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*Inventiones mathematicae*

# The trajectories of particles in Stokes waves

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**Abstract.** Analyzing a free boundary problem for harmonic functions we show that there are no closed particle paths in an irrotational inviscid traveling wave propagating at the surface of water over a flat bed: within a period each particle experiences a backward-forward motion with a slight forward drift.

## 1. Introduction

A Stokes wave is a two-dimensional periodic wave with a symmetric profile that rises and falls exactly once per wavelength, acted on by gravity and traveling at constant speed at the surface of irrotational water above a flat bed. The study of water waves of this type was initiated by formal but far-reaching considerations due to Stokes [31], while the existence of small amplitude Stokes waves was proved via convergent power series expansions by Levi-Civita [20] and Struik [33]. The fact that the Stokes wave problem can be formulated mathematically as a nonlinear free boundary problem for a harmonic function in a planar domain made it possible to use an interplay of harmonic/complex analysis with bifurcation/degree theory to study Stokes waves of large amplitude, that is, waves that can not be regarded as small perturbations of a flat water surface. The substantial amount of analytical theory made available mainly through the work of Toland and collaborators (see [1, 2, 4, 27, 34, 35] and citations therein) unveils to a large extent the fascinating structure of this classical hydrodynamical problem.

The aim of this paper is to describe qualitatively the trajectories of water particles in a Stokes wave. Due to the lack of explicit formulas for the free surface or for the velocity field, the current understanding of

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this fundamental aspect of Stokes waves is quite limited: “In progressive gravity waves of very small aplitude it is well known that the orbits of the particles are either elliptical or circular. In steep waves, however, the orbits become quite distorted, as shown by the existence of a mean horizontal drift or mass-transport in irrotational waves” cf. Longuet-Higgins [23]. Indeed, the leading order analysis of the linearized Stokes problem pursued in the classical and modern literature (see [13,14,17,19,22,25,29,31]) indicates that all particles move on closed orbits – a conclusion apparently supported by photographs with long exposure [14,29,31] and even by films [3]. The correction accounted for steeper/larger waves is suggested by an analysis of the mean energy transport which indicates the presence of an average forward drift [17,36]. In a recent paper [11] it is proved that for the linearized problem there are no closed particle orbits. Our aim is to show that this feature holds for Stokes waves of small and large amplitude. We also provide a geometric description of the actual particle path and thus explain the almost closed elliptic paths recognizable in photographs and visualized on film. While in [11] the analysis relied on the explicit formulas for the velocity field, for the pressure, and for the free surface provided by the linear theory, for a Stokes wave no such information is available and our approach relies on a qualitative study of the specific nonlinear boundary value problem for harmonic functions.

This paper is organized as follows. In Sect. 2 we present the mathematical formulation for Stokes waves and we derive properties of the corresponding water flow. Sect. 3 is devoted to the description of the trajectory of a water particle in a Stokes wave, while Sect. 4 contains some comments on related problems within water wave theory.

## 2. Preliminaries

In this section we present the mathematical problem and we prove some fundamental properties of the velocity field in a Stokes wave.

**2.1. The governing equations.** For most waves propagating on the surface of the sea or in a channel the motion is almost identical in any direction parallel to the crest line. To describe these waves consider a cross section of the flow that is perpendicular to the crest line with Cartesian coordinates $(X, Y)$, the $Y$-axis pointing vertically upwards and the $X$-axis being the direction of wave propagation, while the origin lies on the mean water level. Let $(u(t, X, Y),\ v(t, X, Y))$ be the velocity field of the flow over a flat bed $Y = -d$ and let $Y = \eta(t, X)$ be the water's free surface. The balance between the restoring gravity force and the inertia of the system governs the evolution of the mass of water, so that within the fluid we have the equation of mass conservation

$$u_X + v_Y = 0 \tag{1}$$

and Euler's equation

$$\begin{cases} u_t + uu_X + vu_Y = -P_X, \\ v_t + uv_X + vv_Y = -P_Y - g, \end{cases} \tag{2}$$

where $P(t, X, Y)$ denotes the pressure and $g$ is the gravitational constant of acceleration. On the free surface we have the boundary conditions

$$v = \eta_t + u\eta_X \qquad \text{on} \quad Y = \eta(t, X), \tag{3}$$

and

$$P = P_0 \qquad \text{on} \quad Y = \eta(t, X), \tag{4}$$

where $P_0$ is the (constant) atmospheric pressure, while

$$v = 0 \qquad \text{on} \quad Y = -d, \tag{5}$$

must hold on the flat bed. Assuming no local spin or rotation of a fluid element, as it is the case for water motions starting from rest, the water flow has to be irrotational, that is,

$$u_Y = v_X. \tag{6}$$

The equations (1)–(6) are the governing equations for irrotational water waves [17].

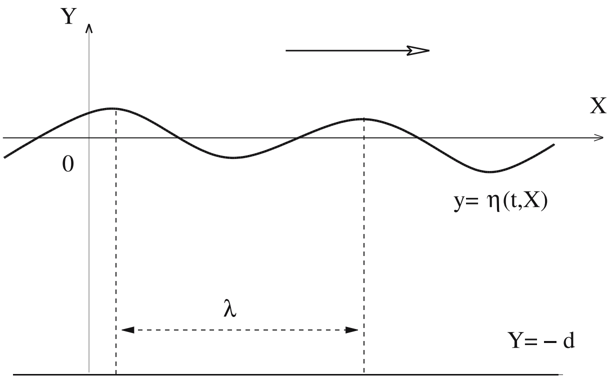


**Fig. 1** A Stokes wave

A Stokes wave is a solution to the governing equations for which $\eta, u, v, P$ are all periodic in the $X$ variable and exhibit an $(t, X)$-dependence in the form of $(X - ct)$, where $c > 0$ is the speed of the wave, with the functions $\eta, u, P$ even and $v$ odd in the variable $(X - ct)$. For such waves it is convenient to eliminate time from the problem by passing to a moving frame

$$x = X - ct, \qquad y = Y. \tag{7}$$

In the moving frame Bernoulli's law holds: the expression $\frac{(u-c)^2+v^2}{2} + g(y+d) + P$ is constant throughout the fluid domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\}.$$

Defining the stream function $\psi(x, y)$ up to a constant by

$$\psi_y = u - c, \quad \psi_x = -v, \tag{8}$$

we see that $\psi$ is harmonic in $\Omega$ in view of (6), whereas (3) and (5) guarantee that $\psi$ is constant on both boundaries of $\Omega$, say $\psi = 0$ on $y = \eta(x)$ while $\psi = m$ on $y = -d$. Using Bernoulli's law, we can express the boundary condition (4) in an alternative form. We thus obtain the following mathematical formulation for Stokes waves: look for a smooth solution $(\eta(x), \psi(x, y))$, periodic and even in the $x$-variable, and such that $\eta$ rises and falls exactly once per period with $\eta'(x) \neq 0$ except at the maximum/minimum, of the free boundary value problem

$$\begin{cases} \Delta\psi = 0 \quad \text{in} \quad -d < y < \eta(x), \\ \frac{|\nabla\psi|^2}{2} + g(y+d) = Q \quad \text{on} \quad y = \eta(x), \\ \psi = 0 \quad \text{on} \quad y = \eta(x), \\ \psi = m \quad \text{on} \quad y = -d, \end{cases} \tag{9}$$

with $m$ and $Q$ physical constants (relative mass flux, respectively hydraulic head). Since $\eta(x) + d > 0$ we must have $Q > 0$. On the other hand, $m < 0$ unless $\psi \equiv 0$. Indeed, experimental data [22] indicates that in general $u < c$, that is, the horizontal motion of individual water particles is slower than the propagation speed of the wave. Using the strong maximum principle for harmonic functions [15] first for $\psi$ and then for $\psi_y$, we infer from $\psi = 0$ on $y = \eta(x)$ that $m < 0$, respectively

$$\psi_y = u - c < 0 \quad \text{in} \quad \Omega, \tag{10}$$

unless $\psi \equiv 0$. Observe that in the moving frame the wave speed $c > 0$ does not appear in the system (9). To recover the wave speed, notice that the mean horizontal velocity per wavelength $\lambda$ at any fixed depth $y_0$ below the wave trough level, $\int_0^\lambda \psi_y(x, y_0)\,dx$, is constant throughout $\Omega$, as one can easily infer by applying the divergence theorem to the vector field $(\psi_x, \psi_y)$ in the rectangular domain bounded above by the segment $y = y_0$ and below by the corresponding segment on the flat bed $y = -d$. This leads naturally to Stokes' definition of the wave speed as the mean velocity in the moving frame of reference in which the wave is stationary,

$$c = -\frac{1}{\lambda}\int_0^\lambda \psi_y(x, -d)\,dx > 0. \tag{11}$$

Several assumptions encompassed in (9), like symmetry or the fact that the free surface is a smooth curve, are not restrictive requirements.

Indeed, assuming the free boundary to be a continuously differentiable curve, a theorem of Lewy [21] ensures that the boundary must be a real-analytic curve and the velocity components have harmonic extensions across it. Moreover, by a result of Spielvogel [30], the free surface has to be the graph of a real-analytic function. The symmetry of the free surface is actually guaranteed if we assume that the wave profile is monotone between crests and troughs [26,7]. The only Stokes waves for which the free surface is not a continuously differentiable curve are the waves of greatest height in which case the curve is symmetric and real-analytic except at the crest where it is just continuous with a corner containing an angle of $2\pi/3$ cf. [2,34].

**2.2. Properties of the velocity field.** The existence of Stokes waves of small and large amplitude (up to the existence of the wave of greatest height) is established using a hodograph transform of (9) cf. [1,2,18,34]. In this paper we assume the existence of a Stokes wave with a non-flat surface profile $\eta$ and we derive some qualitative properties of it, working with the system (9). Without loss of generality, consider Stokes waves of period $2\pi$ with the crest $(0, \eta(0))$ and the trough $(\pi, \eta(\pi))$.

**Lemma 1** [34] *We have*

(12)

$$\psi_x(x, y) < 0, \qquad \frac{d}{dx}\, u(x, \eta(x)) < 0 \quad \textit{for} \quad x \in (0, \pi),\ \ y \in (-d, \eta(x)].$$

*Proof.* For the sake of completeness, we present a short proof. First of all, $\psi_x$ is harmonic in

$$\Omega_0 = \{(x, y) \in \mathbb{R}^2 :\ x \in (0, \pi),\ -d < y < \eta(x)\},$$

with $\psi_x = 0$ on $y = -d$. Since by assumption $\eta'(x) < 0$ on $(0, \pi)$, differentiating the relation $\psi(x, \eta(x)) = 0$ on $(0, \pi)$, we obtain using (10) that $\psi_x < 0$ on the top of $\Omega_0$. The strong maximum principle [15] then forces $\psi_x < 0$ in $\Omega_0$. On the other hand, from (2) we infer by a direct calculation that $P$ is superharmonic:

$$\Delta P = -2\psi_{xy}^2 - 2\psi_{xx}^2 \le 0.$$

Therefore [15] the minimum of $P$ is attained on the flat bed or on the free surface. Since $P_y = -g$ on $y = -d$ by (2) and (5), Hopf's maximum principle [15] ensures that $P > P_0$ below the free surface. Since $\eta'(x) < 0$ on $(0, \pi)$, relation (4) yields by Hopf's maximum principle that $P_x(x, \eta(x)) < 0$ for $x \in (0, \pi)$. But $P_x = (c - u)[u_x + \eta_x u_y]$ on $y = \eta(x)$ in view of (2) and (3), so that $u_x(x, \eta(x)) + \eta'(x) u_y(x, \eta(x)) < 0$ for $x \in (0, \pi)$ in view of (10), which is precisely the missing part of (12). $\square$

**Lemma 2** *The function $u$ decreases strictly as we go from crest to trough along the broken line* $[(0, \eta(0)),\ (0, -d)] \cup [(0, -d),\ (\pi, -d)] \cup [(\pi, -d), (\pi, \eta(\pi))]$.

*Proof.* From Lemma 1 we know that $\psi_x < 0$ in $\Omega_0$. Since $\psi_x = 0$ on $y = -d$, on $x = 0$ and on $x = \pi$, by Hopf's maximum principle [15] we deduce that

$$\psi_{xy}(x, -d) < 0, \qquad x \in (0, \pi), \tag{13}$$

$$\psi_{xx}(0, y) < 0, \qquad y \in (-d, \eta(0)). \tag{14}$$

and

$$\psi_{xx}(\pi, y) > 0, \qquad y \in (-d, \eta(\pi)). \tag{15}$$

Relation (13) yields at once that the function $x \mapsto u(x, -d)$ is strictly decreasing on $(0, \pi)$. For the monotonicity of $u$ on the remaining two open vertical segments, notice that there $v = 0$ so that on $x = 0$ we have $P_y + g = \psi_y \psi_{xx} > 0$ while $P_y + g = \psi_y \psi_{xx} < 0$ on $x = \pi$ in view of (2), (10), and (14), whereas $P + g(y + d) + \frac{\psi_y^2}{2}$ remains constant on both segments by Bernoulli's law. But then $\partial_y\big(\frac{\psi_y^2}{2}\big) = (u - c)u_y < 0$ must hold on the open segment $x = 0$, while $\partial_y\big(\frac{\psi_y^2}{2}\big) = (u - c)u_y > 0$ holds on the open segment $x = \pi$. Using (10) we infer that $u_y(0, y) > 0$ for $y \in (-d, \eta(0))$, whereas $u_y(\pi, y) < 0$ for $y \in (-d, \eta(\pi))$, and the statement follows. □

*Remark* The previous two lemmas in combination with relation (10) show that $u < c$ in the closure $\overline{\Omega_0}$ of $\Omega_0$, unless we deal with a Stokes wave of greatest height (in which case $u = c$ at the crest $(0, \eta(0))$ with $u < c$ at all other points in $\overline{\Omega_0}$ cf. [34]).

**Lemma 3** *The zero level set $\{u = 0\}$ of the function $u$ in $\overline{\Omega_0}$ consists of a curve $\mathfrak{C}$ connecting a point $(x_+, \eta(x_+))$ to a point $(x_-, -d)$ for some $x_-, x_+ \in (0, \pi)$. Each streamline $\psi = \psi_0$ with $\psi_0 \in [m, 0]$ intersects $\mathfrak{C}$ in precisely one point.*

*Proof.* Since (11) ensures that the average of $u$ over the segment $[(0, -d), (\pi, -d)]$ is zero, from Lemma 2 we deduce the existence of a unique point $x_- \in (0, \pi)$ with $u(x_-, -d) = 0$. Furthermore, we have that

$$\begin{cases} u(0, y) > 0, & y \in [-d, \eta(0)], \\ u(x, -d) > 0, & x \in [0, x_-), \\ u(x, -d) < 0, & x \in (x_-, \pi], \\ u(\pi, y) < 0, & y \in [-d, \eta(\pi)]. \end{cases} \tag{16}$$

Since $u(0, \eta(0)) > 0$ while $u(\pi, \eta(\pi)) < 0$, the monotonicity of $u$ along the top boundary of $\Omega_0$ ensured by Lemma 1 guarantees the existence of a unique point $x_+ \in (0, \pi)$ such that $u(x_+, \eta(x_+)) = 0$ while

$$\begin{cases} u(x, \eta(x)) > 0, & x \in [0, x_+), \\ u(x, \eta(x)) < 0, & x \in (x_+, \pi]. \end{cases} \tag{17}$$

The zero level set of the harmonic function $u$ has a simple structure: for a sufficiently small neighborhood $\mathcal{N}(x_0, y_0)$ of a point $(x_0, y_0) \in \Omega_0$ where $u(x_0, y_0) = 0$ there is an integer $n = n(x_0, y_0) \geq 1$ and $n$ analytic curves $\gamma_k : (-1, 1) \to \mathcal{N}(x_0, y_0)$ such that:

(i) $\{u = 0\} \cap \mathcal{N}(x_0, y_0) = \cup_{k=1}^{n} \gamma_k$ where $\gamma_k = \{\gamma_k(t) :\ t \in (-1, 1)\}$;
(ii) $\gamma_k(0) = (x_0, y_0)$ and the angle at $(x_0, y_0)$ between $\gamma_k$ and $\gamma_{k+1}$ is precisely $\frac{\pi}{n}$.

To see this, it suffices to associate to $u$ the analytic function

$$f(z) = u(z) + i\big(v(z) - v(z_0)\big), \qquad z = x + iy,\ z_0 = x_0 + iy_0,$$

chosen such that $f(z_0) = 0$. Clearly $f \not\equiv 0$ so that there exists a unique positive integer $n \geq 1$ with $f(z) = (z - z_0)^n f_1(z)$ in a small neighborhood of $z_0$, where $f_1$ is analytic in that neighborhood and $f_1(z_0)) = Z_0 \neq 0$. Choosing a single-valued branch of $z^{\frac{1}{n}}$ in a neighborhood of $Z_0$, via a conformal transformation we find a function $f_2(z)$ that is analytic in a neighborhood of $z_0$ and satisfies

$$f(z) = [(z - z_0) f_2(z)]^n$$

in that neighborhood. Therefore we can find an analytic homeomorphism $\varphi$ such that $u \circ \varphi(z)$ is precisely the real part of $z^n$ in a neighborhood of $z_0$ and the claims (i)–(ii) follow at once. Extending these local curves to a maximal curve, the properties (i)–(ii) ensure that each maximal curve has its endpoints on the boundary of $\Omega_0$. The mean-value property of harmonic functions shows that these maximal curves do not selfintersect (otherwhise we would get an open set where $u \equiv 0$ hence $u \equiv 0$ throughout $\Omega_0$) and their endpoints are disjoint. The properties (16)–(17) now ensure that there can be only one such maximal curve $\mathcal{C}$. Notice that by (10) and (12) a streamline $\psi = \psi_0$ with $\psi_0 \in (m, 0]$ is the graph of a strictly decreasing smooth function $x \mapsto y(x)$, while $\psi = m$ corresponds to $y = -d$. Since $u(0, y(0)) > 0 > u(\pi, y(\pi))$ by (16), $\mathcal{C}$ intersects $y = y(x)$ in at least one point. This point is unique by the above structural properties of the set $\{u = 0\}$. $\square$

## 3. Particle trajectories

The path $(X(t), Y(t))$ of a particle with location $(X(0), Y(0))$ at time $t = 0$ is given by the solution of the differential system

$$\begin{cases} X' = u(X - ct, Y), \\ Y' = v(X - ct, Y). \end{cases} \tag{18}$$

The corresponding system in the moving frame is the Hamiltonian system

$$\begin{cases} x' = u(x, y) - c, \\ y' = v(x, y), \end{cases} \tag{19}$$

with Hamiltonian function $\psi(x, y)$. The transformation (7) maps solutions of (19) into solutions of (18). Notice that if we do not have a wave of greatest height, then there is some $\varepsilon > 0$ such that $c - u \geq \varepsilon$ on $\Omega$ according to the remark of Sect. 2, so that each solution of (19) starting in $\Omega_0$ intersects the line $x = -\pi$ in finite time in the future and the line $x = \pi$ in finite time in the past. If we deal with a Stokes wave of great height, then this statement holds also true. Indeed, for a solution starting at some point in $\Omega_0$ this follows easily by noticing that a solution of (19) stays on the same level set of $\psi$ and along this level set we have $c - u \geq \varepsilon$ for some $\varepsilon > 0$ in view of the remark in Sect. 2. On the other hand, a solution of (19) starting on the top boundary of $\Omega_0$ reaches the point $(0, \eta(0))$ in finite time since

$$\int_0^{\frac{\pi}{2}} \frac{dx}{c - u(x, \eta(x))} < \infty$$

due to the fact that $c - u(x, \eta(x)) = O(\sqrt{x})$ as $x \downarrow 0$. The last estimate follows from the boundedness of $\eta'(x)$ away from zero as $x \downarrow 0$ and from the inequality

$$(c - u(x, \eta(x)))^2 \leq 2(Q - g[\eta(x) + d]) = O(x) \quad \text{as} \quad x \downarrow 0$$

which is a consequence of the nonlinear boundary condition in (9), with the estimate on the above right-hand side obtained from the mean-value theorem as $Q = g\,[\eta(0) + d]$ in the case of a wave of greatest height. This means that in the case of the wave of greatest height uniqueness fails for the solution of (19) with initial data $(0, \eta(0))$. The physically reasonable solution is not the constant solution as this would mean that particles collide at the crest. Therefore in the moving frame a solution starting on the top of $\Omega_0$ reaches the point $(0, \eta(0))$ in finite time and does not pause there but moves on with a decreasing $x$-coordinate.

**Lemma 4** *Given $y_0 \in [-d, \eta(\pi)]$, let $\theta = \theta(y_0) > 0$ be the time needed for the solution $(x(t), y(t))$ of* (19) *with initial data $(\pi, y_0)$ to intersect the line $x = -\pi$. Then this solution corresponds via* (7) *to a closed particle path if and only if $\theta = \frac{2\pi}{c}$.*

*Proof.* By symmetry we know that the solution intersects the line $x = -\pi$ at the point $(-\pi, y_0)$ so that $y(\theta) = y_0$ while $x(\theta) = -\pi$.

Assume that $\theta = \frac{2\pi}{c}$. Then

$$X(\theta) - X(0) = [x(\theta) + c\theta] - x(0) = 0$$

while

$$(X(\theta) - c\theta) = -\pi = x(0) - 2\pi = X(0) - 2\pi$$

so that the periodicity of the right-hand side of (18) proves sufficiency.

Conversely, if we have a closed path $(X(t), Y(t))$ of (18) of period $\tau > 0$, then $y(0) = y(\tau)$ so that $\tau = n\theta$ for some integer $n \geq 1$. But then, $x(\tau) = x(0) - 2n\pi$ so that $X(0) = X(\tau)$ forces

$$0 = X(\tau) - X(0) = [x(\tau) + c\tau] - x(0) = x(\tau) - x(0) + cn\theta = -2n\pi + cn\theta.$$

Thus $\theta = \frac{2\pi}{c}$ is also necessary. □

Let us now prove the following result.

**Proposition** *There are no closed particle paths in a Stokes wave.*

*Proof.* According to the above considerations it suffices to show that $\theta(y_0) > \frac{2\pi}{c}$ for all $y_0 \in [-d, \eta(\pi)]$.

Let us first choose $y_0 = -d$ and let $(x(t), y(t))$ be the solution of (19) with initial data $(\pi, -d)$. Then $y(t) = -d$ for all $t \geq 0$. Since $u - c < 0$ on $y = -d$, we infer from (19) and from $x(\theta) = -\pi = x(0) - 2\pi$ the identity

$$\int_{-\pi}^{\pi} \frac{dx}{c - u(x, -d)} = \theta.$$

On the other hand, since by Lemma 2 the function $(c - u)$ is monotone on the flat bed, we obtain by the Cauchy-Schwarz inequality that

$$4\pi^2 < \int_{-\pi}^{\pi} \frac{dx}{c - u(x, -d)} \int_{-\pi}^{\pi} (c - u(x, -d))\, dx = 2\pi c \int_{-\pi}^{\pi} \frac{dx}{c - u(x, -d)}$$

in view of (11). We deduce that $\theta(-d) > \frac{2\pi}{c}$.

Consider now the case when $y_0 \in (-d, \eta(\pi)]$. Applying the divergence theorem to the vector field $(\psi_x, \psi_y)$ in the strip

$$\{(x, y) \in \mathbb{R}^2 : \; x \in (-\pi, \pi), \; -d < y < y(x)\},$$

where $y = y(x)$ is the equation of the streamline $\psi = \psi(\pi, y_0)$, we obtain that

$$\int_{-\pi}^{\pi} (c - u(x, -d))\, dx = \int_{-\pi}^{\pi} \sqrt{[c - u(x, y(x))]^2 + v^2(x, y(x))}\, dx.$$

Taking into account (11), we infer that

$$\int_{-\pi}^{\pi} \sqrt{[c - u(x, y(x))]^2 + v^2(x, y(x))}\, dx = 2\pi c$$

and thus Lemma 1 yields

$$\int_{-\pi}^{\pi} [c - u(x, y(x))]\, dx < 2\pi c. \tag{20}$$

On the other hand, since $u - c < 0$ along $y = y(x)$ except perhaps at $(0, y(0))$ in the case of the wave of greatest height with $y_0 = \eta(\pi)$, we obtain

$$\int_{-\pi}^{\pi} \frac{dx}{c - u(x, y(x))} = \theta. \tag{21}$$

Using the Cauchy-Schwarz inequality in combination with (21), we get

$$\begin{aligned}\theta \int_{-\pi}^{\pi} [c - u(x, y(x))]\, dx &= \int_{-\pi}^{\pi} \frac{dx}{c - u(x, y(x))} \int_{-\pi}^{\pi} [c - u(x, y(x))]\, dx \\ &\geq 4\pi^2\end{aligned}$$

so that

$$\theta \geq \frac{4\pi^2}{\int_{-\pi}^{\pi} [c - u(x, y(x))]\, dx} > \frac{2\pi}{c}$$

in view of (20) and the proof is complete. □

The main result of the paper follows now from the following considerations. Along each streamline $y = y(x)$ with $x \in [-\pi, \pi]$ we have

$$u(-\pi, y(-\pi)) = u(\pi, y(\pi)) < 0 < u(0, y(0))$$

with the function $x \mapsto u(x, y(x))$ changing sign at its two roots on $(-\pi, \pi)$. Moreover,

$$v(x, y(x)) = -v(-x, y(-x)) < 0, \qquad x \in (0, \pi),$$

while for any solution of (18) with $X(0) = \pi$ we have

$$\theta > \frac{2\pi}{c}, \qquad X(\theta) = c\theta - \pi > \pi,$$

where $\frac{2\pi}{c}$ is the wave period and $\theta > 0$ is the first time when $Y(\theta) = Y(0)$.

**Theorem** *In a Stokes wave the particle located initially on the bed at $(\pi, -d)$ stays on the flat bed. It first moves to the left, then turns back to the right and before the period elapses it returns to move to the left with its final location after one period to the right of $(\pi, -d)$. A particle located initially at the point $(\pi, y_0)$ above the flat bed moves in a similar way, except that the particle stays above the line $y = y_0$ while a period elapses: the first backward motion and the first part of the subsequent forward motion is upwards, while for the second part of the forward motion and the following backward motion preceding the fulfillment of one period the particle moves downwards. This downward motion continues until the particle reaches the same minimal height $y_0$ above the flat bed at time $\theta(y_0)$ at a point located to the right of $(\pi, y_0)$.*

*Remark* It was known that certain particle trajectories might be as depicted in Fig. 2 – see for example a nice photograph in [24]. In this paper we proved that all trajectories above the flat bed are of this type.

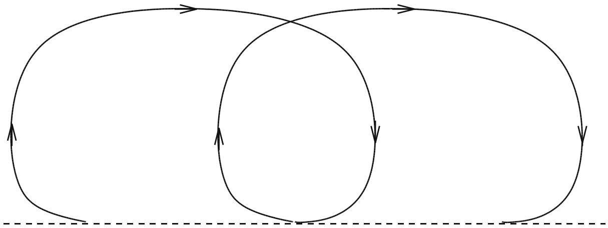

**Fig. 2** The trajectory of a particle located above the flat bed

## 4. Comments

A result similar to the one obtained for finite depth holds true for Stokes waves in water of infinite depth, as the approach pursued above can be carried over. Indeed, the governing equations for irrotational deep-water waves are (1)–(6) with (5) replaced by the condition

$$(5') \qquad (u, v) \to (0, 0) \quad \text{as} \quad y \to -\infty \quad \text{uniformly in} \quad x \in \mathbb{R},$$

expressing that at great depths there is practically no motion. Consequently, the last condition in (9) is replaced by

$$(9') \qquad \nabla\psi \to (0, -c) \quad \text{as} \quad y \to -\infty \quad \text{uniformly in} \quad x \in \mathbb{R},$$

with $\psi < 0$ for $y < \eta(x)$. The results obtained in Lemma 1, Lemma 2, and Lemma 3 continue to hold true as a consequence of the Phragmen-Lindelöf principle [15] with the following obvious modifications: all statements about the flat bed $y = -d$ are replaced by (9′), and in Lemma 3 the curve $\mathfrak{C}$ starts at a point $(x_+, \eta(x_+))$ with $x_+ \in (0, \pi)$, and it extends to $y = -\infty$. From (9′) one can infer that the convergence of $\nabla\psi$ to $(0, -c)$ and that of $\psi(x, y) + cy$ to some constant is (uniformly in $x$) exponentially fast as $y \to -\infty$ (see [34, 8]). In particular, this allows us to apply the divergence theorem to the vector field $\nabla\psi$ in the strip $\{(x, y) \in \mathbb{R}^2 : \; x \in (-\pi, \pi), \; -\infty < y < y(x)\}$, where $x \mapsto y(x)$ is a streamline. The validity of the calculations presented in the Proposition is easily checked (with $y = -d$ being replaced by $y = -\infty$). Therefore all particle trajectories in a deep-water wave are of the type depicted in Fig. 2.

While rigourous results on large amplitude periodic traveling waves with an arbitrary vorticity distribution are available [9, 10] we do not expect that in such rotational flows (with nonvanishing vorticity $\omega = v_x - u_y$) results on the particle trajectories similar to those established here for irrotational flows hold true. Since specific properties of harmonic functions were crucial to our approach, one might think that perhaps different tools could possibly be developed for rotational water waves. However, the only known explicit solution to the governing equations (1)–(5) for water waves, discovered by Gerstner in 1809 [16] and rediscovered by Rankine in 1863 [28] (see the historical survey [12]), has the property that all particles move on circles; Gerstner's wave is in water of infinite depth with nonvanishing vorticity (see [5]). Interestingly, for certain specific nonvanishing vorticities there are even nontrivial three-dimensional explicit solutions to the governing equations for water waves with all particle paths closed [6]. Thus the existence of closed particle trajectories is contingent on the vorticity of the flow.

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## References

1. Amick, C.J., Toland, J.F.: On periodic water-waves and their convergence to solitary waves in the long-wave limit. Philos. Trans. R. Soc. Lond., Ser. A **303**, 633–669 (1981)
2. Amick, C.J., Fraenkel, L.E., Toland, J.F.: On the Stokes conjecture for the wave of extreme form. Acta Math. **148**, 193–214 (1982)
3. Bryson, A.E.: Waves in Fluids. National Committee for Fluid Mechanics Films. Encyclopaedia Britannica Educational Corporation. 425 North Michigan Avenue, Chicago, IL 60611.
4. Buffoni, B., Séré, É., Toland, J.F.: Minimization methods for quasi-linear problems with an application to periodic water waves. SIAM J. Math. Anal. **36**, 1080–1094 (2005)
5. Constantin, A.: On the deep water wave motion. J. Phys. A **34**, 1405–1417 (2001)
6. Constantin, A.: Edge waves along a sloping beach. J. Phys. A **34**, 9723–9731 (2001)
7. Constantin, A., Escher, J.: Symmetry of steady periodic surface water waves with vorticity. J. Fluid Mech. **498**, 171–181 (2004)
8. Constantin, A., Escher, J.: Symmetry of steady deep-water waves with vorticity. European J. Appl. Math. **15**, 755–768 (2004)
9. Constantin, A., Strauss, W.: Exact steady periodic water waves with vorticity. Comm. Pure Appl. Math. **57**, 481–527 (2004)
10. Constantin, A., Sattinger, D., Strauss, W.: Variational formulations for steady water waves with vorticity. J. Fluid. Mech. **548**, 151–163 (2006)
11. Constantin, A., Villari, G.: Particle trajectories in linear water waves. J. Math. Fluid. Mech., to appear.
12. Craik, A.D.D.: The origins of water wave theory. Annu. Rev. Fluid Mech. **36**, 1–28 (2004)
13. Crapper, G.D.: Introduction to Water Waves. Chichester: Ellis Horwood Ltd. 1984
14. Debnath, L.: Nonlinear Water Waves. Boston, MA: Academic Press Inc. 1994
15. Fraenkel, L.E.: An Introduction to Maximum Principles and Symmetry in Elliptic Problems. Cambridge: Cambridge University Press 2000
16. Gerstner, F.: Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile. Ann. Phys. **2**, 412–445 (1809)
17. Johnson, R.S.: A Modern Introduction to the Mathematical Theory of Water Waves. Cambridge: Cambridge Univ. Press 1997
18. Keady, G., Norbury, J.: On the existence theory for irrotational water waves. Math. Proc. Camb. Philos. Soc. **83**, 137–157 (1978)
19. Lamb, H.: Hydrodynamics. Cambridge: Cambridge Univ. Press 1895
20. Levi-Civita, T.: Détermination rigoureuse des ondes permanentes d'ampleur finie. Math. Ann. **93**, 264–314 (1925)
21. Lewy, H.: A note on harmonic functions and a hydrodynamical application. Proc. Am. Math. Soc. **3**, 111–113 (1952)
22. Lighthill, J.: Waves in Fluids. Cambridge: Cambridge Univ. Press 1978
23. Longuet-Higgins, M.S.: The trajectories of particles in steep, symmetric gravity waves. J. Fluid Mech. **94**, 497–517 (1979)
24. Longuet-Higgins, M.S.: Eulerian and Lagrangian aspects of surface waves. J. Fluid Mech. **173**, 683–707 (1986)

25. Milne-Thomson, L.M.: Theoretical Hydrodynamics. London: The Macmillan Co. 1938
26. Okamoto, H., Shōji, M.: The Mathematical Theory of Permanent Progressive Water-waves. River Edge, NJ: World Scientific Publishing Co. Inc. 2001
27. Plotnikov, P.I., Toland, J.F.: Convexity of Stokes waves of extreme form. Arch. Ration. Mech. Anal. **171**, 349–416 (2004)
28. Rankine, W.J.M.: On the exact form of waves near the surface of deep water. Philos. Trans. R. Soc. Lond., Ser. A **153**, 127–138 (1863)
29. Sommerfeld, A.: Mechanics of Deformable Bodies. New York: Academic Press Inc. 1950
30. Spielvogel, E.R.: A variational principle for waves of infinite depth. Arch. Rational Mech. Anal. **39**, 189–205 (1970)
31. Stoker, J.J.: Water Waves. The Mathematical Theory with Applications. New York: Interscience Publ. Inc. 1957
32. Stokes, G.G.: On the theory of oscillatory waves. Trans. Camb. Phil. Soc. **8**, 441–455 (1847)
33. Struik, D.J.: Détermination rigoureuse des ondes irrotationelles périodiques dans un canal à profondeur finie. Math. Ann. **95**, 595–634 (1926)
34. Toland, J.F.: Stokes waves. Topol. Methods Nonlinear Anal. **7**, 1–48 (1996)
35. Toland, J.F.: Stokes waves in Hardy spaces and as distributions. J. Math. Pures Appl. **79**, 901–917 (2000)
36. Yih, C.-S.: The role of drift mass in the kinetic energy and momentum of periodic water waves and sound waves. J. Fluid Mech. **331**, 429–438 (1997)