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On the nonlinear energy transfer in the unidirected adiabatic surface waves

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ABSTRACT

The results of numerical simulation of the adiabatic evolution of waves are presented. The model is based on the fully nonlinear 1D equations of potential waves written in conformal coordinates. It is shown that a wave spectrum is subject to strong fluctuations. Most of such fluctuations are reversible, however a residual effect of the fluctuations causes downshifting of the spectrum. The rate of downshifting depends on nonlinearity.

1. Introduction

The Letter considers the results of numerical simulation of nonlinear one-dimensional unidirected surface waves. The calculations are done using the 1D fully nonlinear potential wave equation written in conformal coordinates and solved by the Fourier-transform method. Initial conditions are assigned as a group of linear waves. According to the general opinion based on the quasi-linear Hasselmann's theory, such waves cannot produce downshifting, i.e., a regular transfer of wave energy from high to low wavenumber modes. The calculations prove that this statement is incorrect. The downshifting develops at any wave steepness, since linear waves quickly obtain nonlinear properties. Finally the mechanism similar to that indicated in the Benjamin–Feir theory leads to formation and growth of the new modes.

2. Calculations

The calculations were performed using the fully nonlinear one-dimensional model of potential waves written in conformal coordinates. All variables are assumed to be nondimensional with the scales constructed with acceleration of gravity g and an arbitrary length scale L. The numerical scheme is based on the Fourier-transform method and the fourth-order Runge–Kutta scheme [1–3]. This approach was described in our publications (see also the analogous work [4]). It was shown in [5,6], that the conformal model is exact. It was proved by simulation of Stokes wave propagation with steepness $ak = 0.43$ (a is an amplitude of Stokes wave and k is a wave number of the first mode) for hundreds of periods without any change of shape. In this Letter the model was used for simulation of deep water wave evolution for very long periods. The wave field in the initial condition was assigned as a superposition of linear waves with random phases and a spectral distribution described by the one-dimensional JONSWAP spectrum for different inverse wave ages $\Omega = U/c_p$ (U – a wind velocity, c_p is a phase velocity of a wave in a peak of spectrum), i.e. for different wave steepness. Since the equations were integrated over thousands of wave periods, details of the initial conditions as well as the specific set of phases were of no significance.

In the course of evolution the wave spectrum was changing due to nonlinear interactions. Contrary to the linear case a value of the integral potential energy E_p in a nonlinear wave field is not a constant, since the potential E_p and kinetic E_k energies fluctuate. However, the total energy $E = E_p + E_k$ is an adiabatic integral invariant. The total energy in a numerical model maintains constant on condition that a spectral domain is very broad and a flux of energy into the high wave number part of the spectrum is not restricted. For a finite size of domain a flux of energy into the truncated part of the spectrum forms. A corresponding decrease of the total energy can be considered as dissipation. To make the process quasi-stationary such a weak loss of energy was compensated by corresponding correction of the total energy E. The procedure of high wavenumber smoothing and maintaining of the total energy was described in the previous papers (for example ([2,5,8]).

The equations were integrated with the total number of modes $M = 1000$, and a number of grid knots $N = 4000$ with a time step $\Delta t = 0.001$ for 5 000 000 steps, which corresponds to 8000 initial peak wave periods $T_p = 2\pi/\omega_p$, where ω_p is the frequency in the maximum of an initial spectrum $S(t = 0, \omega)$ connected with the peak wave number k_p by the dispersion relation $\omega_p = \sqrt{k_p}$. The initial value of k_p was always equal to $k_0 = 100$. The initial spectrum decreases fast for the wavenumbers $k < k_p$, while for $k > k_p$ it was assigned in a wavenumber space up to the wavenumber $k_p + 20$. We do not give more details, because a specific shape of the initial spectrum is of no significance, and only the integral characteristics are important (see Table 1).

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Table 1
Integral characteristics of numerical runs (see text).

No	U/c_p	s_1	s_2	$\Delta_t k$
1	1.0	0.064	0.029	12
2	1.5	0.078	0.036	19
3	2.0	0.092	0.042	30
4	2.25	0.097	0.044	33
5	2.5	0.103	0.047	34
6	3.0	0.113	0.051	42
7	3.5	0.123	0.055	50

The main parameter of JONSWAP spectrum is the so-called 'inverse wave age' U/c_p (U – a wind velocity, $c_p = k_p^{-1/2}$ – a phase velocity in a peak of spectrum) which characterizes the level of sea waves development. The value $U/c_p = 3.5$ corresponds to the case of steep ('young') waves when wind velocity exceeds phase velocity 3.5 times. The value $U/c_p = 1$ corresponds to the case of an 'old sea', when wind velocity is equal to peak phase velocity. In this case waves do not obtain energy from the wind. In our calculation the parameter U/c_p is used as an index for the cases with different wave steepness. Wave steepness is characterized by the two integral parameters:

$$s_1 = \left(\sum_1^M k^2 S(k) \Delta k \right)^{1/2}, \tag{1}$$

$$s_2 = k_p \left(\sum_{k=1}^M S(k) \Delta k \right)^{1/2} \tag{2}$$

where $S(k)$ is the spectral density in the interval $\Delta k = 1$. The parameter s_2 characterizes the steepness of a low-wavenumber part of spectrum, while a parameter s_1 characterizes the steepness created mostly by high wavenumber modes and the local steepness in a physical space. The calculations were made for 6 different values of U/c_p. As seen, the steepness s_2 in our calculation varies from 0.064 to 0.123 (the values corresponding to a developed wave field and the so-called 'young sea' respectively). The runs with a larger initial steepness were terminated by breaking instability, followed by the local steepness approaching infinity in a physical space. The breaking can be prevented by introduction of a breaking parameterization algorithm (see [2]), however, such cases cannot be referred to adiabatic, so, they were excluded from consideration.

The aim of this work was observation of a spectrum evolution. It was found that at the initial stage of development the energy spreads quickly to a high wavenumber part of spectrum forming a spectral 'tail'. The energy moves also to a low wavenumber domain, but it is a slow process. At all stages of the development amplitudes of each mode quickly fluctuate in time even in a peak of spectrum. Such fluctuation can be recognized as a manifestation of the reversible nonlinear interactions. However, an alternative explanation can be based on a purely geometrical consideration. When a spectral resolution is high, the Fourier series represent just an approximation of surface. Since each Fourier coefficient is a product of integration over the entire domain, the small disturbances of surface can misplace the energy from one mode to another (probably, the closest to the initial one). The treatment of a wave field as superposition of linear modes with fixed phases and phase velocities is too straightforward to be correct.

Fast fluctuations of amplitudes obscure the directed evolution of spectrum, but the averaged over wave number spectrum obtain the two-peak structure in all runs: on the low wavenumber slope of the spectrum a new peak starts to grow, while the initial peak weakens. Finally, the spectrum shifts to the lower wavenumbers. Both spectral peaks fluctuate, which is why the largest amplitude can belong to either of the first or second peak alternatively.

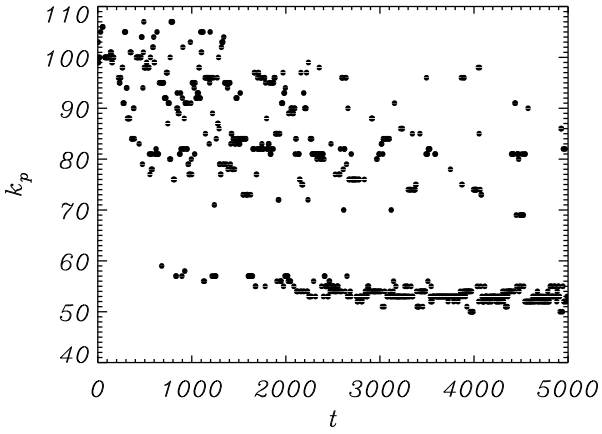

Fig. 1. Wave number k_p in a maximum of spectrum as a function of time (case 6).

It is well seen in Fig. 1 where the location of a spectral maximum is indicated as a function of nondimensional time. The top group of points belongs to the high wavenumber peak, while the bottom group refers to a new peak (see Fig. 5 below). The bottom cloud of points becomes denser with time, since the low wavenumber peak turns into the main one. The scatter reflects fluctuations of the amplitudes. Note that the smooth spectra demonstrated in various investigations are often obtained at a low spectral resolution. Note also that contrary to the wavenumber spectra, the frequency spectra obtained over the long enough periods are always smooth, since the fluctuations of energy at the adjacent wave numbers cause the averaging of spectrum in a frequency space.

As seen from Fig. 1 the formally defined wave number k_p in a maximum of spectra has a very large scatter due to the amplitude fluctuation. The mean spectrum-weighted wavenumber k_w, defined by the following expression

$$k_w = \sum_{k=1}^M kS(k) \left(\sum_{k=1}^M S(k) \right)^{-1} \tag{3}$$

is more convenient.

In Fig. 2 the weighted wave number is represented as a function of a nondimensional time t. For all cases the time was equal approximately to 8000 of initial wave periods. As seen, all spectra move monotonically to the low wave numbers. The rate of this downshifting increases with increase of the initial steepness (given in Table 1), while the steepness s_1 decreases due to the spectrum broadening. It is particularly noticeable for the high initial steepness (cases 1 and 2).

The level of energy was maintained in all numerical experiments with an accuracy of 6 digits. The reviewer of this Letter suggested that downshifting can be produced by this input of energy. This statement can hardly be correct since an input of energy in every spectral bin is proportional to the energy contained in this bin. To validate this statement three additional runs for cases 1, 2 and 4 (Table 1) were performed for $M = 2048$ with no correction of energy. In this case the truncation area was shifted far towards the high wave numbers. The results are shown in Fig. 3. As seen in panel a, the total energy decreases to 50–60% off the initial value. Respectively, the integral steepness decreases also two times (panel c). This is why a rate of downshifting became much lower than that for the same cases 1, 2 and 4 with a permanent level of energy. However, the downshifting appears in these cases as well. A systematic downshifting was not reproduced in the one-dimensional wave field simulations carried out on the basis of the nonlinear Schrödinger's equation and Zakharov's equation [9]. Both approaches assume a weak nonlinearity of wave fields and use a

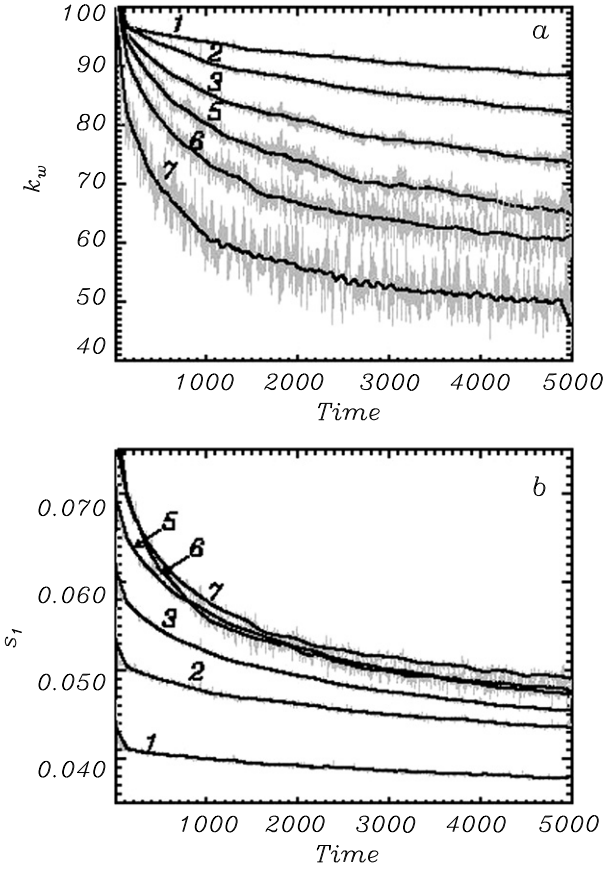

Fig. 2. Results of calculations with a number of modes $M = 1000$ and with correction of total energy: a – dependence of a weighted wavenumber k_w (Eq. (3)) on a time t; b – dependence of the integral steepness s_1 on a time t. In both frames fluctuating grey curves show actual dependence; solid curves are the product of the moving averaging with a window width equal to 41.

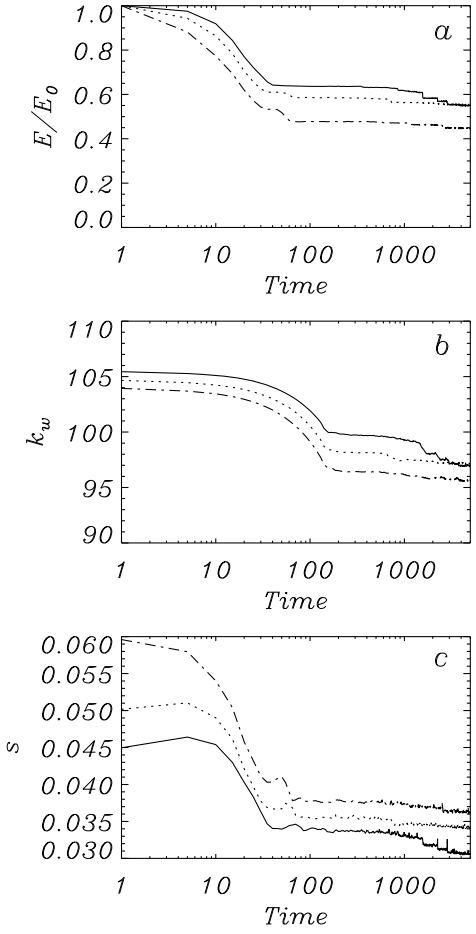

Fig. 3. Results of the calculations with a number of modes $M = 2048$ with no correction of the total energy: a – dependence of the total wave energy on a time t; b – dependence of the spectrum-weighted wavenumber k_w (Eq. (3)) integral steepness s_1 on a time t; c – dependence of the integral steepness s_1 on a time t.

number of simplifying hypotheses. Probably, the wave field simulated in this investigation was not steep enough to show the downshifting (see Fig. 4). It is also quite possible that the simplified 1D equations in no way can reproduce the downshifting.

The total shift of the weighted frequency $\Delta k_w/k_0$ over the entire period of integration as a function of the initial steepness s_1 is shown in Fig. 4.

As seen, the downshifting in a unidirected adiabatic wave field can be quite significant. The three lowermost points in Fig. 4 were obtained with no correction of total energy. As seen, the rate of downshifting for these cases is much smaller than that for the cases with large steepness, because wave energy dissipates due to a flux to a truncated part of spectrum. As a result the averaged steepness becomes small.

The wave spectrum $S(k)$ and a rate of spectrum change due to the nonlinear interactions $S_n(k)$, are shown in Fig. 5. Spectra $S(k)$ are averaged over 5 successive intervals, while spectra $S_n(k)$ are a simple difference between successive averaged wave spectra

$$S_n(k) = (\Delta t)^{-1}\big(S(k, t + \Delta t) - S(k, t)\big). \tag{4}$$

The wave spectra have a multi-peak structure most of the time. It can be explained by specific initial conditions: all energy was assigned to the unidirected modes with no angle spreading. The angle spreading decreases the energy of modes directed along the x-axis. The spectral peak in the initial condition was probably too high for 1D simulations. Anyway, a presence of an additional peak does not change the results qualitatively. The spectrum of the rate of the nonlinear interaction $S_n(k)$ reflects a tendency for a wave spectrum evolution: it is mostly positive on the low-wavenumber slopes of the wave spectrum and negative on the high-wavenumber slopes of the wave spectrum, which results in shifting of the spectral energy to the left.

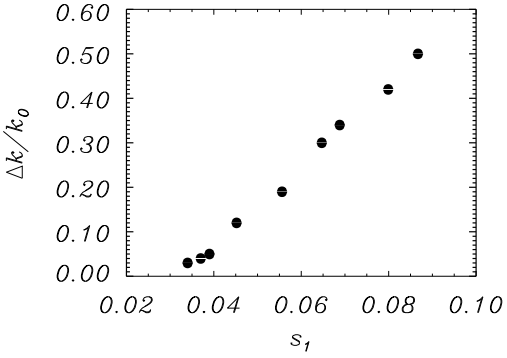

Fig. 4. Dependence of the total change of the weighted wavenumber k_w for the entire period of integration (normalized by the initial wavenumber k_0) on the initial integral steepness s_1.

3. Conclusions

The Letter represents the results of numerical modeling of the multi-mode unidirected adiabatic waves evolution performed with use of the precise 1D fully nonlinear model. It is shown, that due to the nonlinear interaction the irreversible nonlinear interactions

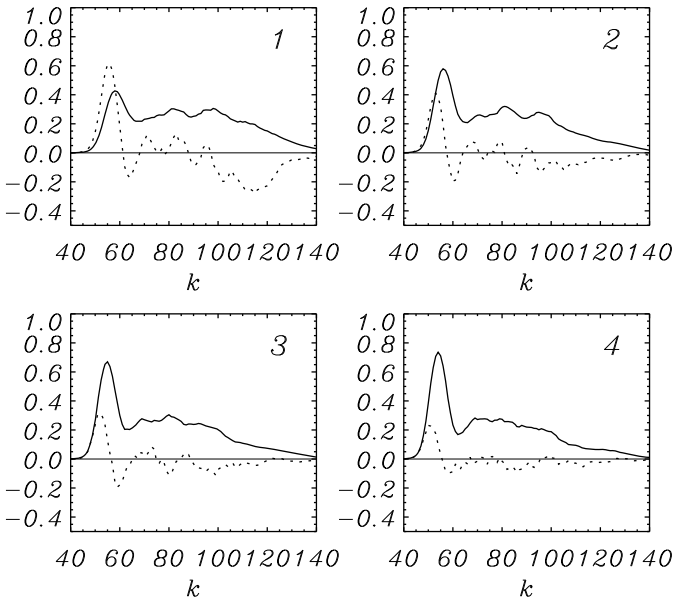

Fig. 5. The averaged over the consecutive periods of length $\Delta t = 1000$ wave spectra, $S(k)$ (solid curves) and the spectrum of the nonlinear interaction rate $S_n(k)$ (dotted curves). The spectral density below $k = 40$ is very close to 0.

and downshifting develop. The rate of downshifting increases with increase of nonlinearity. This conclusion contradicts to the Hasselmann's results. The Hasselmann's theory is based on the numerous simplifying assumptions. Representing a wave field as a superposition of linear modes with random phases is the most restricting assumption. In the Benjamin and Feir investigation [6] it was shown that keeping just the first Stokes 'correction to a harmonic wave resulted in developing instability. Later it was demonstrated that assigning of nonlinear modes was not necessary: initially 1D harmonic waves took a Stokes-like shape. Then, the B.-F. instability developed, and a fully random wave field was generated at the nonlinear stage [7]. Hence, an absence of the 1D interactions is inherent to linear waves only. It follows from the Hasselmann's integral that in a 2-dimensional wave field all the interactions between modes running in the same direction are missing. Evidently, the inaccuracies of nonlinear interaction calculations with Hasselmann's integral grow with a narrowing of spectrum. Our calculations confirm that the unidirected wave interactions have probably the same intensity as those between the two-dimensional waves. This effect is important for many practical problems, especially for a wave forecasting problem.

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