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Particle trajectory and mass transport of finite-amplitude waves in water of uniform depth

Hsien-Kuo Chang [a], *, Jin-Cheng Liou [b], Ming-Yang Su [c]

[a] Department of Civil Engineering, National Chiao Tung University, Hsinchu, Taiwan
[b] Harbor & Coastal Engineering Department, China Engineering Consultants Inc., Taipei, Taiwan
[c] Oceanographer from US Naval Research Laboratory (retired)

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Abstract

A set of governing equations in Lagrangian form is derived for propagating gravity waves in water of uniform depth. The Lindstedt–Poincaré perturbation method is used to obtain approximations up to fifth order. Recognizing the Lagrangian frequency to be a position function for all particles is a key to find these higher-order approximations. The present solution has zero pressure at the free surface and satisfies exactly the dynamic boundary condition. Under the present approximations, the Lagrangian frequency is composed of two parts. The first part is constant for all particles and equivalent to the term in the fifth-order Stokes' wave theory [J.D. Fenton, A fifth-order Stokes theory for steady waves, J. Waterway, Port, Coastal Ocean Eng. 111 (1985) 216–234]. The second part is a function of the depth. All the particles move as open (nonclosed) loops and have mean drift displacements that decrease exponentially with the water depth. Thus, a new fourth-order mass transport velocity is found.

Keywords: Particle trajectory; Mass transport; Lagrangian approach; Stokes wave

1. Introduction

If a small neutrally buoyant float is placed in a wave tank and its trajectory traced as waves pass by, a small mean motion that is called the mass transport velocity in the direction for the waves can be observed. The closer to the water surface, the greater the tendency for this net motion [1]. Although the mass transport velocity is often week, its persistence can result in the transport of bottom sediments. There are two approaches for examining this mass transport: The Eulerian frame, using a fixed point to measure the mean flux of mass, or the Lagrangian frame, which involves moving with the water particles [2]. In general, the Eulerian method is more convenient in mathematical manipulations than the Lagrangian method. Thus the Eulerian approach is more frequently used in solving fluid dynamic problems [3]. However, the trajectory of particles and mass transport under a wave motion using Eulerian approach is hard to describe.

* Corresponding author. Fax: +886 3 5131487.
E-mail address: hkc@faculty.nctu.edu.tw (H.-K. Chang).

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Since Stokes [1] proposed the well-known Stokes wave theory in the Eulerian system, subsequently some works [4–6] have been carried out in the Eulerian system. One advantage of the Lagrangian formulation is that the total acceleration is linear and the free surface equation is independent of time [7–10]. The low-order Lagrangian approximations have also been applied with some successes to surface gravity waves [11–14]. Osborne et al. [15] finds that the Korteweg–de Vries equation in Lagrangian coordinates can be effectively used in describing the evolution of a nonlinear integral system. The problem of gravity waves in a horizontal bed is a classic but fundamental wave motion. It is an interesting nonlinear problem. In spite of more complicated operations and few techniques developed for the Lagrangian approach, being able to describe the particle motion, is chosen in the present paper for solving this problem.

Miche [16] uses a perturbation technique to solve the Lagrangian equations for first- and second-order surface gravity waves. To the first-order approximation, his results yield a wave profile identical to that of Gerstner's trochoidal wave, whereas such an agreement is not achieved until the third-order approximation in the Eulerian system. Moe et al. [17] develop a second-order theory for the wave motion in a finite water depth in a manner similar to that by Miche [16]. Their results are in manageable algebra up to second order, and predict quite well the behavior of water waves in wave tanks. All the solutions mentioned above do not satisfy exactly the irrotational condition. Buldakov et al. [18] developed an asymptotic formulation for nonlinear water waves in Lagrangian coordinates and obtained the fifth-order approximation for regular traveling waves in deep water and the third-order approximation for standing Faraday waves. Considering the Lagrangian wave frequency varying with water depth, Chen [19] obtains a third-order Lagrangian solution for gravity waves, but his solution cannot be transformed into the existing Eulerian solution, such as Stokes [1] or Fenton [6]. This difficultly has been overcome specifying a condition of surface elevation shown in Section 3.2 and resolved using successive Taylor's expansion in an accompanied paper.

Obtaining the particle trajectory from the Eulerian solutions involves integrating the particle velocity about its mean position over time. Up to now, an approximation up to the third-order is not available for the particle trajectory of nonlinear gravity waves due to the failure of transformation from the Eulerian solution to the Lagrangian solution [20]. Using Taylor expansions about a fixed-point of the velocity of the Eulerian solution and then taking the time average over one wave period to find the mass-transport velocity is extended to the second-order so far [2,20].

A fifth-order Lagrangian approximation by a perturbation technique is derived in this paper to investigate the particle trajectory and mass transport velocity of gravity waves. The nonlinear wave frequency dependence is accounted for and taken as a function of the Lagrangian variables. These steps are essential to obtain the solution in Section 3. The mathematical validity and numerical check are carried out to verify the accuracy of this approximation in Section 4. The wave profile, mass-transport velocity and particle trajectory in Lagrangian form are presented in detail in Section 5.

2. Problem formulation in the Lagrangian system

All dependent variables will be expressed in terms of Lagrangian variables ([illegible]) to designate a label for individual particles rather than an initial position. Variables ([illegible]) denote Cartesian spatial coordinates with the positive [illegible]-axis oriented vertically upwards, referenced from the mean water level. For periodic waves, variables ([illegible]) may be viewed as the average horizontal and vertical particle motion without a horizontal drift over a wavelength or period [17]. Under the assumption that the fluid density, ρ, is constant, the continuity equation can be stated as [21]

$$\frac{1}{\rho}\frac{[illegible]}{[illegible]} = 0 \tag{1}$$

in which [illegible] is the Jacobian of the transformation between ([illegible]) and ([illegible]) variables. Introducing the transformation between the Eulerian coordinates and the Lagrangian coordinates proposed by Euler [22] and Truesdell [23], gives the derivatives

$$\frac{\partial}{\partial [illegible]} = \frac{1}{[illegible]}\frac{\partial ([illegible])}{\partial ([illegible])} \tag{2a}$$

$$\frac{\partial}{\partial [illegible]} = \frac{1}{[illegible]}\frac{\partial ([illegible])}{\partial ([illegible])} \tag{2b}$$

The irrotational flow condition is equivalent to zero average angular velocities of two mutually perpendicular line elements. Using this definition in the Eulerian system in association with the above formula, the irrotational condition in the Lagrangian system is expressed as

$$\frac{1}{\ }\frac{<(\ )}{<(\ )} - \frac{1}{\ }\frac{<(\ )}{<(\ )} = 0, \tag{3}$$

If viscosity is neglected, the momentum equations become [24]

$$+ \quad = - \quad - \frac{1}{\rho}, \tag{4a}$$

$$+ \quad = - \quad - \frac{1}{\rho}, \tag{4b}$$

where is the gravitational acceleration and subscripts denote partial derivatives with respect to the variables and is the total pressure.

In the Gerstner's trochoidal wave any particle moves in a circle whose center is ( ). The spatial mean level corresponding to any particle trajectory over a wave-length, i.e. the level with respect to the same amount of water elevated as depressed at any time, is found to be below the center of the generating circle by Milne and Thomson [25] and Constantin [26]. The physical definition of spatial mean level is given by Milne and Thomson [25]

$$= \frac{1}{\ } \int_0 (\ (\ )\,\mathrm{d}\ , \tag{5}$$

in which is fixed and is the wave lenght. For a regular train of irrotational gravity waves in a uniform water depth any particle at a specified mean level is expected to remain at the same specified mean level after it advances for a wavelength. Thus it is expected that $=$ by the definition of for a wave motion. Furthermore, the free surface can be specified as $= 0$.

These equations must be subjected to the boundary conditions

$$(\ (\ ) = 0, \quad = 0, \tag{6}$$

and

$$(\ (\ ) = 0, \quad = - , \tag{7}$$

where is the water depth. It should be mentioned that the qualities and do not stand for the initial coordinates of a particle, but are simply labeling variables serving to identify a particle. Eq. (6) is the dynamic boundary condition of zero pressure at the free surface when $= 0$ is specified, and (7) is the bottom boundary condition of zero vertical velocity.

The position of a particle departing from equilibrium at any time by a perturbation motion (′) and the hydrostatic pressure separated from the total pressure are written as

$$(\ (\ ) = \ + \ '(\ (\ ), \tag{8a}$$

$$(\ (\ ) = \ + \ '(\ (\ ), \tag{8b}$$

$$(\ (\ ) = -\rho \ + \ (\ (\ ), \tag{8c}$$

Eqs. (8a) and (8b) perform a diffeomorphism from the still water region to the water region, bounded below the rigid bed and above the free water surface [26]. Substituting (8a–c) into (1) and (3)–(7) yields

$$' + ' + \frac{<(\ ')}{<(\ )} + \frac{<(\ ')}{<(\ )} = 0, \tag{9a}$$

$$' - ' + \frac{<(\ ')}{<(\ )} - \frac{<(\ ')}{<(\ )} = 0, \tag{9b}$$

$$\frac{<^2 '}{<^2} + \frac{1}{\rho}\frac{<}{<} + \frac{<'}{<} + '\,' + '\,' = 0, \tag{9c}$$

$$\frac{<^2\ '}{<^2}+\frac{1}{\mathbf{r}}\frac{<}{<}+\frac{<\ '}{<}+{}_{\upsilon}\ '\ {}_{\upsilon}\ '+\ '\ \ '=0 \tag{9d}$$

$$\int_0^{l}\ '\ 1+{}_{\upsilon}\ '\ \mathrm{d}\ =0 \tag{9e}$$

$$=0\qquad =0 \tag{9f}$$

$$'=0\qquad =- \tag{9g}$$

Each of (9a–d) is divided into linear and nonlinear terms. The first two terms in (9a,b) are linear and the Jacobians denoting the products of variables and their derivatives are nonlinear. The first three terms in (9c,d) relating the particle acceleration to external conservative force are linear, and the other two products showing external force due to area deformation are nonlinear parts. The present treatment makes the physical meanings of the governing equations for this problem clearer than the original equations, (1), (3) and (4a,b).

3. Perturbation approximations

Eqs. (9a–e) are nonlinear, and no exact theoretical solutions have been found yet. Ursell [10] proves that the mass transport velocity, denoting a slow drift in the direction of wave propagation and first shown by Stokes [1], is an increasing function of water elevation. Skjelbreia [4] found from experimental observation that the trajectory of a particle is not closed. A particle rotates around the mean position and takes a little more time to move forward. Longuet-Higgins [27] described the difference between the Eulerian period and the Lagrangian period when a regular wave of wavelength propagating with a velocity associated with a mean horizontal velocity or “Stokes drift” is considered. As measured at a fixed vertical line, the apparent period is the Eulerian period that is defined as . The other measurement following a fixed particle gives the Lagrangian period defined as due to the horizontal drift velocity at the free surface. Longuet-Higgins [27] showed that the Lagrangian wave period of particles at the free surface differs from the Eulerian wave period by as much as 38% in deep water. The existence of a slow drift associated with the passage of gravity waves over the surface of an inviscid fluid is proved by Ursell [10] when the wave motion is irrotational. The drift is in the direction of propagation and decreases steadily from the surface towards the bottom. This result indicates that the drift velocity varies with the water level. Based on the above results, it is reasonable to assume that the Lagrangian period is a function of the designated position of each individual particle.

The Lindstedt–Poincaré perturbation technique that can yield uniformly valid expansions is chosen to find the approximations [28]. The wave motion is periodic in time and space. Thus the perturbation variables can be expressed by a series of which each term is a power of n-th order in terms of the dimensionless perturbation parameter and has sinusoidal functions with an argument σ where $\sigma = 2\pi$ is the Lagrangian frequency.

$$_{\upsilon}\ '=\sum_{n=1}^{\infty}{}_{\upsilon n}\ (\ (\ ;\sigma\ ) \tag{10a}$$

$$'=\sum_{n=1}^{\infty}{}_{n}\ (\ (\ ;\sigma\ ) \tag{10b}$$

$$=\sum_{n=1}^{\infty}{}_{n}\ (\ (\ ;\sigma\ ) \tag{10c}$$

$$\sigma=\sum_{n=0}^{\infty}\sigma_{n}\ (\ ) \tag{10d}$$

where the quantities of ${}_{\upsilon n}$, ${}_{n}$, ${}_{n}$ and ρ_n is by an order O n. Inserting (10a–d) into (9) and collecting all terms of equal power result in the governing equations. The zero-order terms balance out exactly.

3.1. First-order approximation

To account for the nonlinear dependence of the wave frequency, σ is explicitly exhibited in the differential equations. Introducing the transformation $= \sigma$ and collecting terms of order O , the first-order governing equations are obtained as

$$\sigma_0 \, {}_1 + {}_1 + \sigma_0 \, {}_1 + \sigma_0 \, {}_1 + \sigma_0 \sigma_0 \, {}_1 + \sigma_0 \, {}_1 = 0 \tag{11a}$$

$$\sigma_0 \, {}_1 - {}_1 + \sigma_0 \, {}_1 - \sigma_0 \, {}_1 + \sigma_0 \sigma_0 \, {}_1 - \sigma_0 \, {}_1 = 0 \tag{11b}$$

$$\sigma^2_0 \, {}_1 + \frac{1}{} \, {}_1 + {}_1 + \sigma_0 \left(\frac{1}{} \, {}_1 + {}_1 \right) = 0 \tag{11c}$$

$$\sigma^2_0 \, {}_1 + \frac{1}{} \, {}_1 + {}_1 + \sigma_0 \left(\frac{1}{} \, {}_1 + {}_1 \right) = 0 \tag{11d}$$

$$\int_0 {}_1 \, \mathrm{d} = 0 \tag{11e}$$

$${}_1 = 0 \qquad = 0 \tag{11f}$$

$${}_1 = 0 \qquad = - \tag{11g}$$

The nonlinear parts of (9a–d) are of higher orders and are dropped from these equations to yield these first-order governing equations. The last terms on the left-hand side of (11a,b), depending on the time , should be set to zero due to the nonresonance assumption. Thus, we get $\sigma_0 = \sigma_0 = 0$. When the wave crest begins from $= 0$ and $= 0$, a trial solution for ${}_1$ and ${}_1$ is found

$${}_1 = -A \cosh + \sin - \tag{12a}$$

$${}_1 = A \sinh + \cos - \tag{12b}$$

where $= 2$ is the wavenumber. Eqs. (12a,b) satisfy exactly both the governing equations (11a,b,e) and the bottom boundary conditions. Setting $= 0$ in (12b) refers to the particle position at the free surface. Therefore, letting ${}_0 = A \sinh$ be the usual amplitude of the surface elevation with a dimension of length and substituting (12a,b) into (11c,d) associated with the free surface boundary condition, yield

$$\sigma^2_0 = \tanh \tag{13}$$

and

$${}_1 = -A \frac{\sinh}{\cosh} \cos - \tag{14}$$

Eq. (13) is the dispersion relation, the same as that of the first-order Stokes wave theory in the Eulerian system, so $\sigma_0 = \sigma_0 \equiv \sigma_0$. Eq. (14) is the solution for the wave dynamic pressure which decreases with water elevation as a function of sinh and satisfies the condition of zero pressure at the free surface. The variables with dimension in the solutions (12a,b) can be nondimensionalized by a scaling with the wave length. Thus

$$0 \approx \tilde{}_1 = {}_0 \cos - \sigma$$

where the tilde denotes a dimensionless variable and we have $= {}_0$. The dimensionless procedure applied to high order solution yields $\approx {}_0 \approx G/2$ where G is the wave height [29].

3.2. Second-order approximation

Collecting terms of order O ^{2} and using the fact that $\sigma_0 = \sigma_0 = 0$, the second-order governing equations are obtained. The term $\rho_0 \, \sigma_1 \, {}_1 + \sigma_1 \, {}_1$ in the continuity equation and the term $\rho_0 \, \sigma_1 \, {}_1 - \sigma_1 \, {}_1$ in the irrotationality equation should be set to zero for the steady-wave of permanent form moving with a constant speed. Thus, $\rho_1 = \rho_1 = 0$ since both υ_1 and ${}_1$ are nonzero. Substituting the first-order approximation into the second-order continuity and irrotationality equations produces two nonhomogeneous equations

$$\rho_0 \upsilon_2 + {}_2 = A^2 \hbar^2 \rho_0 \sin 2\hbar - \tag{15a}$$

$$\rho_0 {}_2 - \upsilon_2 = -A^2 \hbar^2 \rho_0 \sinh 2\hbar + \tag{15b}$$

To satisfy the bottom boundary condition, the general solution for (15a,b) should include a harmonic solution for the homogeneous equation and a particular solution for the nonhomogeneous equation that can be assumed in the form of

$$\upsilon_2 = [-M_{222} \cosh 2\hbar + + {}_{202}] \sin 2\hbar - - M_{211} \cosh \hbar + \sin \hbar - + {}_{220} \cosh 2\hbar + \rho_0 \tag{16a}$$

$$_2 = M_{222} \sinh 2\hbar + \cos 2\hbar - + M_{211} \sinh \hbar + \cos \hbar - \tag{16b}$$

where M_{222} and M_{211} are the coefficients of the harmonic solution, and ${}_{202}$ and ${}_{220}$ are the coefficients of a particular solution. When (16a,b) is substituted into (15a,b), the coefficients ${}_{202}$ and ${}_{220}$ become

$$_{202} = \frac{1}{4} A^2 \hbar \tag{17a}$$

$$_{220} = \frac{1}{2} A^2 \hbar \tag{17b}$$

In order to satisfy the mean water level, a second-order vertical correction is required in (16b) thus yielding

$$_2 = M_{222} \sinh 2\hbar + \cos 2\hbar - + M_{211} \sinh \hbar + \cos \hbar - + \frac{1}{4} A^2 \hbar \sinh 2\hbar + \tag{18}$$

Substituting the first- and second-order solutions for υ and into the momentum equation in the υ-direction and integrating over , the second-order dynamic pressure is given by

$$_2 = \left\{ \frac{\rho_0^2}{2\hbar} \left[-2M_{222} \coth \hbar \sinh 2\hbar + + 4M_{222} \cosh 2\hbar + - \frac{3}{2} A^2 \hbar \right] \cos 2\hbar - + \frac{1}{\hbar} [-M_{211} \coth \hbar \sinh \hbar + \rho_0^2 + \rho_0 M_{211} \rho_0 + 2A \rho_1 \cosh \hbar +] \cos \hbar - \right\} + {}_2 \tag{19}$$

In (19), pb_2 is an integration constant to be determined. Substituting (19) into the momentum equation in the -direction and then integrating with respect to yields

$$pb_2 = -\frac{1}{4} A^2 \rho_0^2 [\coth \hbar \sinh 2\hbar + - \cosh 2\hbar +] + pc_2 \tag{20}$$

Applying the zero pressure condition at the free surface yields

$$M_{222} = \frac{3}{8} \frac{A^2 \hbar}{\sinh^2 \hbar} \tag{21}$$

$$\rho_1 = 0 \tag{22}$$

$$pc_2 = \frac{1}{4} A^2 \rho_0^2 \tag{23}$$

Except for M_{211}, all the coefficients in the second-order approximation are found. This coefficient M_{211} would be uniquely determined by an extra condition. This condition is here chosen to consist equating the Lagrangian solutions to the Eulerian solutions.

We review the existing Stokes wave theories in the Eulerian system and classify them into three kinds. The first kind, such as Isobe et al. [5] and Fenton [6], has a perturbation parameter, $\hbar G$ 2. In their expressions for the surface profile, the sum of the coefficients of equal orders for all odd harmonic components is zero. This feature can be used as an additional condition for determining the coefficient M_{211}. In the second and third kinds, the perturbation parameter is $k\ G$, where G is about one half of the wave height. Skjelbreia and Hendrickson's [4] fifth-order Stokes wave theory belongs to the second kind. The coefficient in the fundamental component of the surface profile is only $k\ G$ without any higher order terms. The third kind, like Dingemans's [29] third-order wave theory, points no amount of higher order terms in the fundamental component of the velocity potential.

Each of these three kinds of Stokes wave theories can provide an additional condition to determine M_{211}. Fenton [6] obtains a fifth-order Stokes theory for periodic waves, which uses the actual wave steepness as an expansion parameter. For application of Fenton's [6] theory of which the wavelength is initially unknown only one nonlinear equation must be solved. However the other two kinds will require the solution of two or three simultaneous equations. Fenton's [6] theory is shown to be quite accurate for waves shorter than 10 times the water depth. Therefore, Fenton's [6] theory became popular for calculating dynamic properties and shoaling of gravity waves [30,31]. In this paper, the first kind of Stokes' wave theory is chosen to demonstrate the procedure for finding M_{211}. Solutions can also be found for the other two kinds of Stokes wave theories following the same procedure. These two kinds of solutions have also been derived by the authors to provide alternative expressions in the Lagrangian system.

In order to find M_{211}, we need to transform the present Lagrangian expression for the wave surface elevation into the Eulerian expression. Let the Lagrangian phase function be $\sigma\ l = k - \rho l$ and the Eulerian phase function be $\sigma = k_{\upsilon} - \rho$, where ρ is the Eulerian frequency. Both phase functions can be related by a phase difference, ν, as

$$\sigma\ l = \sigma + \nu \tag{24}$$

The phase difference can be written to first order as

$$\nu = k - {}_{\upsilon} + \ldots = -k_{\upsilon 1} = k\ A \cosh k + \sin \sigma\ l \tag{25}$$

where $\ldots = \rho - \rho l\ k$ equals zero for the first order approximation. Using a Taylor expansion of $\sigma\ l$ about σ, in (8b) with (12b), leads to

$$k = k + k\ A \sinh k + \cos\sigma - \nu \sin\sigma + k M_{222} \sinh 2k + \cos 2\sigma + k M_{211} \sinh k + \cos\sigma$$
$$+ \frac{1}{4} A^2 k^2 \sinh 2k + \tag{26}$$

Eq. (26) shows a vertical shift and higher order components for the wave profile in the Lagrangian form. Setting $= 0$ in (26), we obtain an alternative expression of the surface profile in the Eulerian form. Collecting the coefficients of order O ^{2} in all odd harmonics components and then setting the sum to zero yields

$$k M_{211} \sinh k = 0 \tag{27}$$

Therefore, the coefficient M_{211} is finally obtained

$$M_{211} = 0 \tag{28}$$

From time to $+ l$, a particle travels a distance that is called the drift or mass transport. The horizontal and vertical components, denoted by ξ_{υ} and ξ, respectively, are given by

$$\xi_{\upsilon} = {}_{\upsilon} ((+ l - {}_{\upsilon} ((= \frac{1}{2} A^2 k \cosh 2k + \rho_0\ l \tag{29a}$$
$$\xi = ((+ l - ((= 0 \tag{29b}$$

The value of (29a) being nonzero implies that a particle will move a horizontal distance ξ_{υ} during time l. Dividing ξ_{υ} by l gives the drift velocity of a particle,

$$\ldots_2 = \frac{1}{2} A^2 k \cosh 2k + \rho_0 \tag{30}$$

Eq. (29b) being zero means that a particle stays at the original elevation after it marches for a period of l.

The second-order Lagrangian solutions are assembled as follows

$$_{\upsilon 2} = A^2 k \left[-\frac{3}{8 \sinh^2 k} \cosh 2k + + \frac{1}{4} \right] \sin 2k - + \frac{1}{2} A^2 k \cosh 2k + \rho_0 \tag{31a}$$

$$_2 = \frac{3}{8} \frac{A^2 k}{\sinh^2 k} \sinh 2k + \cos 2k - + \frac{1}{4} A^2 k \sinh 2k + \tag{31b}$$

$$_2 = A^2 k \left\{ \left[\frac{3\cosh 2k(\ \ +\ )}{4\cosh k\ \ \sinh k} - \frac{3\sinh 2k(\ \ +\ )}{8\sinh^2 k} - \frac{3}{4}\tanh k \right] \cos 2(\ \ -\ ) + \frac{1}{4}\left[\tanh k\ \ \cosh 2k(\ \ +\ ) - \sinh 2k(\ \ +\ ) + \tanh k\ \ \right] \right\}, \tag{31c}$$

$$\rho_1 = 0. \tag{31d}$$

Each of the above $_2$, $_2$, and $_2$ has a second harmonics that propagates with the same speed as the fundamental component. The second term on the right-hand side of (31a) is an aperiodic function increasing linearly in time, implying that a particle marches forward continuously and horizontally in time and does not complete a closed loop like the first-order approximation. The solution for $_2$ includes a term that is a function of only, independent of time and is a second-order vertical correction decaying with depth. The dynamic pressure of the second-order is also zero at the free surface. The third term on the right-hand side of (31c) is a part of the dynamic pressure depending on only and varying with elevation. No correction on the second-order frequency is found. The third- and higher-order are listed in Appendixes A and B for easy reading.

4. Accuracy verification of these approximations

Fenton [6] introduces a convenient numerical method that is a variant of the procedure known as extrapolation to the limit to check new theoretical results. When a perturbed approximation is substituted into a nonlinear governing equation, a residual error of an order O(^{m}) occurs. It is assumed that the residual error can be given by

$$( ) = \ \ ^m + O(\ ^{m+1}), \tag{32}$$

where and m are independent of . The value of m can be obtained from the ratio of the errors computed for two values of by the expression

$$m = \frac{\log(\ _2/\ _1)}{\log(\ _2/\ _1)} + O(\ _1,\ _2), \tag{33}$$

in which $_1$ is the first value of used; and $_2$ is the second. Fenton's [6] procedure is followed in this paper to give an error order index. If such error order indices are greater than 5, the derived approximation is correct up to fifth order. This approximation is obtained by imposing the surface boundary condition for finding the undetermined coefficients. Therefore, the present approximation satisfies only both boundary conditions, but does not satisfy all the governing equations. With the conditions for $/\ = 0.2$, $G/\ = 0.009$, $_1 = k G/2$, $_2 = 2\ _1$, the error order index for each component in every governing equation is computed. The particles were examined at elevation $= 0$ and $= -\ /4$ at time $= 0$ and $= /2$. The error order index for the odd component error is 7 and that for the even components is 6. Therefore, this check confirms that all approximation coefficients are correct up to fifth order.

For an incompressible fluid the invariance condition on the volume of a Lagrangian particle, that is the Jacobian of and with respect to and which fix the position of a particular water particle before the passage of a wave must be independent of time [3,24]:

$$\frac{\partial(\ ,\ )}{\partial(\ ,\ )} = \frac{\partial(\ _0,\ _0)}{\partial(\ ,\ )}, \tag{34}$$

where $_0$ and $_0$ are the initial positions of the particles to which and refer. Eq. (34) is an alternative expression in Lagrangian form is for the mass conservation for an incompressible fluid. Setting the proposed fifth-order approximation at $= 0$ and differentiating $_0$ and $_0$ with respect to and gives the Jacobian, $_0$, for the initial. If the proposed approximations, and , are directly differentiated with respect to and and inserted into (34), we have the Jacobian, , at time . Both and $_0$ are found alike. Alternatively, following the map of changing variables $(\ ,\ ) \mapsto (\ + \rho/k,\ )$ by Constantin [26] has the Jacobians of the transformations, defined by $(\ ,\ ) \mapsto (\ + \ '/\ _0,\ + \ '/\ _0)$ and $(\ ,\ ) \mapsto (\ + \ '/\ ,\ + \ '/\ )$ where $/\ _0 = k$, independent of time and equivalent up to O(^{6}). These results show that the present approximation satisfying (13a) should also satisfy (34) for the same physical interpretation. The detailed comparison can be seen in Appendix C.

The other possible form of the equation of continuity for incompressible flow is $= 1$ when the fluid particle is identified with the coordinates (,) either at initial time or in the undisturbed position [8,24,25]. The position of a particle departing from equilibrium (,) at any time by disturbed components ($'$, $'$) is considered in the present paper for a periodic wave. Then the size of a physical element in undisturbed water and the same element in waves must be equal, i.e. $= 1$ [16,17]. For the proposed fifth-order approximation, both $_0$ and are found simultaneously to approach to unity with a sixth-order error. That $= 1$ is also used to solve some wave-motion problems [18,32,33].

5. Additional results and discussions

5.1. Wave angular frequency

Longuet-Higgins [27] shows that the Lagrangian wave period of the particle at the free surface differs from the Eulerian wave period, i.e.

$$\rho(0) = \rho - k(0) \tag{35}$$

where $\rho(0)$ is the Lagrangian frequency for the particle at the free surface, ρ the Eulerian frequency and (0) the mass transport of the particle at the free surface. Substituting (30), (42), (54e) and (55d) into (35) and setting $= 0$, we obtain

$$\rho \equiv \rho_0 + \omega_2 + \omega_4, \tag{36a}$$

$$\omega_2 = \rho_0 k^2 {}_0^2 \frac{2 + 7S^2}{4(1 - S)^2}, \tag{36b}$$

$$\omega_4 = \rho_0 k^4 {}_0^4 \frac{4 + 32S - 116S^2 - 400S^3 - 71S^4 + 146S^5}{32(1 - S)^5}, \tag{36c}$$

where $S = 1/\cosh 2kh$. The Eulerian frequency obtained is equivalent to that of Fenton's [6] fifth-order Stokes wave theory. The Lagrangian–Eulerian wave frequency relation (35) is applicable only to the particles at the free surface. However, using the present Lagrangian wave frequency and the mass transport velocity at different elevation, we obtain a more general Eulerian wave frequency, (36), for all particles at different elevations. If neglecting the drift velocity, all particles at any location will have a constant period, as predicted by the Stokes wave theory in the Eulerian system.

5.2. Particle trajectory

The Lagrangian solution gives an expression of particle position at any time. The wave parameters $= 0.2$ and $G = 0.08$ is set for the particle trajectory computation at three levels $= 0$, $= -0.075$ and $= -0.15$. The results are shown in Fig. 1 for time duration of five times (0), which is the wave period of a particle at the free surface. These particle trajectories display nonclosed loops of different shapes and magnitudes at different depths. Both the horizontal and vertical excursions of a particle are functions of elevation. The vertical excursion increases with elevation and becomes zero at the bottom. Thus the particle trajectory near the mean water level displays comparable excursions in the horizontal and vertical components. However, near the bottom, the trajectory shows a thin and flat loop due to the larger horizontal excursion and smaller vertical excursion. The nonclosed particle trajectory is shown in Fig. 1 to give a drift, after one wave period. The mass-transport velocity will be discussed in detail in the next subsection. For the same time duration, particles near the bottom describe more loops than those near the surface because particles at different elevations have different frequencies. For this computation the fifth-order solution has a greater horizontal drift than the third-order solution at the free surface. Conversely, the fifth-order solution has a smaller horizontal shift than the third-order at elevations of $= -0.075$ and $= -0.15$.

5.3. Mass transport velocity

The present Lagrangian solution contains aperiodic terms in the even order solutions that relate directly to the mass-transport velocity. Longuet-Higgins [34] and Ursell [10] have presented a rigorous proof for the mass transport velocity to have a zero net transport of water. Longuet-Higgins [34] presents an exact theory for the mass transport velocity but only for the particles at the free surface in deep water. A comparison of the present solution with

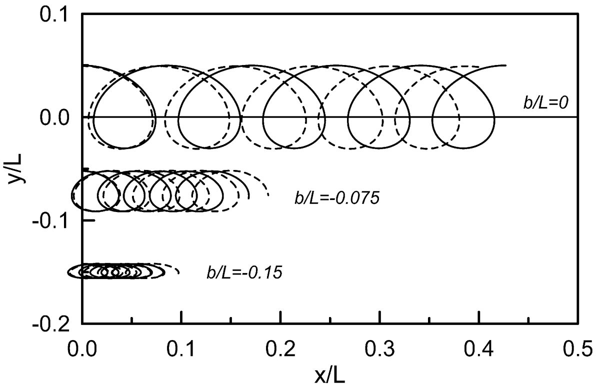

Fig. 1. The trajectories of particles at three elevations for the wave condition $h/L = 0.2$ and $H/L = 0.08$ ($---$, third-order; —, fifth order approximations).

Table 1
A comparison of the mass-transport velocity at the free surface of an irrotational gravity wave in deep water

$kH/2$	U/c	$kU/(\varepsilon\omega_0)$	Error (%)
0.1	0.01005	0.01005	0
0.2	0.04090	0.04080	0.24
0.3	0.09558	0.09405	1.6
0.35	0.13491	0.13001	3.63
0.4	0.18797	0.17280	8.07
0.42	0.21779	0.19196	11.86
0.44316	0.29882	0.21568	27.82

Longuet-Higgins [34] result, for deep water ($h \to \infty$) and $y = 0$, is listed in Table 1. The second column is the non-dimensionalized mass transport velocity obtained by Longuet-Higgins [31]. The third column is the present result. The last column is the relative error of both results. Even for a rather steep wave ($kH/2 \approx 0.35$), the present solution deviates from Longuet-Higgins exact solution only by 3.63%. For waves close to the limiting Stokes wave, the present solution has a relative error of 27.82%.

If the total horizontal transport is assumed to be zero for the case of wave experiments in a tank, then a modified mass-transport velocity, U^*, is given by

$$U^* = \varepsilon^2 U_2 - \frac{\varepsilon_0^2 \omega_0}{2}\coth kh + \varepsilon^4 U_4 + \frac{\varepsilon_0^4 k^2 \omega_0 (21\cosh kh + 9\cosh 3kh + 5\cosh 5kh + \cosh 7kh)}{256\, \sinh^7 kh}. \tag{37}$$

The first two terms above are the second order mass-transport velocity corrections which are the same as those obtained by Longuet-Higgins [20]. The last two terms are the fourth-order mass transport velocity correction, found in the present paper. Fig. 2 shows the mass-transport velocity profile for a wave of parameters $h/L = 0.2$ and $H/L = 0.114$. Both the second-order and the fourth-order mass transport velocities display monotonous decay from the surface to the bottom.

Differentiating (37) with respect to y yields

$$\frac{dU^*}{dy} = A^2 k^2 \omega_0 \sinh 2k(y+h) + \frac{A^4 k^4 \omega_0}{64\sinh^4 kh}\big[-(24 + 7\cosh 2kh + 20\cosh 4kh + 3\cosh 6kh)\sinh 2k(y+h) + 16(8\cosh 2kh + \cosh 4kh)\sinh 2k(y+h)\cosh 2k(y+h)\big]. \tag{38}$$

The first term above is the gradient of the second-order mass transport velocity and has a positive value for all particles from surface to bottom. This indicates that the second-order mass transport velocity is a nondecreasing function of distance from the bottom and has a zero gradient at the bottom. The other terms are the fourth-order corrections. Since

$$\cosh 2k(y+h) \geqslant 1,$$

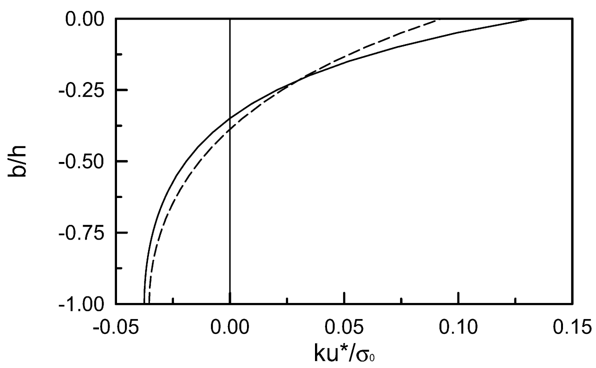

Fig. 2. The mass-transport velocity profiles of zero mass flux for the wave condition $h/L = 0.2$ and $H/L = 0.114$ $---$, second-order solution; —, fourth-order solution).

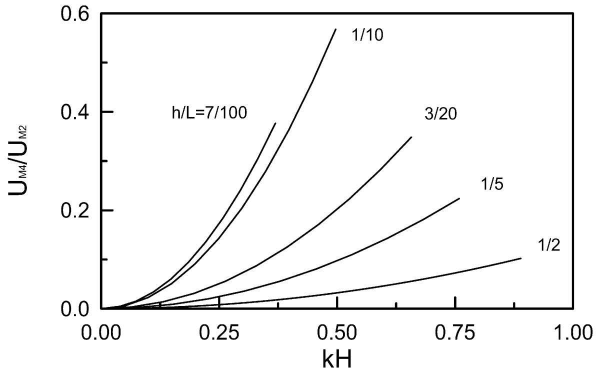

Fig. 3. The ratio of the fourth-order to second-order mass transport velocities of particles at the free surface.

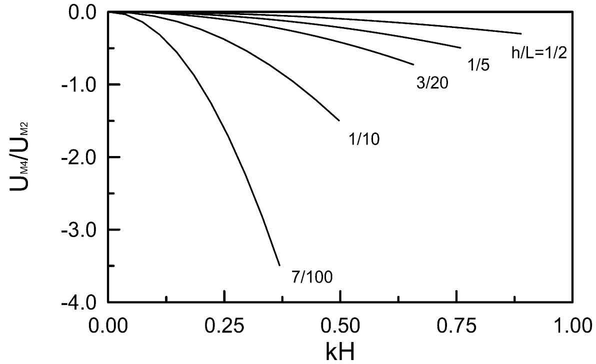

Fig. 4. The ratio of the fourth-order to the second-order mass transport velocity of the particle at the bottom.

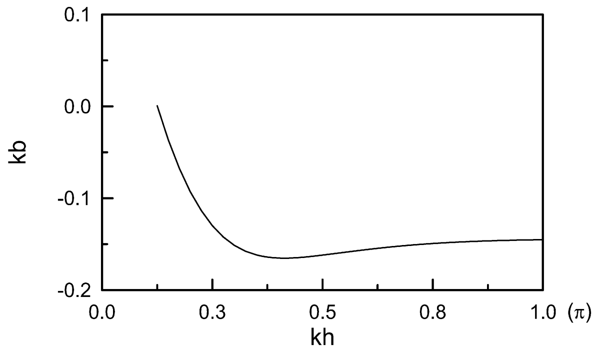

Fig. 5. The elevation of zero fourth-order mass transport corrections.

and retaining this minimum value of $\cosh 2k(h + b)$, it can be proved that (38) is positive, implying that the fourth-order mass transport velocity has also a positive gradient in any depth for all waveheights. These features agree with the theorem that the mass transport velocity decreases with increasing depth, as proposed by Ursell [10]. The nonlinear interactions among particles near the surface are stronger due to the larger particle velocity which enhances mass transport velocity. Thus, the fourth-order solution has a larger mass-transport velocity at the surface than the second-order solution. On the other hand, the fourth-order solution has a smaller mass-transport velocity than the second-order near the bottom in accordance with the conservation of mass.

Fig. 3 shows the ratios of the fourth-order mass transport velocity of surface particles to the second-order one at several water depths. At each water depth, the limiting wave is determined by Miche's [16] criterion. Fig. 3 shows that the mass transport velocity ratio increases with the wave height. In spite of the fact that this ratio is about 0.102 for the limiting wave in deep water ($h/L = 1/2$), this ratio may exceed 50% for large waves at $h/L = 1/10$. For short waves at $h/L > 7/100$, the mass transport velocity ratio decreases with water depth. On the contrary, this ratio increases with water depth for long waves at $h/L < 7/100$. Therefore, the ratio being zero indicates a mass transport velocity without fourth-order corrections. Letting the fourth-order corrections on the mass transport velocity to be zero in (37), the criterion $h/L \approx 0.0627$ is found.

For bottom particles, the ratio of the fourth-order mass transport velocity to the second-order is shown in Fig. 4. The mass transport velocity ratio at all water depths is negative and monotonically decreases with both the wave height and the water depth.

In Fig. 2, there is one elevation at which the second-order mass transport velocity is equal to the fourth-order one. This indicates that $\bar{u}_4$ vanishes at this elevation which can be found from (37) as

$$kb = -kh + \frac{1}{2}\cosh^{-1}\Bigg(\frac{1}{32(8\cosh 2kh + \cosh 4kh)}\big(48 + 14\cosh 2kh + 40\cosh 4kh + 6\cosh 6kh + \big(6(3310 + 1040\cosh 2kh + 3401\cosh 4kh + 872\cosh 6kh + 190\cosh 8kh + 40\cosh 10kh + 3\cosh 12kh)\big)^{1/2}\big)\Bigg) \tag{39}$$

and shown in Fig. 5. The elevation with zero fourth-order mass transport correction is independent of the wave height. In deep water, this elevation is about -0.1451 and reaches a minimum value of -0.1651 at $kh = 0.425$. Beyond this value, the elevation rises rapidly to the surface as the water depth decreases.

6. Conclusions

The Lagrangian description of particle motions within propagating gravity waves over a horizontal bed is investigated in the present paper. The governing equations in the Lagrangian system are derived from the continuity equation and the irrotationality equation in the Eulerian system through a Lagrangian–Eulerian transformation. Recognizing that the particle frequency is a function of position, unlike the Eulerian solution in usual perturbation techniques, is crucial to obtain the fifth-order approximation.

Four governing equations of the fifth-order solution are numerically checked following Fenton [6] up fifth-order. The particle frequency in the Lagrangian solution consists of two parts. The first part is a constant and is equivalent to the frequency of the Stokes wave theory in the Eulerian system, while the second part is a function of the particle elevation. Both parts are modified only in the odd order approximations. The Lagrangian and Eulerian wave frequencies for all particles at any elevation obey a relationship which is more general than the expression given by Longuet-Higgins [27] for surface particles. The particle frequency near the surface is lower than that near the bottom. Thus, the particles near the surface move longer distance over one cycle than those near the bottom.

The Lagrangian wave profile is shown to have a vertical shift and higher order components. These terms originate from the fact that the wave profiles for the odd-order approximations lie below the higher order approximations, and that the even-order approximations require modifications in the vertical direction for the conservation of mass. The present approximation satisfies the dynamic boundary condition of zero pressure at the free surface. The dynamic pressure decreases with depth as a hyperbolic function and is more accurate than those of previous Eulerian formulations, and consequently is a more accurate description of the nonclosed-loop particle trajectory, in which a particle marches forward a horizontal distance over each wave period with largest excursions at the free surface. While the previous mass-transport velocity shows only a second-order correction, the fifth-order solution can give an additional fourth-order mass-transport velocity correction. The fourth-order mass transport velocity of surface particles has only slight disparity from Longuet-Higgins's exact solution for deep water by 3.64%, even for fairly steep wave ($kH/2 \approx 0.35$). The fourth-order mass transport velocity is proved rigorously to decay monotonically from the surface to the bottom and has a zero gradient at the bottom.

For the case of zero horizontal transport, the mass transport velocity of surface particles is positive, and become negative for bottom particles. The fourth-order mass transport velocity is larger than the second-order one by over 50% for steep waves at $H/L = 1/10$. The mass transport velocity at the free surface increases with the wave height. On the other hand, the second-order mass transport velocity of bottom particles is larger than the fourth-order one. The effects of the water depth and the wave height on the mass transport velocity can be evaluated by the present solution for irrotational gravity waves.

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Appendix A. Third-order approximation

Following the same method as Section 3.2, collecting terms of order $O(\varepsilon^3)$ in the governing equations associated with the first- and second-order approximations yields the third-order governing equations. The terms that increase linearly with time being set to zero gives

$$\frac{\partial \sigma_2}{\partial a} = 0, \tag{40a}$$

$$\frac{\partial \sigma_2}{\partial b} = -A^2k^3\sigma_0 \sinh 2k(b+h). \tag{40b}$$

Eq. (40a) shows that σ_2 is independent of a. Integrating equation (40b) over b leads to the second-order Lagrangian frequency

$$\sigma_2 = -\frac{1}{2}A^2k^2\sigma_0 \cosh 2k(b+h) + \omega_2. \tag{41}$$

It consists of two parts: the first varying monotonically with the elevation and reaching a maximum value of $-A^2k^2\sigma_0/2$ at the bottom, the second, ω_2, being an undetermined constant for all particles.

Substituting the first- and second-order approximations into the continuity equation and the irrotationality equation yields

$$\sigma_0\left(\frac{\partial x_3}{\partial a} + \frac{\partial y_3}{\partial b}\right) = \frac{A^3k^3\sigma_0}{\sinh^2 kh}\left[\frac{1}{16}\bigl(10+2\cosh 2k(b+h)\bigr)\cosh 3k(b+h)\sin k(a-\sigma t) - \frac{3}{8}\bigl(-7+\cosh 2k(b+h)\bigr)\cosh k(b+h)\sin 3k(a-\sigma t)\right], \tag{42a}$$

$$\sigma_0\left(\frac{\partial y_3}{\partial a} - \frac{\partial x_3}{\partial b}\right) = \frac{A^3k^3\sigma_0}{\sinh^2 kh}\left[-\frac{1}{16}\bigl(26+10\cosh 2k(b+h)\bigr)\sinh 3k(b+h)\cos k(a-\sigma t) - \frac{1}{8}\bigl(5+\cosh 2k(b+h)\bigr)\sinh k(b+h)\cos 3k(a-\sigma t)\right]. \tag{42b}$$

A trial solution for x_3 and y_3 is given as

$$x_3 = \bigl[-\beta_{333}\cosh 3k(b+h) + \alpha_{313}\cosh k(b+h)\bigr]\sin 3k(a-\sigma t) + \bigl[\alpha_{331}\cosh 3k(b+h) - \beta_{311}\cosh k(b+h)\bigr]\sin k(a-\sigma t), \tag{43a}$$

$$y_3 = \bigl[\beta_{333}\sinh 3k(b+h) + \beta_{313}\sinh k(b+h)\bigr]\cos 3k(a-\sigma t) + \bigl[\beta_{331}\sinh 3k(b+h) + \beta_{311}\sinh k(b+h)\bigr]\cos k(a-\sigma t), \tag{43b}$$

which satisfies the bottom boundary condition and is compatible with the functions on the right-hand side of (42a,b). For solving these four coefficients of the particular solution, inserting (43a,b) into (42a,b) gives

$$\alpha_{313} = \frac{1}{48}\frac{A^3k^2}{\sinh^2 kh}(17 - 2\cosh 2kh), \tag{44a}$$

$$\alpha_{331} = -\frac{1}{16}\frac{A^3k^2}{\sinh^2 kh}(11 + 4\cosh 2kh), \tag{44b}$$

$$\beta_{313} = -\frac{3}{16}\frac{A^3k^2}{\sinh^2 kh}, \tag{44c}$$

$$\beta_{331} = \frac{1}{16}\frac{A^3k^2}{\sinh^2 kh}(7 + 2\cosh 2kh). \tag{44d}$$

No vertical third-order correction is required, after we apply the condition of the mean water level depth. β_{333} and β_{311} remain to be determined. Substituting (43a,b) and (44a–d) into the third-order momentum equations, and then integrating with respect to a and b, respectively, leads to the third-order dynamic pressure with an integral constant being a function of time only

$$
\begin{aligned}
{}_3 = \ & \Big\{\Big[3 M_{333} \tanh \hbar \cosh 3\hbar + -M_{333}\sinh 3\hbar + \\
&+ \frac{1}{8}\frac{A^3\hbar^2}{\sinh 2\hbar}(-25+4\cosh 2\hbar)\cosh\hbar + + \frac{3}{16}\frac{A^3\hbar^2}{\sinh^2\hbar}\sinh\hbar + \Big]\cos 3\hbar - \rho \\
&+ \Big[\frac{9}{8}\frac{A^3\hbar^2}{\sinh 2\hbar}\cosh 3\hbar + - \frac{1}{16}\frac{A^3\hbar^2}{\sinh^2\hbar}(7+2\cosh 2\hbar)\sinh 3\hbar + \\
&+ \left(-\frac{1}{4}\tanh\hbar\,(A^3\hbar^2 - 4M_{311}) + \frac{2A\omega_2\rho_0}{\hbar}\right)\cosh\hbar + \\
&- M_{311}\sinh\hbar + \Big]\cos\hbar - \rho \Big\} + pc_3 \qquad (45)
\end{aligned}
$$

Here, there are four undetermined coefficients, M_{333}, M_{311}, ω_2 and pc_3. Applying zero pressure at the free surface to (45) yields

$$M_{333} = -\frac{1}{64}\frac{A^3\hbar^2}{\sinh^4\hbar}(-11+2\cosh 2\hbar), \qquad (46)$$

$$\omega_2 = \frac{1}{16}\frac{A^2\hbar^2\rho_0}{\sinh^2\hbar}(8+\cosh 4\hbar), \qquad (47)$$

$$pc_3 = 0. \qquad (48)$$

Using the phase difference, ν, in (24) written up to third order and expanding about up to third order, the surface profile to third order becomes

$$
\begin{aligned}
\hbar = \ & \hbar + \hbar A\sinh\hbar + \left(\cos - \nu\sin - \frac{1}{2}\nu^2\cos\right) + \hbar M_{222}\sinh 2\hbar + \cos 2 - 2\nu\sin 2 \\
&+ \frac{1}{4}A^2\hbar^2\sinh 2\hbar + + \hbar M_{333}\sinh 3\hbar + \cos 3 + \hbar M_{311}\sinh\hbar + \cos \\
&+ \hbar M_{313}\sinh\hbar + \cos 3 + \hbar M_{331}\sinh 3\hbar + \cos \qquad (49)
\end{aligned}
$$

and

$$\nu = -\hbar\,{}_{1} + {}_{2} - {}_{2} = \hbar A\cosh\hbar + \sin + \hbar M_{222}\cosh 2\hbar + \sin 2 - \hbar\,{}_{202}\sin 2. \qquad (50)$$

Setting $= 0$ in (49), the additional condition for the surface profile gives

$$M_{311} = -\frac{1}{64}\frac{A^3\hbar^2}{\sinh^4\hbar}(13+2\cosh 2\hbar + 10\cosh 4\hbar + 2\cosh 6\hbar). \qquad (51)$$

Finally, the third-order Lagrangian solutions are listed as follows

$$ {}_3 = \sum_{=0}^{1}\sum_{m=0}^{1} {}_{3\,2+1\,2m+1}\cosh(2+1)\hbar + \sin(2m+1)\hbar - \rho, \qquad (52a)$$

$$ {}_3 = \sum_{=0}^{1}\sum_{m=0}^{1} M_{3\,2+1\,2m+1}\sinh(2+1)\hbar + \cos(2m+1)\hbar - \rho, \qquad (52b)$$

$$
\begin{aligned}
{}_3 = \ & \sum_{=0}^{1}\sum_{m=0}^{1}\big[{}_{3\,2+1\,2m+1}\cosh(2+1)\hbar + \\
&+ {}_{3\,2+1\,2m+1}\sinh(2+1)\hbar + \big]\cos(2m+1)\hbar - \rho, \qquad (52c)
\end{aligned}
$$

$$\rho_2 = \sum_{=0}^{1} {}_{3\,2}\cosh 2\hbar + \rho_0. \qquad (52d)$$

The above coefficients $_{3,m}$, $_{3,m}$, $_{3,m}$, $_{3,m}$ and $_{2,}$ are listed below. The third-order solutions for $_3$, $_3$ and $_3$ are periodic functions that are combinations of both the fundamental and third harmonics, but have no time-dependent terms like the second-order solution. This implies that neither third-order modification of the time-averaged horizontal drift nor a vertical correction on the mean level is needed. However, the Lagrangian frequency still has a second-order correction.

In order to simplify the algebraic expressions, we define $_{m} = \sinh m k$ and $_{m} = \cosh m k$, where m is the power order of the hyperbolic sine/cosine function and is the multiplier of the product k.

$$_{311} = \frac{A^3 k^2}{64 \quad ^4 1} \; 13 + 2 \quad 2 + 10 \quad 4 + 2 \quad 6, \tag{53a}$$

$$_{313} = \frac{A^3 k^2}{48 \quad ^2 1} \; 17 - 2 \quad 2, \tag{53b}$$

$$_{331} = -\frac{A^3 k^2}{16 \quad ^2 1} \; 11 + 4 \quad 2, \tag{53c}$$

$$_{333} = \frac{A^3 k^2}{64 \quad ^4 1} \; -11 + 2 \quad 2, \tag{53d}$$

$$_{311} = -\; _{311}, \tag{53e}$$

$$_{313} = -\frac{3A^3 k^2}{16 \quad ^2 1}, \tag{53f}$$

$$_{331} = \frac{A^3 k^2}{16 \quad ^2 1} \; 7 + 2 \quad 2, \tag{53g}$$

$$_{333} = -\; _{333}, \tag{53h}$$

$$_{311} = -\frac{A^3 k^2}{64 \quad 1 \quad ^3 1} \; 51 - 40 \quad 2 + 16 \quad 4, \tag{53i}$$

$$_{313} = \frac{A^3 k^2}{16 \quad 1 \quad 1} \; -25 + 4 \quad 2, \tag{53j}$$

$$_{331} = \frac{9A^3 k^2}{16 \quad 1 \quad 1}, \tag{53k}$$

$$_{333} = -\frac{3A^3 k^2}{64 \quad 1 \quad ^3 1} \; -11 + 2 \quad 2, \tag{53l}$$

$$_{311} = \frac{A^3 k^2}{64 \quad ^4 1} \; 13 + 2 \quad 2 + 10 \quad 4 + 2 \quad 6, \tag{53m}$$

$$_{313} = \frac{3A^3 k^2}{16 \quad ^2 1}, \tag{53n}$$

$$_{331} = -\frac{A^3 k^2}{16 \quad ^2 1} \; 7 + 2 \quad 2, \tag{53o}$$

$$_{333} = \frac{A^3 k^2}{64 \quad ^4 1} \; -11 + 2 \quad 2, \tag{53p}$$

$$_{20} = \frac{A^2 k^2}{16 \quad ^2 1} \; 8 + \quad 4, \tag{53q}$$

$$_{22} = -\frac{1}{2} A^2 k^2. \tag{53r}$$

Appendix B. Fourth-order and fifth-order approximations

Following the same derivation procedures as above, the fourth-order approximations in Lagrangian form are

$$_4 = \sum_{=0}^{2}\sum_{m=1}^{2} \; 4\,2\,(2_m \cosh 2 \;\; + \sin 2_m \;\; - \;\; + \sum_{=1}^{2} \; 4\,2\,(0 \cosh 2 \;\; + \rho_0 \qquad (54a)$$

$$_4 = \sum_{=1}^{2}\sum_{m=1}^{2} 4\,2\,(2_m \sinh 2 \;\; + \cos 2_m \;\; - \;\; + \sum_{=1}^{2} 4\,2\,(0 \sinh 2 \;\; + \qquad (54b)$$

$$_4 = \sum_{=0}^{2}\sum_{m=0}^{2} 4\,2\,(2_m \cosh 2 \;\; + \cos 2_m \;\; - $$

$$+ \sum_{=1}^{2}\sum_{m=0}^{2} 4\,2\,(2_m \sinh 2 \;\; + \cos 2_m \;\; - \qquad (54c)$$

$$\rho_3 = 0 \qquad (54d)$$

$$_4 = \sum_{=1}^{2} \; 4\,2\,(0 \cosh 2 \;\; + \rho_0 \qquad (54e)$$

Eq. (54d) shows that the fourth-order approximation, like the second-order approximation, has no frequency correction. The expressions of $_4$ and $_4$ are periodic functions made of both the second and fourth harmonics and contain terms depending linearly on time and elevation. The last term on the right-hand side of (54a) is a linearly time-dependent term denoting a correction for the fourth order horizontal drift for a particle motion time-averaged over one wave period. The last two terms on the right-hand side of $_4$, depending upon the elevation, show a fourth order mean elevation correction. These two corrections, occurring only in the even order approximations, are new results from the present derivations.

The fifth-order approximation expressions are

$$_5 = \sum_{=0}^{2}\sum_{m=0}^{2} \; 5\,2 \; + 1\,2_m{+}1 \cosh\, 2 \;\; + 1 \;\; + \sin\, 2_m + 1 \;\; - \qquad (55a)$$

$$_5 = \sum_{=0}^{2}\sum_{m=0}^{2} 5\,2 \; + 1\,2_m{+}1 \sinh\, 2 \;\; + 1 \;\; + \cos\, 2_m + 1 \;\; - \qquad (55b)$$

$$_5 = \sum_{=0}^{2}\sum_{m=0}^{2}\big[\; 5\,2 \; + 1\,2_m{+}1 \cosh\, 2 \;\; + 1 \;\; + $$

$$+ \; 5\,2 \; + 1\,2_m{+}1 \sinh\, 2 \;\; + 1 \;\; + \big] \cos\, 2_m + 1 \;\; - \qquad (55c)$$

$$\rho_4 = \sum_{=0}^{2} \; 4\,2 \; \cosh 2 \;\; + \rho_0 \qquad (55d)$$

The number of terms in the fifth-order approximation increases rapidly. However, $_5$ and $_5$ are composed of the odd harmonics and have no time-averaged horizontal corrections and no vertical mean elevation corrections. The fifth-order dynamic pressure satisfies the free surface boundary condition and decays with water elevation. Eq. (55d) still has a fourth-order frequency correction that consists of one term varying with mean elevation and a constant for all particles and a constant correction term, that is identical to the term obtained by Fenton [6]. The coefficients in the fourth and fifth-order approximations are too numerous to be listed in this paper.

Appendix C. The invariance of Jacobian for periodic waves

For an incompressible fluid the invariance condition on the volume of a Lagrangian particle, that is the Jacobian of $_{\upsilon}$ and with respect to and , must be independent of time [3,24]. The existing first- and second-order wave theories [16,17] in Lagrangian form are first examined for the invariance of Jacobian.

1. For the first-order approximation:

The Lagrangian first-order approximation was given by Miche [16] and Moe and Arntsen [17] as follows:

$$_{\upsilon} = \ + \partial \frac{\cosh \ - }{\sinh} \sin \rho \ - \qquad (56a)$$

and

$$= \ - \partial \frac{\sinh \ - }{\sinh} \cos \rho \ - \qquad (56b)$$

Inserting (56a) and (56b) into (34) yields the Jacobian

$$= 1 - \frac{^2 \partial^2}{2} \frac{\cos 2 \ \rho \ - \ + \cosh 2 \ - }{\sinh^2} \qquad (57)$$

If we take the zeroth-order approximation under the condition of no wave, i.e. $_{\upsilon} =$ and $=$, thus the Jacobian exactly equals 1. (57) indicates a Jacobian approaching to unity with a second-order error. For deep water (56a) and (56b) are degenerated into the Gerstner's wave theory [24–26] and (57) has $= 1 - \partial^2$ that is time independent and satisfies exactly the invariance of Jacobian. However, the particle motion in Gerstner's waves having nonzero vorticity is regarded rotational. Accordingly the Gerstner's wave theory is not commonly used in wave mechanics when water waves are generally considered irrotational.

2. For the second-order wave theory:

The symbolic notations in Miche's second-order solution are changed into the present ones to have $_{\upsilon}$, , , G , , $\theta \rightarrow$, , ∂ , , , . Thus the expressions of Miche's solution [16] can be rewritten as

$$_{\upsilon} = \ + \partial \frac{\cosh \ - }{\sinh} \sin \rho \ - \ - \partial^2 \left(1 - \frac{3\cosh 2 \ - }{2\sinh^2}\right) \frac{\sin 2 \ \rho \ - }{4\sinh^2} + \partial^2 \qquad (58a)$$

and

$$= \ - \partial \frac{\sinh \ - }{\sinh} \cos \rho \ - \ - \partial^2 \left(1 + \frac{3\cos 2 \ \rho \ - }{2\sinh^2}\right) \frac{\sinh 2 \ - }{4\sinh^2} \qquad (58b)$$

Introducing (58a) and (58b) in (34) the Jacobian becomes

$$= 1 - \frac{^3 \partial^3}{8} \frac{\cosh \ - }{\sinh^5} \cos \rho \ - \ [\cosh 2 \ + \cosh 2 \ - \ - 2 \ 6 - 7\cos 2 \ \rho \ - $$
$$+ \cosh 2 \ \cos 2 \ \rho \ - \ - 5\cosh 2 \ - \ ] - \frac{\partial^3}{\sinh} \frac{\sin \rho \ - }{\sinh \ - } \frac{\mathrm{d}}{\mathrm{d}} + \mathrm{O}(^4 \partial^4) \qquad (59)$$

If $= 0$ is set in (58a) and (58b) the Jacobian for the initial time is then obtained as

$$_0 = 1 - \frac{^3 \partial^3}{8} \frac{\cosh \ - }{\sinh^5} \cos \ [\cosh 2 \ + \cosh 2 \ - \ - 2 \ 6 - 7\cos 2 \ + \cosh 2 \ \cos 2$$
$$- 5\cosh 2 \ - \ ] + \mathrm{O}(^4 \partial^4) \qquad (60)$$

Comparing (60) with (59) shows that (60) can be achieved substituting $= 0$ in (59) and $\approx \ _0 \approx 1 + \mathrm{O} \ ^3$.

3. For the proposed approximation:

The proposed approximation can be rewritten as

$$_{\upsilon} \ = \ + _{\upsilon} \ + \qquad (61a)$$

$$= \ + \qquad (61b)$$

and

$$\rho \quad = \rho_0 + \rho_3 \quad + \rho_5 \qquad (61c)$$

where $= _2 + _4$ is the mass transport velocity of a particle, and and denote the parts of the proposed approximation in a periodic function with an argument $= - \rho$. Inserting (61a)–(61c) into (34) yields the Jacobian that can be separated into two determinants after some arrangement

$$= \begin{vmatrix} 1+ & \\ & 1+ \end{vmatrix} + \begin{vmatrix} 1+ & \rho + \\ & \rho \end{vmatrix} \qquad (62)$$

where

$$\begin{vmatrix} 1+ & \\ & 1+ \end{vmatrix} = 1 + \quad + \quad + \quad - \qquad (63a)$$

and

$$\begin{vmatrix} 1+ & \rho + \\ & \rho \end{vmatrix} = \rho - \quad + \quad \rho - \quad \rho \qquad (63b)$$

Substituting the proposed fifth-order approximation into (63a) yields a value approaching to unity with an error of O ^{6}. Inserting the proposed approximation into (63b) and collecting the coefficients of each term on the right-hand of (63b) gives a consequence that the value of ρ is the same as that of at each order and the value of ρ is identical for that of ρ at each order. Thus (62) is independent of time up to O ^{6}. Thus both and $_0$ are equivalent and approach to unity with a sixth-order error.

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