S})$ then equation (32) can be written

$$\omega^2 \mathscr{T}(\mathbf{S}, \mathbf{S})-\mathscr{K}(\mathbf{S}, \mathbf{S})-\Delta\mathscr{D}(\mathbf{S}, \mathbf{S}) = 0. \tag{33}$$

If $\omega = {}_n\omega_l$ and $\mathbf{S}$ is a vector in the $(2l+1)+(2l'+1)$ dimensional space ${}_n\mathscr{S}_l \oplus {}_{n'}\mathscr{S}_{l'}$, then the functional equation (33) is stationary to first order in an arbitrary small variation in $\mathbf{S}$. Taking first variations in equation (33) (including a small variation in $\mathbf{\Omega}$ away from $\Omega = 0$), treating Δ as a small perturbation and retaining the rotational potential (a term of second order in Ω/ω) leads to

$$\delta\omega^2\, \mathscr{T}(\mathbf{S}, \mathbf{S}) = \delta\mathscr{V}(\mathbf{S}, \mathbf{S})-\omega^2\delta\mathscr{T}(\mathbf{S}, \mathbf{S})+2\omega\mathscr{W}(\mathbf{S}, \mathbf{S})+\mathscr{P}(\mathbf{S}, \mathbf{S})$$
$$+\Psi(\mathbf{S}, \mathbf{S})+\Delta\mathscr{D}(\mathbf{S}, \mathbf{S}). \tag{34}$$

Now choose an orthonormal basis of the subspace ${}_n\mathscr{S}_l \oplus {}_{n'}\mathscr{S}_{l'}$ consisting of $2l+1$ orthonormal basis vectors $\{{}_n\mathbf{S}_l^m, m = -l, \ldots, 0, \ldots l\}$ of ${}_n\mathscr{S}_l$ and $2l'+1$ orthonormal basis vectors $\{{}_{n'}\mathbf{S}_{l'}^m, m = -l', \ldots, 0, \ldots, l'\}$ of ${}_{n'}\mathscr{S}_{l'}$. Relabel the elements of this basis by $\mathbf{S}_1, \ldots, \mathbf{S}_q$, where $\mathbf{S}_1, \ldots, \mathbf{S}_p$, are in ${}_n\mathscr{S}_l$ and $\mathbf{S}_{p+1}, \ldots, \mathbf{S}_q$ are in ${}_{n'}\mathscr{S}_{l'}$. Then any element $\mathbf{S}$ of ${}_n\mathscr{S}_l \oplus {}_{n'}\mathscr{S}_{l'}$ can be written in the form

$$\mathbf{S} = \sum_{j=1}^{q} a^j\, \mathbf{S}_j.$$

Now, as in Section 2, equation (34) can be readily reduced to an eigenvalue problem for a Hermitian matrix $\tilde{R}$. The dimension of $\tilde{R}$ in this case is the dimension of ${}_n\mathscr{S}_l \oplus {}_{n'}\mathscr{S}_{l'}$, that is $(2l+1)+(2l'+1)$.

$$\tilde{R}_{ij}\, a^j = (\delta\omega^2)\, a^i \tag{35}$$

where

$$\tilde{R}_{ij} = \delta\mathscr{V}(\mathbf{S}_i, \mathbf{S}_j)-\omega^2\, \delta\mathscr{T}(\mathbf{S}_i, \mathbf{S}_j)+2\omega\mathscr{W}(\mathbf{S}_i, \mathbf{S}_j)$$
$$+\mathscr{P}(\mathbf{S}_i, \mathbf{S}_j)+\Psi(\mathbf{S}_i, \mathbf{S}_j)+\Delta\mathscr{D}(\mathbf{S}_i, \mathbf{S}_j). \tag{36}$$

The inclusion of the rotational potential term $\Psi(\mathbf{S}, \mathbf{S})$ is necessary in order to assure that the matrix $\tilde{R}$ is Hermitian.

As before, the eigenvalues of $\tilde{R}$ represent the first-order correction to the eigenfrequencies of the perturbed non-SNREI Earth model, while the eigenvectors of $\tilde{R}$ serve to characterize the zeroeth order eigenfunctions associated with the new eigenvalues. Note that in the case of two quasi-degenerate multiplets ${}_n\mathscr{S}_l$ and ${}_{n'}\mathscr{S}_{l'}$, the zeroeth order eigenfunctions of the perturbed non-SNREI Earth model will in general be elements of ${}_n\mathscr{S}_l \oplus {}_{n'}\mathscr{S}_{l'}$. The quasi-degenerate theory reduces to the ordinary separate theory for the two multiplets unless there is some element $\mathbf{S}$ of ${}_n\mathscr{S}_l$ and some element $\mathbf{S}'$ of ${}_{n'}\mathscr{S}_{l'}$ such that $\mathscr{W}(\mathbf{S}, \mathbf{S}') \neq 0$ or $\delta\mathscr{V}(\mathbf{S}, \mathbf{S}') \neq 0$ or $\delta\mathscr{T}(\mathbf{S}, \mathbf{S}') \neq 0$ or $\mathscr{P}(\mathbf{S}, \mathbf{S}') \neq 0$ or $\Psi(\mathbf{S}, \mathbf{S}') \neq 0$. On an elliptical, rotating Earth this means that the quasi-degenerate theory need only be applied to a few cases. Not only must $\Delta = {}_{n'}\omega_{l'}{}^2 - {}_n\omega_l{}^2$ be very small compared to spacing between adjacent eigenfrequencies, but also if ${}_n\mathscr{S}_l$ is of the form ${}_n\mathscr{S}_l^S$, then ${}_{n'}\mathscr{S}_{l'}$ must be of the form ${}_{n'}\mathscr{S}_{l\pm1}^T$ or ${}_{n'}\mathscr{S}_{l\pm2}^S$, and if ${}_n\mathscr{S}_l$ is of the form ${}_n\mathscr{S}_l^T$ then ${}_{n'}\mathscr{S}_{l'}$ must be of the form ${}_{n'}\mathscr{S}_{l\pm1}^S$ or ${}_{n'}\mathscr{S}_{l\pm2}^T$. The most interesting case is the quasi-degeneracy of a poloidal multiplet of order l with a toroidal multiplet of order $l \pm 1$ since this is affected by both rotation and ellipticity to first order. In this case the eigenfunctions to zeroeth order are of the form $\boldsymbol{\sigma}_l + \boldsymbol{\tau}_{l\pm1}$; each of these eigenfunctions is associated with a squared eigenfrequency ${}_n\omega_l{}^2 + {}_n(\delta\omega^2)_l^m$ where ${}_n(\delta\omega^2)_l^m$ is computed by determining the eigenvalues in equation (35). Fig. 3 is a diagram showing the eigenfrequencies ${}_n\omega_l^S$ and ${}_n\omega_l^T$ of SNREI Earth model 1. The same diagram for models G1 and Q1 would look very similar. It can be seen that there are for this model several pairs of multiplets to which the quasi-degenerate perturbation theory must be applied. In particular, over a broad range of angular order numbers (from about $l = 10$ to about $l = 25$), the fundamental poloidal eigenfrequency ${}_0\omega_l^S$ is very nearly equal to the fundamental toroidal eigenfrequency ${}_0\omega_{l+1}^T$. For example the degenerate eigenfrequencies ${}_0\omega_{11}^S$ and ${}_0\omega_{12}^T$ are equal to within 0·15 per cent; the degenerate eigenfrequencies ${}_0\omega_{19}^S$ and ${}_0\omega_{20}^T$ are equal to within 0·001 per cent. When the ordinary perturbation theory described in Section 3 of this paper is applied to fundamental modes ${}_0S_l$ and ${}_0T_l$ in this range, the second-order rotational splitting parameter ${}_0\alpha_l^r\,(\Omega/{}_0\omega_l)^2$ generally turns out to be one or two orders of magnitude larger than the first-order parameter ${}_0\beta_l^r(\Omega/{}_0\omega_l)$. Also the first-order rotational displacement field ${}_0(\mathbf{S}_l{}^r)_l^m$ in general turns out to be of about the same order as the zeroeth order displacement field ${}_0\mathbf{S}_l^m$. This was taken as an indication that it would be necessary or at least more convenient to use a quasi-degenerate perturbation theory to examine

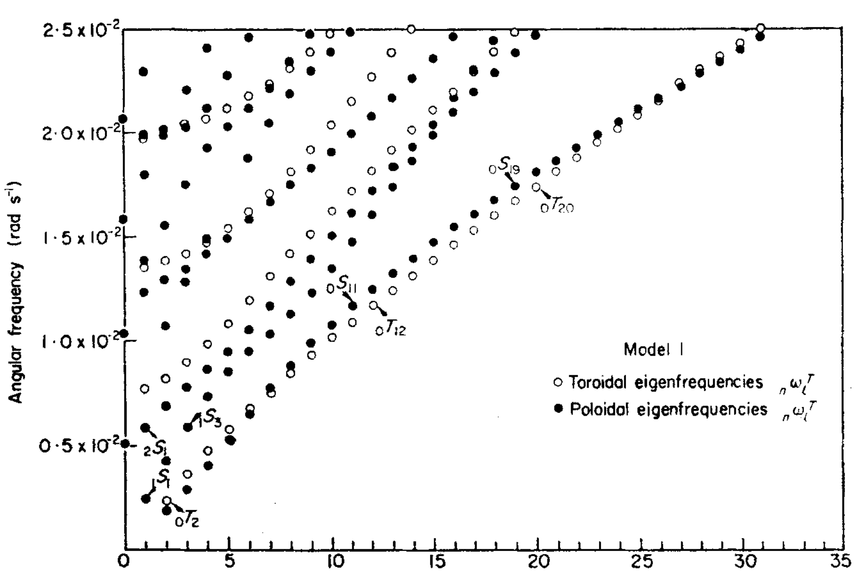

FIG. 3. Normal mode eigenfrequencies ${}_n\omega_l{}^S$ and ${}_n\omega_l{}^T$ for SNREI Earth model 1.

the effect of rotation and ellipticity on these multiplets. This is the reason that Tables 1–3 were not extended to include fundamental modes with angular order $l > 10$. To zeroeth order in ε_a and in $(\Omega/{}_0\omega_l^S)$ or $(\Omega/{}_0\omega_l^T)$, the fundamental modes for $10 < l < 25$ will be of the form $\boldsymbol{\sigma}_l + \boldsymbol{\tau}_{l+1}$, partly of poloidal nature and partly of toroidal nature. Presumably unless the near degeneracy of ${}_0\omega_l^S$ and ${}_0\omega_{l+1}^T$ is severe (as in the case of ${}_0S_{11}$, ${}_0T_{12}$ and ${}_0S_{19}$, ${}_0T_{20}$ for this Earth model), any given mode will be primarily of one nature, so that one can speak of a primarily poloidal split multiplet and a primarily toroidal split multiplet. It can in fact be shown that in the limit when Δ is large compared to the ordinary perturbation parameters ${}_n\alpha_l$, ${}_n\beta_l$, ${}_n\gamma_l$, and ${}_{n'}\alpha_{l'}$, ${}_{n'}\beta_{l'}$, ${}_{n'}\gamma_{l'}$, the quasi-degenerate perturbation theory reduces to the ordinary theory (which gives rise to a purely poloidal split multiplet and a purely toroidal split multiplet, if the first and higher order displacement fields are neglected).

A forthcoming paper will discuss the results of the application of the quasi-degenerate perturbation theory to the fundamental poloidal and toroidal modes in the range $10 < l < 25$. It is likely that the quasi-degenerate theory for rotation and ellipticity alone will not suffice for many if not all of the multiplet pairs in this range because of the perturbation by the lateral inhomogeneities in the crust and upper mantle. If the effect of these lateral inhomogeneities is to be taken into account, the computations become much more involved; in general, it will no longer suffice to consider individually a single poloidal and toroidal multiplet pair.

The eigenfrequencies for the modes ${}_0T_2$ and ${}_1S_1$ have never been accurately measured in any of the existing data, presumably because they have never been sufficiently excited, but for Earth models 1, G1, and Q1, the two degenerate eigenfrequencies ${}_0\omega_2{}^T$ and ${}_1\omega_1{}^S$ are within about 10 per cent of each other. For these three SNREI Earth models, the spacing is not close enough to necessitate the use of a quasi-degenerate theory, but there may be other SNREI Earth models close in some sense to models 1, Q1, or G1 for which the eigenfrequencies ${}_0\omega_2{}^T$ and ${}_1\omega_1{}^S$ are almost exactly equal. This question is being pursued. Since the modes ${}_0T_2$ and ${}_1S_1$ have probably never been clearly observed, it is not clear whether a quasi-degenerate theory will in fact be necessary for the real Earth.

Fig. 3 also reveals the close spacing between eigenfrequencies for model 1 of the modes ${}_1S_3$ and ${}_2S_1$ (within 0·5 per cent). These modes are coupled to first order by ellipticity but not by rotation. It can however be shown that in the second order approximation, rotation also will act to couple these two modes. Slichter (1967) has resolved three peaks near the theoretical eigenfrequencies of ${}_1S_3$ and ${}_2S_1$ which he identifies as belonging to $m = -2$, $m = -1$ and $m = +2$ of multiplet ${}_1S_3$. A quasi-degenerate theory, correct to first order in the ellipticity and to second order in rotation, for poloidal modes of order l and $l \pm 2$ is also presently being pursued.

The fact that many normal modes (in particular fundamental modes in the range $l = 10$ to 25) will not be even to zeroeth order of a purely poloidal or a purely toroidal nature will certainly tend to hamper the problem of mode identification.

6. Summary and conclusions

Before attempting to use the Earth's normal mode data to investigate the interior properties of the Earth, it is essential to consider the various factors which may affect the precision of the data. Because of the very large extent of the non-uniqueness in the inverse problem and because of the relative mathematical simplicity, it is customary to include only SNREI Earth models in inverse problem calculations. In this case the fact that the real Earth is not a SNREI Earth model may be looked upon as one of the factors contaminating the raw data. It is necessary to correct the contaminated raw data for variations away from zero angular rotation, sphericity and isotropy. In this paper, Rayleigh's principle is used to provide an explicit scheme for computing first-order corrections to the theoretical eigenfrequencies of an arbitrary SNREI Earth model due to slow angular rotations and small asphericities and anisotropies. For the lower order fundamental normal modes, it is expected that the Earth's rotation and ellipticity are the dominant perturbing effects. The computed rotational and elliptical splitting parameters depend upon the properties ρ_0, κ, μ of the unperturbed SNREI Earth model. In this paper, the eigenfrequencies ${}_n\omega_l^S$ and ${}_n\omega_l^T$ and the associated $2l+1$ dimensional eigenspaces were computed for three different SNREI Earth models, and then Rayleigh's principle and second-order rotational perturbation theory were used to determine the corrections to the eigenfrequencies, correct to first order in the ellipticity ε_a and to second-order in the rotation. The degeneracy of any multiplet ${}_nS_l$ or ${}_nT_l$ is in general completely removed; to zeroeth order the eigenfunctions of a rotating elliptical Earth without geographical variations in properties can be characterized by a single spherical harmonic Y_l^m. The first-order effect of ellipticity and the second-order effect of rotation not only act to shift the entire multiplet but also cause the splitting of a multiplet to be asymmetrical. It is pointed out that another effect of rotation, ellipticity and lateral inhomogeneities is to give rise to the presence of small amplitude first-order displacement fields. In particular there will be poloidal fields at toroidal eigenfrequencies and toroidal fields at poloidal eigenfrequencies.

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APPENDIX A

First-order computation of eigenfrequency shift due to ellipticity and rotation

Correct to first order in ε_a and in $(\Omega/{}_n\omega_l^S)$ or $(\Omega/{}_n\omega_l^T)$, the shift ${}_n(\delta\omega)_l^m$ in the eigenfrequency ${}_n\omega_l^S$ or ${}_n\omega_l^T$ associated with a normal mode ${}_nS_l^m$ or ${}_nT_l^m$ may be written as

$$ {}_n(\delta\omega)_l^m = \frac{1}{2{}_n\omega_l}\,[\delta\mathscr{V}({}_n\mathbf{S}_l^m, {}_n\mathbf{S}_l^m) - {}_n\omega_l{}^2\,\delta\mathscr{T}({}_n\mathbf{S}_l^m, {}_n\mathbf{S}_l^m)] + i\Omega({}_n\mathbf{S}_l^m, \hat{\mathbf{z}}\times{}_n\mathbf{S}_l^m). \qquad (23) $$

If the mode is poloidal, then ${}_n\mathbf{S}_l^m$ may be written

$$ {}_n\mathbf{S}_l^m = \hat{\mathbf{r}}\,{}_nU_l(r)\,Y_l^m + \nabla_1\,[{}_nV_l(r)\,Y_i^m] $$

and if the mode is toroidal, then ${}_n\mathbf{S}_l^m$ may be written

$$ {}_n\mathbf{S}_l^m = -\hat{\mathbf{r}} \times \nabla_1 [{}_nW_l(r)\, Y_l^m]. $$

Backus & Gilbert (1961), and Pekeris, Alterman & Jarosch (1961) have evaluated $i\Omega({}_n\mathbf{S}_l^m, \hat{\mathbf{z}} \times {}_n\mathbf{S}_l^m)$ and have shown that for a rotating, spherical Earth model (for which $\delta\mathscr{V}({}_n\mathbf{S}_l^m, {}_n\mathbf{S}_l^m) = \delta\mathscr{T}({}_n\mathbf{S}_l^m, {}_n\mathbf{S}_l^m) = 0$) ${}_n(\delta\omega)_l^m/{}_n\omega_l = m\,{}_n\beta_l^r\,(\Omega/{}_n\omega_l)$ where for a toroidal mode ${}_n\beta_l^r = \dfrac{1}{l(l+1)}$, and for a poloidal mode

$$ {}_n\beta_l^r = \frac{{}_nM_l + 2{}_nQ_l}{{}_nL_l + l(l+1){}_nM_l} $$

where

$$ {}_nL_l = \int_0^a r^2 \rho_0(r)\, [{}_nU_l(r)]^2\, dr $$

$$ {}_nM_l = \int_0^a r^2 \rho_0(r)\, [{}_nV_l(r)]^2\, dr $$

$$ {}_nQ_l = \int_0^a r^2 \rho_0(r)\, [{}_nU_l(r)\, {}_nV_l(r)]\, dr. $$

It remains to evaluate the ellipticity contribution. For convenience in what follows, the subscripts n and l and superscript m will be dropped from ${}_n\omega_l$, ${}_n\mathbf{S}_l^m$, ${}_nU_l(r)$, ${}_nV_l(r)$, and ${}_nW_l(r)$, and it is to be understood that a single normal mode ${}_nS_l^m$ or ${}_nT_l^m$ is under consideration. The frequency shift $\delta\omega$ due to the ellipticity may be written, in terms of components in a Cartesian axis system, as

$$ \begin{aligned} \delta\omega \int [\rho_0 S^2]\, dV \\ = \frac{1}{2\omega} \int dV\, [&\delta\kappa K + \delta\mu M + \delta\rho_0\, S_i S_j^* \partial_i \partial_j \phi_0 \\ &+ \rho_0 S_i S_j^* \partial_i \partial_j \delta\phi_0 + \delta\rho_0\, \partial_j \phi_0 (S_i \partial_i S_j^* - S_j \partial_i S_i^*) \\ &+ \rho_0\, \partial_j \delta\phi_0 (S_i \partial_i S_j^* - S_j \partial_i S_i^*) + \delta\rho_0 (S_i^* \partial_i \phi_1 + S_i \partial_i \phi_1^*) - \omega^2 \delta\rho_0 S^2] \end{aligned} \tag{41} $$

where $\delta\rho_0$, $\delta\phi_0$, $\delta\kappa$, $\delta\mu$ are given in equations (21) and (22). Since ϕ_0 is spherically symmetric for a SNREI Earth model, equation (41) may be reduced to

$$ \begin{aligned} \delta\omega \int_V [\rho_0 S^2]\, dV \\ = \frac{1}{2\omega} \int_V dV\, [&\delta\kappa K + \delta\mu M + 4\pi G \rho_0 S_r^2 \\ &+ \Lambda \partial_r \phi_0\, \delta\rho_0 + 2\delta\rho_0\, \mathbf{S}^*, \nabla\phi_1 + \rho_0\, \mathbf{S}.\nabla(\mathbf{S}^*, \nabla\delta\phi_0) - \rho_0 (\nabla.\mathbf{S})(\mathbf{S}^*, \nabla\delta\phi_0) - \omega^2 \delta\rho_0 S^2] \end{aligned} $$

where S_r is the $\hat{\mathbf{r}}$ component of $\mathbf{S}$, and where

$$ \Lambda = \mathbf{S}.\nabla S_r^* - S_r \nabla.\mathbf{S}^* - 2r^{-1} S_r^2 $$

(Backus & Gilbert 1967).

For a toroidal mode, $\nabla.\mathbf{S} = 0$ and $S_r = 0$, thus $\Lambda = 0$. In fact for a toroidal mode, equation (42) reduces to

$$\delta\omega \int [\rho_0 S^2]\, dV = \frac{1}{2\omega} \int dV [\delta\mu M + \delta\rho_0(-\omega^2 S^2)]. \tag{43}$$

Recall that $\frac{1}{2}\mu M$ is the shear energy density in a SNREI Earth model and that

$$M = \tfrac{1}{2} tr[(\boldsymbol{\Delta})^\dagger . (\boldsymbol{\Delta})],$$

where $\boldsymbol{\Delta}$ is the strain deviator tensor

$$\boldsymbol{\Delta} = \tfrac{1}{2}[(\nabla\mathbf{S}) + (\nabla\mathbf{S})^T] - \tfrac{1}{3}(\nabla.\mathbf{S})\,\mathbf{I}$$

and where the dagger † denotes the conjugate transpose. Denote for simplicity

$$\delta\mu = \mu_1(r)\, P_2{}^0(\cos\theta), \qquad \delta\rho_0 = \rho_1(r)\, P_2{}^0(\cos\theta),$$

$$\delta\kappa = \kappa_1(r)\, P_2{}^0(\cos\theta), \qquad \delta\phi_0 = g_1(r)\, P_2{}^0(\cos\theta);$$

then

$$\mu_1(r) = \tfrac{2}{3} r\varepsilon(r)\, \mu'(r), \quad \rho_1(r) = \tfrac{2}{3} r\varepsilon(r)\, \rho_0'(r), \quad \kappa_1(r) = \tfrac{2}{3} r\varepsilon(r)\, \kappa'(r),$$

and

$$g_1(r) = \tfrac{2}{3} r\varepsilon(r)\, \phi_0'(r) - \tfrac{1}{3}\Omega^2 r^2.$$

The evaluation of the integrals in equation (43) is vastly simplified by making use of the tangent tensor representation theorem (Backus 1966) and many consequent formulae from Backus (1967). Some applications of Gauss' theorem as well as several formulae from Backus (1967) show that equation (43) can be reduced to

$$_n(\delta\omega)_l^m\, [l(l+1)\, {}_nJ_l] = \frac{1}{2_n\,\omega_l} \int_0^a dr\, r^2\, [\mu_1(r)\, {}_nM_l(r) + \rho_1(r)\, {}_nR_l(r)] \tag{44}$$

where

$$_nM_l(r) = A_l^m\, [-l^2(l+1)^2\, r^{-2}\, W^2] + B_l^m\, [r^{-2}(rW' - W)^2 + (2l(l+1) - 8)\, r^{-2}\, W^2]$$

$$_nR_l(r) = B_l^m\, [-\omega^2\, W^2]$$

$$_nJ_l = \int_0^a r^2\, \rho_0(r) [{}_nW_l{}^2(r)]\, dr$$

and where

$$A_l^m = \int_s P_2{}^0\, Y_l^m\, Y_l^{m*}\, d\Omega = \frac{l(l+1) - 3m^2}{(2l-1)(2l+3)}$$

$$B_l^m = \int_s P_2{}^0\, \mathbf{B}_l^m . \mathbf{B}_l^{m*}\, d\Omega = \int_S P_2{}^0\, \mathbf{C}_l^m . \mathbf{C}_l^{m*}\, d\Omega = [l(l+1) - 3]\, A_l^m.$$

If a discontinuity in κ, μ or ρ_0 located at $r = b$ is moved to $r = b + h(\theta)$ where $h(\theta) = -\frac{2}{3} b\varepsilon(b)\, P_2{}^0(\cos\theta)$, then similar methods may be used to show that the effect on an eigenfrequency of a toroidal mode is given by

$$_n(\delta\omega)_l^m\, [l(l+1)\, {}_nJ_l] = \frac{1}{3_n\,\omega_l}\, b^3\, \varepsilon(b)\, [\mu\, {}_nM_l + \rho_0\, {}_nR_l]_-^+ \tag{45}$$

where for any function $f(r)$, the symbol $[f]_-^+$ denotes the limit as ε tends to zero through positive values of $[f(b+\varepsilon) - f(b-\varepsilon)]$.

The volume integrals in equation (42) for a poloidal mode are more difficult to compute, but similar methods may be used to show that equation (42) reduces to

$$_n(\delta\omega)_l^m\,[_nL_l+l(l+1)\,_nM_l] = \frac{1}{2_n\,\omega_l}\int_0^a r^2\,dr[\kappa_1(r)\,_nK_l(r)+$$

$$\mu_1(r)\,_nM_l(r)+\rho_1(r)\,_nR_l(r)+g_1(r)\,_nG_l(r)] \quad (46)$$

where

$$_nK_l(r) = A_l^m(U'+F)^2$$

$$_nM_l(r) = A_l^m\,[\tfrac{1}{3}(2U'-F)^2-l^2(l+1)^2\,r^{-2}\,V^2]$$
$$+B_l^m\,[r^{-2}(rV'-V+U)^2+\big(2l(l+1)-8\big)\,r^{-2}\,V^2]$$

$$_nR_l(r) = A_l^m\,[8\pi G\rho_0\,U^2-\phi_0{}'\,U(F+2r^{-1}\,U)+2U\phi_1{}'$$
$$-\omega^2\,U^2+\frac{4\pi G}{r^3}\int_0^r dr\,r^2\big(4U^2-3l(l+1)\,UV\big)]$$
$$+B_l^m\,[r^{-1}\,V(U\phi_0{}'+2\phi_1)-\omega^2\,V^2+\frac{4\pi G}{r^3}\int_0^r dr\,r^2\,\rho_0\,UV]$$

$$_nG_l(r) = \rho_0\,A_l^m\,[l(l+1)(r^{-1}\,UV'-10r^{-2}\,UV)+18r^{-2}\,U^2$$
$$+l^2(l+1)^2\,r^{-2}\,V^2-l(l+1)\,r^{-1}\,V(U'+F)]$$
$$+\rho_0\,B_l^m\,[-l(l+1)\,r^{-2}\,V^2+r^{-1}\,V(U'+F)+4r^{-2}\,UV-r^{-1}\,UV']$$

where

$$F(r) = \frac{1}{r}\,[2U-l(l+1)\,V].$$

If a discontinuity in κ, μ or ρ_0 located at $r = b$ is moved to $r = b+h(\theta)$ the effect on a poloidal eigenfrequency is similarly given by

$$_n(\delta\omega)_l^m\,[_nL_l+l(l+1)\,_nM_l] = \frac{1}{3_n\,\omega_l}\,b^3\,\varepsilon(b)[\kappa_n\,K_l+\mu\,_nM_l+\rho_0\,_n\tilde{R}_l]_-^+,$$

where

$$_n\tilde{R}_l(r) = A_l^m\,[-\omega^2\,U^2-\phi_0{}'U(F+2r^{-1}\,U)]+B_l^m\,[-\omega^2\,V^2+r^{-1}\,V(U\phi_0{}'+2\phi_1)].$$

In equations (45) and (47), U, ϕ_0, $\phi_0{}'$, ϕ_1 are continuous at $r = b$ while V, W, U', V', W' and $\phi_1{}'$ may be discontinuous.

Note that since A_l^m and B_l^m are both proportional to the quantity $l(l+1)-3m^2$, the frequency shift $_n(\delta\omega)_l^m/_n\omega_l$ may be written in the form $(_n\alpha_l^e+m^2\,_n\gamma_l^e)\,\varepsilon_a$ and that $_n\alpha_l^e = -\tfrac{1}{3}l(l+1)\,_n\gamma_l^e$.

APPENDIX B

Evaluation of the effect of a rotational potential

The perturbing effect of the rotational potential can be computed by treating ψ as a $\delta\phi_0$ term in Rayleigh's principle, equation (41)

$$\delta\omega\int_V dV\,[\rho_0\,S^2] = \frac{1}{2\omega}\int_V dV\,[\rho_0\,S_i\,S_j{}^*\,\partial_i\,\partial_j\,\psi+\rho_0\,\partial_i\,\psi(S_i\,\partial_i\,S_j{}^*-S_j\,\partial_i\,S_i{}^*)] \quad (48)$$

where ψ, the rotational potential, is equal to

$$\psi(r) = -\tfrac{1}{2}[\Omega^2 r^2 - (\mathbf{\Omega}.\mathbf{r})^2] = -\tfrac{1}{2}\Omega^2 r^2 \sin^2\theta. \tag{3}$$

This may be reduced, using Gauss' Theorem, to

$$\delta\omega \int_V dV[\rho_0 S^2] = \int_V dV[\mathbf{S}.\nabla\rho_0 + 2\rho_0 \nabla.\mathbf{S}]\,[(\hat{\mathbf{z}}\times\mathbf{r}).(\hat{\mathbf{z}}\times\mathbf{S}^*)]. \tag{49}$$

For a toroidal mode, the frequency shift ${}_n(\delta\omega_2{}^P)_l^m$ due to the rotational potential is zero, as expected. For a poloidal mode, the expression (49) for ${}_n(\delta\omega_2{}^P)_l^m$ is easily evaluated.

$$\frac{{}_n(\delta\omega_2{}^P)_l^m}{{}_n\omega_l}[{}_nL_l + l(l+1)\,{}_nM_l] = [C_l^m\,{}_nP_l - A_l^m\,{}_nS_l]\,(\Omega/{}_n\omega_l)^2 \tag{50}$$

where A_l^m is given in equation (44), and where

$$C_l^m = \frac{2[l(l+1)-1+m^2]}{(2l-1)(2l+3)}$$

and where

$${}_nP_l = \int_0^a dr\, r^2\, U\,[r\rho_0'\,U + 2r\,\rho_0 D]$$

$${}_nS_l = \int_0^a dr\, r^2\, V\,[r\rho_0'\,U + 2r\,\rho_0\,D]$$

where

$$D(r) = U' + \frac{2U}{r} - \frac{l(l+1)}{r^2}V.$$

If the SNREI Earth model ρ_0, κ, μ is such that there is a discontinuity in ρ_0 at $r = b$, then a surface integral contribution of the following form must be added to equation (50)

$$\frac{{}_n(\delta\omega_2{}^P)_l^m}{{}_n\omega_l}[{}_nL_l + l(l+1)\,{}_nM_l] = \left(\frac{\Omega}{{}_n\omega_l}\right)^2 b^3[C_l^m\,\rho_0\,U^2 - A_l^m\,\rho_0\,UV]_-^+. \tag{51}$$

Note that ${}_n(\delta\omega_2{}^P)_l^m$ is of the form

$$({}_n\alpha_l{}^r + m^2\,{}_n\gamma_l{}^r)\;(\Omega/{}_n\omega_l)^2.$$

APPENDIX C

Interference with measurements of dissipation

The observed dissipative properties of the Earth may be conveniently described by giving the Q of each normal mode. The Q of a normal mode is defined as

$$Q^{-1} = \frac{1}{2\pi}\left(\frac{\Delta E}{E}\right) \tag{52}$$

where $\Delta E/E$ is the fraction of the total energy of oscillation which is dissipated as heat in a single cycle. Equation (52) may be used to measure Q; an alternative method of measurement is to utilize an amplitude spectrum of the decaying signal.

In an amplitude spectrum whose peaks are broadened only by the dissipation, the Q of a free oscillation whose angular frequency is ω may be given by

$$Q^{-1} = \frac{\Delta\omega}{\omega} \tag{53}$$

where $\Delta\omega$ is the width of the peak at the half-power level. Measurements of the Q's of some of the Earth's normal modes have been attempted by both methods and by several investigators (Benioff, Press & Smith 1961; Alsop, Sutton & Ewing 1961; Nowroozi 1968).

The rotation and ellipticity of the Earth interfere with both types of measurements; with the first by introducing beats due to the interference of the $2l+1$ modes in a single multiplet, and with the second by broadening the spectral peak of a multiplet because of splitting. It is easiest to see the effect of ellipticity and rotation by considering the second method of measurement, the measurement of spectral peak widths at the half-power level.

In an Earth in which the dissipation process is not spherically symmetric, the individual Q's of the members of any given multiplet will not all be the same. At the present time, however, the errors and uncertainties in the measurement of Q are so great that it is still convenient to speak of the Q of a multiplet (by this one means some kind of average over the individual Q's in the multiplet). Consider a spectral peak, produced by an unresolvable multiplet and centred at angular frequency ω. It is clear that one cannot determine the Q of this multiplet by merely measuring the spectral peak width since part of the width is produced by the splitting. However, if the splitting is known for a given multiplet, then it is possible to assign bounds to the actual Q of the multiplet in terms of the measured Q. In fact the Q which one determines by a direct application of formula (53) is a lower bound Q_{lb} to the actual Q of the multiplet since it may be that only one element of the multiplet is excited. On the other hand, if all members of the multiplet (or at least both end members)

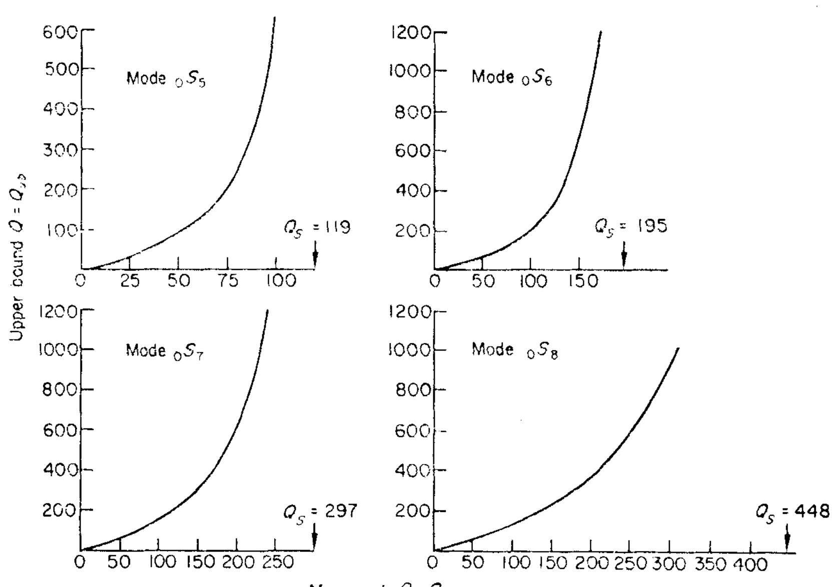

FIG. 4. Graphs for four poloidal modes showing how the upper bound on Q is related to the measured Q. Splitting parameters $_0\beta_l$ and $_0\gamma_l$ used are those for model 1. Q_s is given by $Q_s^{-1} = (\Delta\omega)_s/\omega$.

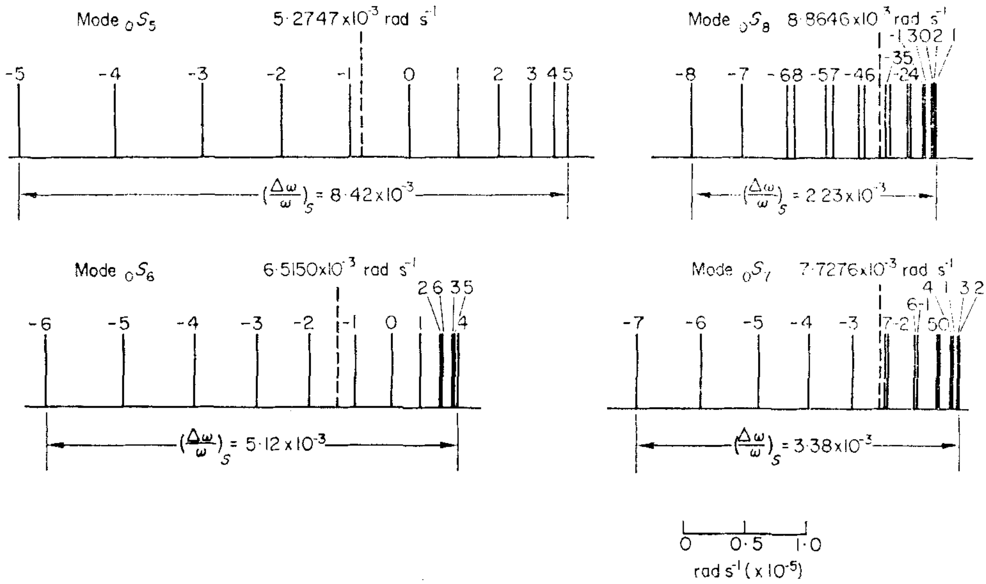

FIG. 5. Schematic indication of line spacing for the four poloidal multiplets of Fig. 4. The splitting parameters are those of model 1.

are excited to about the same level, then the measured width includes the total splitting width; it is thus possible to assign an upper bound Q_{ub} to the actual Q of the multiplet. If $(\Delta\omega)_s$ is the total splitting due to rotation and ellipticity, and if $(\Delta\omega)_m$ is the measured spectral peak width at the half-power level (the measured Q is $Q = (\Delta\omega)_m/\omega$), then an upper bound Q_{ub} is given by (Backus & Gilbert, personal communication)

$$Q_{ub} = \frac{(\Delta\omega)_m}{\omega} - \frac{(\Delta\omega)_s}{\omega}. \tag{54}$$

The total splitting of a given multiplet may be computed from the coefficients in Table 1.

In Fig. 4, graphs are provided which allow the determination for four fundamental poloidal modes of Q_{ub} in terms of the measured Q (the measured $Q = (\Delta\omega)_m/\omega$ is Q_{lb}). The rotational and elliptical splitting parameters of model 1 were used in preparing these graphs. The total width was not assumed to be merely $l^2|{}_n\gamma_l^e|\,\varepsilon_a$, as is the case for the higher order modes. Fig. 5 is a schematic indication of the line spacing for model 1 for the same four multiplets. Note that as l increases the splitting looks less and less like a Zeeman type splitting as the effect of ellipticity begins to dominate that of rotation. Similar graphs for other modes listed in Tables 1–3 could of course be easily constructed.

It is seen that the effects of ellipticity and rotation can cause a serious uncertainty in the estimation of the Q's of the Earth's normal modes. The actual bounds are not exact since the computed splitting parameters for rotation and ellipticity depend upon the Earth model used, and since splitting due to effects other than rotation and ellipticity was neglected.