On estimating the mean energy of sea waves from the highest waves in a record

By D. E. Cartwright

National Institute of Oceanography, Wormley, Surrey

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An analysis is made of the probability distribution of the largest values attained by a stationary random variable $f(t)$ over a period of time containing several oscillations. Exact computations are made and asymptotic formulae are derived for the expectation and standard error of the first, second and third greatest maxima in terms of $\sqrt{m_0}$, the r.m.s. deviation of $f(t)$ about its mean value, on the assumption that successive waves are uncorrelated; an analysis is also made of the corrections necessary to allow for mutual correlation when $f(t)$ has a narrow spectrum. The results are applied to measurements from a 24 h record of ocean waves containing some 10000 oscillations.

1. Introduction

When considering a continuous record of some randomly oscillating quantity, such as the height of the sea surface above a fixed point, one often wishes to find a single statistic which will characterize the ensemble of wave heights occurring over a given period of time. Among those which have been used are the mean or r.m.s. wave height, the mean of the highest one-third or one-tenth wave heights, the standard deviation of the record, or, simplest to estimate in practice, the height of the largest wave occurring in the record. For sea waves it is well known that the expectancies of all the above statistics bear simple ratios to one another, as shown by Longuet-Higgins (1952), who derived theoretical values which were in fair agreement with observations. The maximum wave height in a very long stationary record would therefore be equivalent to any of the other more laboriously estimated parameters. Indeed, the idea of estimating the variance of a population from the largest in a sample is quite familiar (see, for example, Kendall 1945, pp. 217–18). However, in practice, stationarity does not hold for more than a few hundred waves, and so only a limited length of record is available; this introduces sampling errors for which confidence limits must be established.

In what follows we study first the probability distribution of the highest of N waves, so obtaining the expectation, standard error, and confidence limits of the estimate. Then similar analyses are made for the second and third highest waves, whose sampling errors are somewhat lower. In the next section, the effect of mutual correlation of adjacent waves on the above quantities is considered, and finally the results are applied to an actual record containing some 10000 ocean waves. The measurements being in good agreement with theory, the latter can be confidently used to estimate the mean energy of any wave system (or the standard deviation of any stationary record), and also the probable errors in such an estimate. Although the discussion will be in terms of sea waves, the whole treatment applies equally well to any related random oscillation, such as the motions or stresses experienced by a ship at sea.

The literature on the theory of extreme values in general is fairly extensive; a recent summary with bibliography is given by Gumbel (1954). However, practically all the work has been done in application to extreme values of very large samples, usually extending over years of measurements, and so only the leading terms in asymptotic expressions have been considered. Further, there is no adequate treatment based on the special probability distribution appertaining to waves, or taking into account correlation of successive members. A fresh approach from first principles for this particular problem is therefore desirable.

In the case where the spectrum of the wave record is very narrow, the expected and most probable values of the maximum of N waves have been derived by Longuet-Higgins (1952). However, as pointed out by that author, his results may be modified in two ways. First, if the spectrum is not narrow, the assumed (Rayleigh) distribution of the waves themselves no longer holds exactly. To allow for an arbitrary spectral shape, we shall therefore consider the more general distribution derived by Rice (1945) and analyzed by Cartwright & Longuet-Higgins (1956); this distribution strictly applies not to wave heights measured from crest to trough, but to crest heights measured from the mean level.

Secondly, if the spectrum is, on the contrary, very narrow, then consecutive crests have a large correlation, so that the effective value of N is reduced. Watson (1954) has shown that the effect of correlation on the distribution of extreme values in general is rather small for large N, provided certain conditions hold, but we shall extend Watson's argument to estimate the magnitude of the effect, which is desirable when N is not very large. Nevertheless, it will be found that for a wide range of conditions it is sufficiently accurate to work in terms of the Rayleigh distribution considered by Longuet-Higgins, with some simple correction factors to allow for the two effects just mentioned.

2. The distribution and statistical parameters of the largest of N uncorrelated crest heights

We begin by considering the general case of mutual independence, since it is least complicated. When the effect of correlation is considered later (§ 4), it will be expressed as a small correction to these results. The case of zero correlation is not without practical significance, since it applies, for example, to a set of waves chosen randomly at reasonably large intervals of time.

Suppose the wave profile is represented by the stationary random function of time $f(t)$, whose mean value is taken to be zero, and whose mean square deviation taken over all time is m_0, and let x denote a crest height in terms of $m_0^{\frac{1}{2}}$. That is,

$$x = f(t')/m_0^{\frac{1}{2}},$$

where $$\frac{\mathrm{d}}{\mathrm{d}t} f(t') = 0, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t') \leqslant 0 \quad \text{and} \quad m_0 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [f(t)]^2 \,\mathrm{d}t.$$

(In the case of sea waves, m_0 represents of course the mean energy per unit area of surface.)

If $q_1(x)$ denotes the probability that any given crest shall exceed x, then the probability that the maximum of N mutually independent crests shall be less than x is

$$p_N(x) = [1-q_1(x)]^N. \tag{1}$$

The expected value of the maximum, or the first moment of the distribution, is given by

$$M_1(N) = \int_{-\infty}^{\infty} x\,\mathrm{d}p_N(x) = \int_0^{\infty} [1-p_N(x)]\,\mathrm{d}x - \int_{-\infty}^{0} p_N(x)\,\mathrm{d}x, \tag{2}$$

after integrating by parts, since $q(x)$ decreases monotonically from 1 at $-\infty$ to 0 at ∞.

Similarly, the rth moment, provided it exists, is given by

$$M_r(N) = \int_{-\infty}^{\infty} x^r\,\mathrm{d}p_N = \int_0^{\infty} rx^{r-1}[1-p_N(x)]\,\mathrm{d}x - \int_{-\infty}^{0} rx^{r-1}p_N(x)\,\mathrm{d}x. \tag{3}$$

The standard error $D(N)$ is of course derived from these moments by means of the relation

$$D^2(N) = M_2(N) - M_1^2(N).$$

Using the notation of Cartwright & Longuet-Higgins (1956), the most general formula for $q_1(x)$ can be written as

$$q_1(x) = q_1(x,\epsilon) = (2\pi)^{-\frac{1}{2}}\left[\int_{x/\epsilon}^{\infty} \mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t + (1-\epsilon^2)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}x^2}\int_{-\infty}^{x(1-\epsilon^2)^{\frac{1}{2}}/\epsilon} \mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t\right], \tag{4}$$

where ϵ is a fixed positive number never exceeding 1, which depends on the shape of the energy spectrum of $f(t)$, and tends to zero as the spectral width decreases.

(a) *Computed values*

Exact calculation of the moments of the general distribution (4) can only be made by numerical integration. Values of M_1, M_2 and D have been computed on the DEUCE electronic computor at the National Physical Laboratory, Teddington, by Mr G. F. Miller and are tabulated in Appendix 1 for $\epsilon = 0\ (0{\cdot}1)\ 1{\cdot}0$ and $N = 2^m$, with $m = 0\,(1)\,15$. From these tables, statistics for most practical values of ϵ and N can be obtained by interpolation. In these tables the last digit given may be in error by not more than one unit.

The probability functions $p_N(x)$ were also computed, and the functions for $\epsilon = 0$ and $0{\cdot}9$ are shown graphically in figures 1 *a* and *b*. For some analytical purposes (see Cartwright & Longuet-Higgins 1956) it is convenient to approximate to $p_N(x)$ with the asymptotic expression

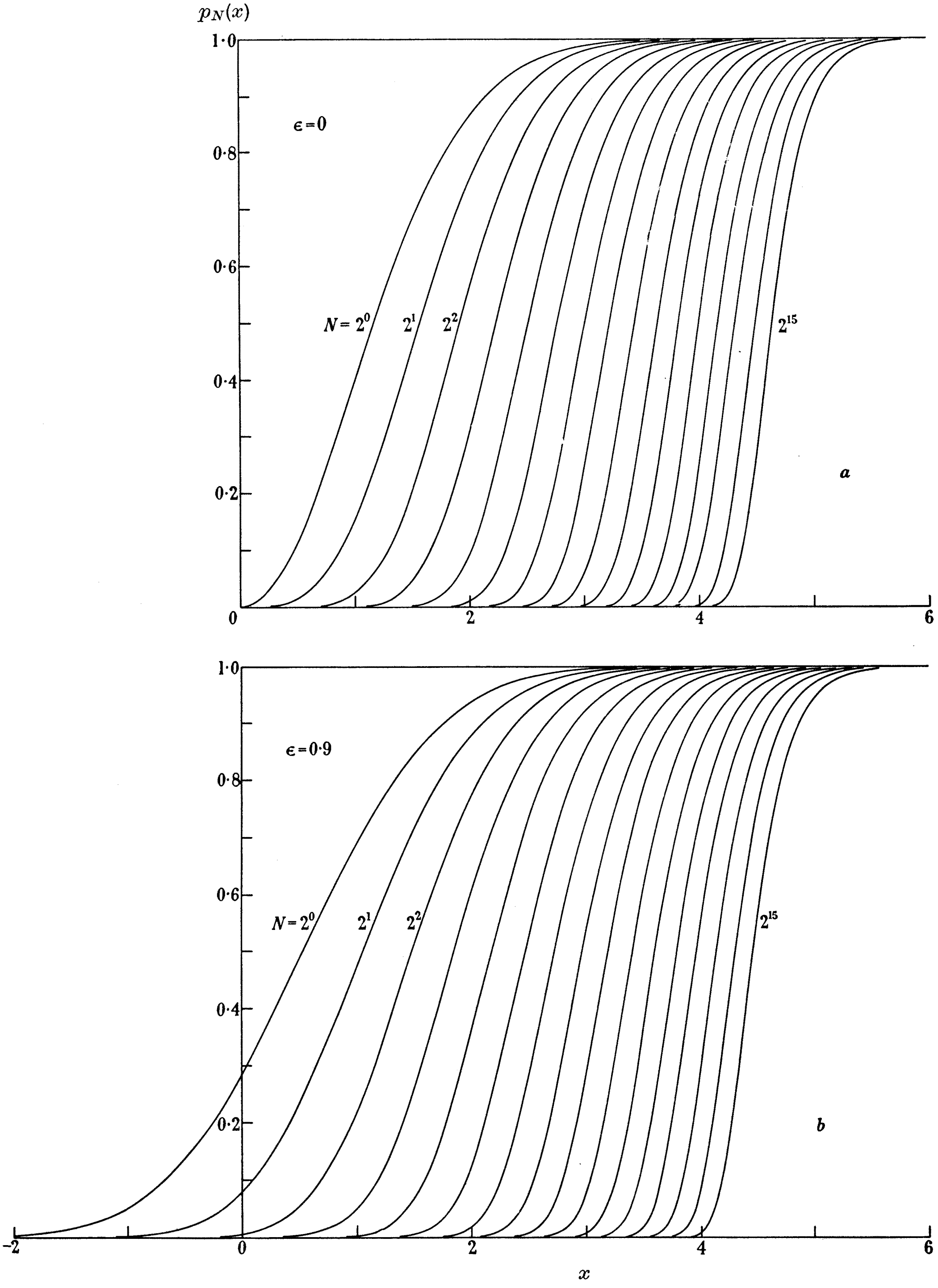
$$p_N(x) \sim \exp\left[-N(1-\epsilon^2)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}x^2}\right] \tag{5}$$

for large N; the difference between the asymptotic and exact expressions are also shown graphically in figures 1 *c* and *d* for the same values of ϵ as in 1 *a* and *b*.

Figures 2 *a* and *b* show the upper and lower confidence limits at 5 and 1 % probability, respectively. They represent the solutions of the equations

$$p_N(x) = 0{\cdot}025, 0{\cdot}975;\ 0{\cdot}005, 0{\cdot}995$$

and were obtained by interpolation from the computed values of $p_N(x)$.



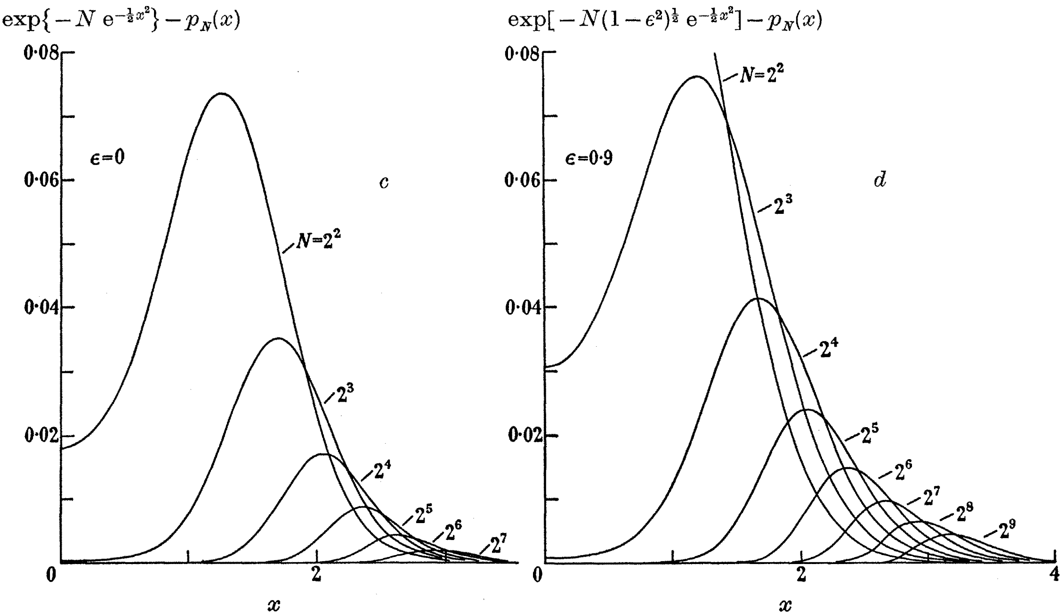
FIGURES 1 *a* and *b*. Probability functions $p_N(x)$ (equations (1) and (4)) for various values of $N = 2^m$ and (*a*) $\epsilon = 0$, (*b*) $\epsilon = 0{\cdot}9$.

Figure 3 shows the standard deviation relative to the mean, D/M_1, plotted against N, for most of the available values of ϵ.

(b) *Analytical formulae*

For analytical purposes we require a simplifying approximation for the general formula (4). Figures 1 *c* and *d* show that the asymptotic formula (5) is quite good for a wide range of values of ϵ, and large values of $N(1-\epsilon^2)^{\frac{1}{2}}$. We shall therefore approximate to the distribution of the maximum of N wave crests by replacing $q_1(x, \epsilon)$ by $q_1(x, 0)$ and N by $N(1-\epsilon^2)^{\frac{1}{2}}$. We can therefore restrict discussion to the simple case

$$q_1(x) = q_1(x,0) = \begin{cases} e^{-\frac{1}{2}x^2} & (x \geq 0), \\ 1 & (x \leq 0), \end{cases}$$



FIGURES 1 *c* and *d*. Difference between exact and asymptotic formulae for the functions $p_N(x)$, with (*c*) $\epsilon = 0$, (*d*) $\epsilon = 0{\cdot}9$.

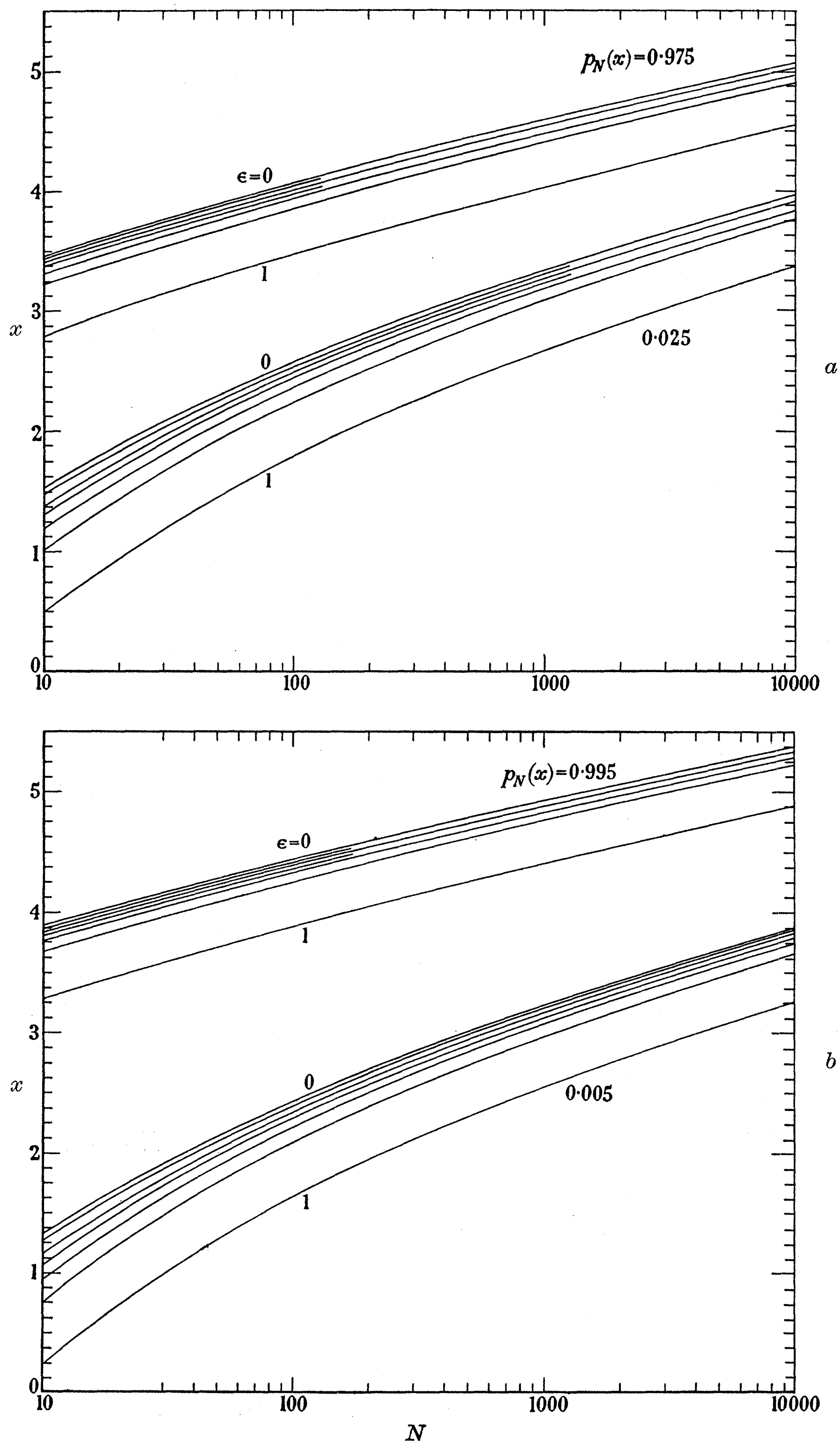
provided $\epsilon < 1$. The case $\epsilon = 1$ is unlikely to arise in practice; in fact it corresponds to a normal distribution, for which the theory of extreme values is well known—see, for example, Tippett (1925).

From (3) we have

$$M_r(N) = r\int_0^\infty x^{r-1}[1-(1-e^{-\frac{1}{2}x^2})^N]\,dx. \tag{6}$$

On expanding the internal bracket binomially and integrating term by term, we may write this as follows

$$M_r(N) = 2^{\frac{1}{2}r}\left(\frac{r}{2}\right)!\left[N - \frac{N(N-1)}{2^{\frac{1}{2}r}.2\,!} + \frac{N(N-1)(N-2)}{3^{\frac{1}{2}r}.3!} \cdots + \frac{(-1)^N}{(N-1)^{\frac{1}{2}r}} - \frac{(-1)^N}{N^{\frac{1}{2}r}}\right]. \tag{7}$$



FIGURES 2*a* and *b*. Upper and lower confidence limits for $p_N(x)$ (*a*) at 5 % level, (*b*) at 1 % level. ($\epsilon = 0, 0{\cdot}4, 0{\cdot}6, 0{\cdot}7, 0{\cdot}8, 0{\cdot}9, 1$).

As it stands, (7) gives no useful information about the behaviour of the moments $M_r(N)$, and is unsuitable for calculation when N is greater than about 20. A more useful result can be obtained as follows. From (7), or on integrating by parts from (6), we have the reduction formula

$$M_r(N) - M_r(N-1) = (r/N)\, M_{r-2}(N), \tag{8}$$

and on summation with respect to N,

$$M_r(N) = r \sum_{s=1}^{N} (1/s)\, M_{r-2}(s), \tag{9}$$

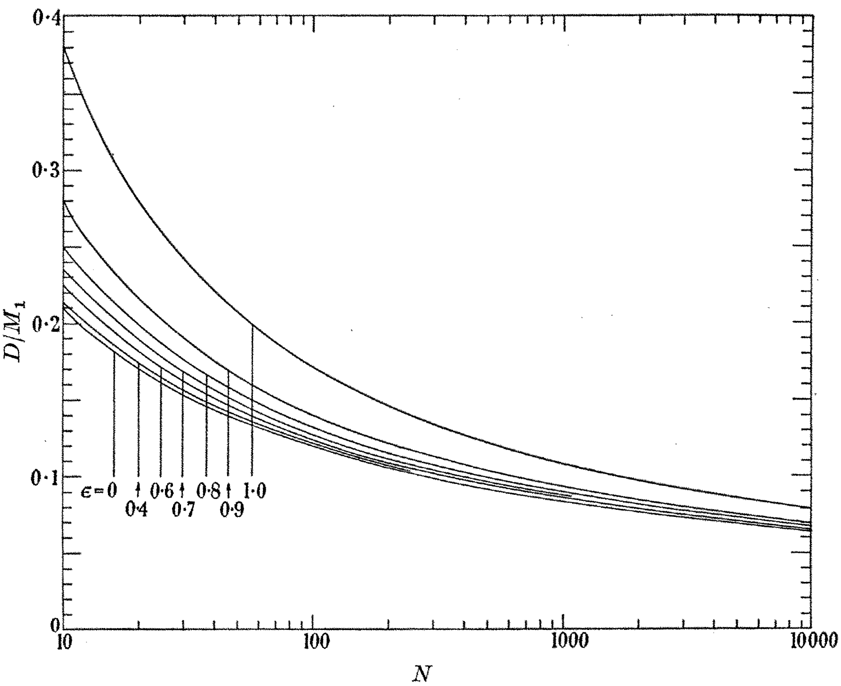


FIGURE 3. Curves of standard deviation of x relative to its expectation.

since from (6), $M_r(0) \equiv 0$. We also have

$$M_0(N) = N - \frac{N(N-1)}{2!} + \frac{N(N-1)(N-2)}{3!} \dots + (-1)^N N - (-1)^N$$
$$= 1 - (1-1)^N \equiv 1$$

and so

$$M_2(N) = 2 \sum_{1}^{N} 1/s,$$

$$M_4(N) = 2^2 . 2! \sum_{s=1}^{N} \Big(1/s \sum_{1}^{s} 1/t\Big)$$

$$= 2^2 . 2! \sum_{1}^{N} \sum_{1}^{N} 1/st \quad (s \geqslant t)$$

and in general

$$M_{2r}(N) = 2^r r! \sum_{s_1=1}^{N} \sum_{s_2=1}^{N} \dots \sum_{s_r=1}^{N} (s_1 s_2 \dots s_r)^{-1} \quad (s_1 \geqslant s_2 \geqslant s_3 \dots \geqslant s_r). \tag{10}$$

These formulae can be expressed as series of powers and products of the functions

$$S_n(N) = \sum_{s=1}^{N} s^{-n},$$

from which asymptotic formulae for large N can be derived by means of the relation (Whittaker & Watson 1952, p. 235)

$$S_1(N) = \ln N + \gamma + \tfrac{1}{2}N^{-1} + O(N^{-2}),$$

and by replacing $S_n(N)$ $(n > 1)$ by $S_n(\infty)$ with errors of order N^{-n+1} (in the above formula γ is Euler's constant, 0·5772...).

Thus

$$\begin{aligned} M_2(N) &= 2S_1(N) \sim 2(\ln N + \gamma), \\ M_4(N) &= 2^2[S_1^2(N) + S_2(N)] \sim 2^2[(\ln N)^2 + 2\gamma \ln N + \gamma^2 + \tfrac{1}{6}\pi^2], \\ M_6(N) &= 2^3[S_1^3(N) + 3S_1(N)\,S_2(N) + 2S_3(N) \\ &\sim 2^3[(\ln N)^3 + 3\gamma(\ln N)^2 + 3(\gamma^2 + \tfrac{1}{6}\pi^2)\ln N + \gamma^3 + \tfrac{1}{2}\gamma\pi^2 + 2S_3], \end{aligned} \tag{11}$$

where $S_3 = S_3(\infty) = 1{\cdot}20206\ldots$.

It appears difficult to formulate the general result for the $2r$th moment by this method and one certainly cannot use it for the odd moments, since M_1 is not expressible in exact terms. However, the asymptotic formulae for all moments may be obtained directly by a different method as shown in Appendix 2. (The author is indebted to Dr Longuet-Higgins for pointing out this method to him.) The result is

$$M_r(N) \sim (2\theta)^{\frac{1}{2}r}\left[1 + \tfrac{1}{2}rA_1\theta^{-1} + \frac{(\frac{1}{2}r)\,(\frac{1}{2}r-1)}{2!}A_2\theta^{-2} + \ldots\right], \tag{12}$$

where $\theta = \ln N$,

$$\begin{aligned} A_1 &= \gamma = 0{\cdot}5772\ldots, \\ A_2 &= \gamma^2 + \tfrac{1}{6}\pi^2 = 1{\cdot}9781\ldots, \\ A_3 &= \gamma^3 + \tfrac{1}{2}\gamma\pi^2 + 2S_3 = 5{\cdot}4449\ldots, \\ A_4 &= \gamma^4 + \gamma^2\pi^2 + 8\gamma S_3 + (3/20)\,\pi^4 = 23{\cdot}5615\ldots. \end{aligned}$$

Equation (12), which is correct to an order $(\ln N)^{\frac{1}{2}r}N^{-1}$, is seen to agree with the previous results (11) derived for the even-order moments. The $2r$th moments also accord with rth moments about the mean derived for a variable proportional to x^2 by Fisher & Tippett (1928). When r is odd, the series in (12) does not terminate, or even ultimately converge, but any finite number of terms will approach the true value asymptotically as $\ln N$ increases, as shown in Appendix 2.

In particular, we have for the first moment

$$M_1(N) \sim (2\theta)^{\frac{1}{2}}\,(1 + \tfrac{1}{2}A_1\theta^{-1} - \tfrac{1}{8}A_2\theta^{-2} + \tfrac{1}{16}A_3\theta^{-3} - \ldots). \tag{13}$$

This is in accordance with the result of Longuet-Higgins (1952), where the first two terms only were obtained by another method.

By formally squaring the series (13) and subtracting from that for M_2, we obtain the following expression for the variance D^2

$$D^2(N) = M_2 - M_1^2 \sim (2\theta)^{-1}(B_0 + B_1\theta^{-1} + B_2\theta^{-2} + \ldots) \quad (14)$$

with

$$B_0 = \tfrac{1}{6}\pi^2 = 1\cdot6449\ldots,$$

$$B_1 = -\tfrac{1}{6}\gamma\pi^2 - S_3 = -2\cdot1515\ldots,$$

$$B_2 = \tfrac{1}{6}\gamma^2\pi^2 + 2\gamma S_3 + (13/288)\pi^4 = 6\cdot3327\ldots.$$

(c) *Discussion of results so far obtained*

Table 1 compares the asymptotic formulae for M_1, M_2 and D for $\epsilon = 0$ and 0·9 with exact values taken from Appendix 1. For $\epsilon = 0$, the formulae obtained above were used directly, with the first four terms only of (13) and the first two terms only

TABLE 1. COMPARISON OF EXACT AND ASYMPTOTIC FORMULAE FOR MOMENTS

N	$\epsilon=0\cdot0$ exact	$\epsilon=0\cdot0$ asymptotic	$\epsilon=0\cdot9$ exact	$\epsilon=0\cdot9$ asymptotic
		M_1		
2^3	2·276	2·283	1·894	1·971
2^4	2·559	2·562	2·221	2·226
2^5	2·817	2·819	2·510	2·508
2^6	3·054	3·057	2·773	2·770
2^7	3·275	3·277	3·013	3·011
		M_2		
2^3	5·436	5·313	3·927	3·653
2^4	6·761	6·700	5·200	5·039
2^5	8·117	8·086	6·524	6·425
2^6	9·488	9·472	7·874	7·811
2^7	10·866	10·858	9·240	9·198
2^8	12·249	12·245	10·613	10·584
2^9	13·633	13·631	11·991	11·970
2^{10}	15·018	15·017	13·372	13·357
		D		
2^5	0·428	0·384	0·470	0·396
2^6	0·398	0·368	0·432	0·387
2^7	0·374	0·352	0·402	0·371
2^8	0·353	0·337	0·377	0·355
2^9	0·336	0·323	0·356	0·340
2^{10}	0·320	0·310	0·338	0·325
2^{11}	0·307	0·299	0·323	0·313
2^{12}	0·295	0·289	0·309	0·301

of (14). The results for $\epsilon = 0\cdot9$ were obtained similarly, but with $N(1-\epsilon^2)^{\frac{1}{2}}$ in place of N. It is seen that the asymptotic formulae for M_1 are very accurate, being within $\frac{1}{2}$% for $\epsilon = 0$, $N = 8$, and for $\epsilon = 0\cdot9$, $N = 16$. (It should also be remarked that Longuet-Higgins (1952) showed that the first two terms alone of M_1 give an error of only 2% for $\epsilon = 0$, $N = 20$.) The formulae for M_2 are slightly less accurate, but are within 1% for $\epsilon = 0$, $N = 16$, or $\epsilon = 0\cdot9$, $N = 64$. The relative error in D is considerably higher, 10% for $\epsilon = 0$, $N = 64$, or $\epsilon = 0\cdot9$, $N = 128$. This is to be expected, however, since in forming equation (14) the leading terms of M_1 and M_2 vanish, leaving terms of relatively higher order but smaller magnitude. In all cases accuracy increases steadily with increasing N and decreasing ϵ.

It is interesting to note that although the asymptotic formulae developed above do not apply to the limiting case $\epsilon = 1$, the moments of the normal distribution $\epsilon = 1$ shown in Appendix 1 do not differ very greatly from the corresponding values for $\epsilon = 0{\cdot}9$, to which the asymptotic formulae developed from the Rayleigh distribution with $N(1-\epsilon^2)^{\frac{1}{2}}$ in place of N give a good approximation.

Having established numerically the closeness of the asymptotic formulae to the exact values, for most values of ϵ and from fairly small values of N upwards, we may now use the asymptotic formulae (12) with confidence to discuss the general behaviour of the moments $M_r(N)$. In what follows N_ϵ denotes $N(1-\epsilon^2)^{\frac{1}{2}}$.

As would be expected, all the moments about the origin increase steadily with N_ϵ, but since $M_r(N) \sim (\ln N_\epsilon)^{\frac{1}{2}r}$ the rate of increase is slow, being slower for the larger values of N_ϵ and the smaller values of r. On the other hand, the variance, or second moment about the mean (14), is seen to decrease like $(\ln N_\epsilon)^{-1}$, showing that the distribution becomes confined within a steadily narrowing range as N_ϵ increases. This is also clear from the steady steepening of the probability curves $p_N(x)$ in figures 1*a* and *b* as N increases, and the narrowing of the interval between the confidence limits in figures 2*a* and *b*. While D is decreasing with $(\ln N_\epsilon)^{-\frac{1}{2}}$, the expectation M_1 is increasing with $(\ln N_\epsilon)^{\frac{1}{2}}$, so that the relative standard error (in the estimate of $m_0^{\frac{1}{2}}$ for example), plotted in figure 3, decreases like $(\ln N_\epsilon)^{-1}$; in fact

$$\frac{D}{M_1} \sim \frac{\pi/\sqrt{24}}{\ln N_\epsilon} = \frac{0{\cdot}641}{\ln N_\epsilon}.$$

This result contrasts with the sampling error of a direct measure of the r.m.s. value of $f(t)$, the error being ultimately proportional to $N^{-\frac{1}{2}}$ (see Rice 1945 and Tucker 1957). Therefore the latter error decreases more rapidly than the error of the estimate obtained from the maximum wave. A numerical comparison between these sampling errors will be made in § 3, after studying the effect of correlation.

The leading terms in the asymptotic series for the 3rd and 4th moments about the mean, m_3 and m_4, also decrease with increasing N_ϵ, as would be expected. They are

$$m_3 = M_3 - 3M_2M_1 + 2M_1^3 \sim (2\ln N_\epsilon)^{-\frac{3}{2}}\, 2S_3,$$
$$m_4 = M_4 - 4M_3M_1 + 6M_2M_1^2 - 3M_1^4 \sim (2\ln N_\epsilon)^{-2}\,(6S_4 + 3S_2^2).$$

It follows that the coefficients of skewness, β_1, and kurtosis, β_2, tend to the finite limits

$$\left.\begin{aligned} \beta_1 &= m_3^2/m_2^3 \sim (2S_3)^2/S_2^3 = 1{\cdot}299\ldots, \\ \beta_2 &= m_4/m_2^2 \sim 3 + 6S_4/S_2^2 = 3 + 12/5 = 5{\cdot}4, \end{aligned}\right\} \tag{15}$$

indicating that the distribution becomes positively skew and more peaked than the normal distribution (for which $\beta_1 = 0$, $\beta_2 = 3$). For this reason the lower confidence limits are nearer to the mean than the upper limits (figures 2*a*, *b*). However, the differences between the respective limits are remarkably close (within 2 or 3 %) to the values which would be appropriate to a normal distribution with the same standard deviation ($3{\cdot}92D$ and $5{\cdot}15D$, respectively). The departure from normality of similar distributions of extreme values was first remarked by Tippett (1925), and Fisher & Tippett (1928) showed that the same numerical values as in (15) apply to the limiting distribution of extreme values of a normal population; this of course corresponds to our $\epsilon = 1$, and it appears that the same limits will hold for all values of ϵ.

3. The second and third highest waves

It is a well-known fact that the second or third, etc., largest individual is often a more reliable measure of the parent population than the largest. We shall, therefore, now investigate the reduction in standard error in the second and third highest waves from that of the highest of N waves, again at this stage assumed mutually uncorrelated.

If $M'_r(N)$ denotes the rth moment about the origin of the distribution of the second highest crest, it can be shown by the same argument as before that

$$M'_r(N) = M_r(N) - N[M_r(N) - m_r(N-1)]. \tag{16}$$

If, as in § 2, we approximate to $q_1(x, \epsilon)$ by $q_1(x, 0)$, with N_ϵ in place of N, then on using equation (8), (16) becomes

$$M'_r(N) = M_r(N) - rM_{r-2}(N). \tag{17}$$

This gives a very simple result for the second moment, namely

$$M'_2(N) = M_2(N) - 2 = 2\left(\sum_1^N 1/s - 1\right) \sim 2(\ln N + \gamma - 1).$$

To obtain the first moment from (17) requires an expression for M_{-1}, or the mean reciprocal of the maximum wave. Although the derivation of the asymptotic formulae for $M_r(N)$ (Appendix 2) is valid only for $r \geqslant 0$, it can be shown that the same general formula (12) can be used for negative values of r also. Thus

$$M_{-1} \sim (2\theta)^{-\frac{1}{2}}\,(1 - \tfrac{1}{2}A_1\theta^{-1} + \tfrac{3}{8}A_2\theta^{-2} - \ldots)$$

and $\quad M'_1(N) = M_1(N) - M_{-1}(N)$

$$\sim (2\theta)^{\frac{1}{2}}\,[1 - \tfrac{1}{2}(1-\gamma)\,\theta^{-1} + \tfrac{1}{8}(2\gamma - A_2)\,\theta^{-2} - \tfrac{1}{16}(3A_2 - A_3)\,\theta^{-3} + \ldots]. \tag{18}$$

If $D'(N)$ denote the standard error of the second highest wave, then

$$[D'(N)]^2 \sim (2\theta)^{-1}\{(\tfrac{1}{6}\pi^2 - 1) + [\tfrac{1}{6}(1-\gamma)\,\pi^2 + \gamma - S_3]\,\theta^{-1} - \ldots\}$$

$$= (2\theta)^{-1}\,(0{\cdot}6449 + 0{\cdot}07060\theta^{-1} - \ldots). \tag{19}$$

Equation (19) shows that D' tends to lower values than D, while from (18) it is seen that M'_1 is only slightly less than M_1, and tends ultimately to be equal to M_1. For large N, therefore, the standard error relative to the mean is less than that for the highest wave by a factor approaching the value

$$(1 - 6/\pi^2)^{\frac{1}{2}} = 0{\cdot}626\ldots.$$

Figure 4 shows the exact value of D'/D, for values of N up to 10^5, and it is seen that the limit is approached rather slowly.

Similarly, the rth moment of the distribution of the third highest wave in a sample of size N is given by

$$M''_r(N) = M_r(N) - \tfrac{3}{2}rM_{r-2}(N) + \tfrac{1}{2}r(r-2)\,M_{r-4}(N). \tag{20}$$

For the second moment this formula gives immediately

$$M''_2(N) = 2\sum_1^N 1/s - 3 \sim 2(\ln N + \gamma - \tfrac{3}{2}).$$

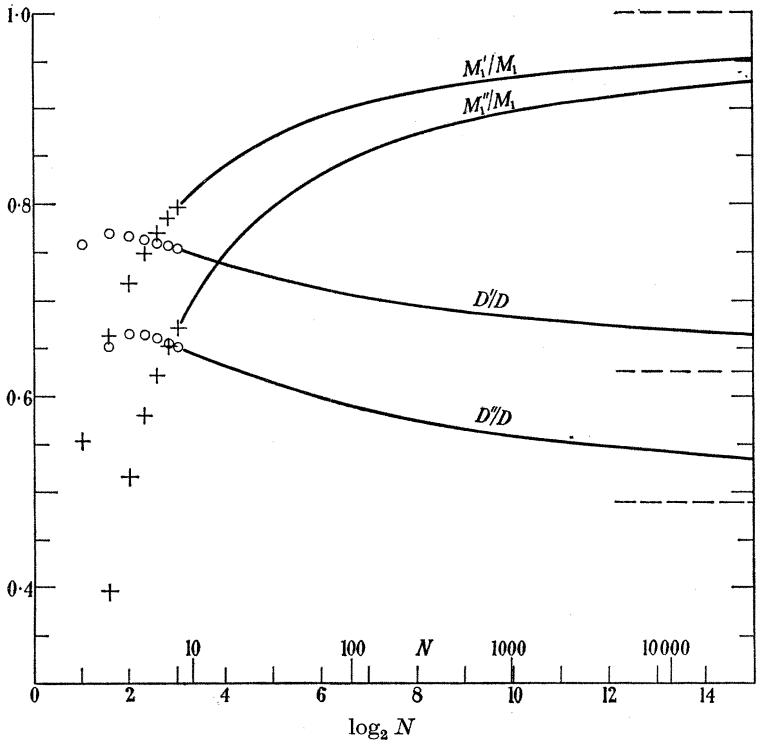


FIGURE 4. Ratios of means and standard deviations of second and third highest maxima to those of the highest maximum, for $\epsilon = 0$. (The circles and crosses represent the computed points for small integral values of N and the curves, if continued, would pass through them. The horizontal broken lines are asymptotic values for large N.)

The asymptotic series for the mean M''_1 and variance, D'', are respectively

$$M''_1(N) \sim (2\theta)^{\frac{1}{2}}\,[1 - (\tfrac{3}{4} - \tfrac{1}{2}\gamma)\,\theta^{-1} - \tfrac{1}{8}(1 - 3\gamma + A_2)\,\theta^{-2} + \tfrac{1}{32}(6\gamma - 9A_2 + 2A_3)\,\theta^{-3} - \ldots] \tag{21}$$

and $$(D'')^2 \sim (2\theta)^{-1}\,[(\tfrac{1}{6}\pi^2 - \tfrac{5}{4}) - (S_3 + \tfrac{3}{4} + \tfrac{1}{6}\gamma\pi^2 - \tfrac{1}{4}\pi^2 - \tfrac{5}{4}\gamma)\,\theta^{-1} + \ldots]$$
$$= (2\theta)^{-1}\,(0{\cdot}3949 + 0{\cdot}2874\theta^{-1} - \ldots). \tag{22}$$

Equation (22) shows a still greater reduction in standard error for large N, bearing a limiting ratio to $D(N)$ of

$$(1 - 15/2\pi^2)^{\frac{1}{2}} = 0{\cdot}490\ldots.$$

The exact values of D'/D, D''/D, M'_1/M_1 and M''_1/M_1 for $\epsilon = 0$ are shown in figure 4. For $N \leqslant 16$ points on the curve were computed directly, using equations (7), (17) and (20), and for greater values of N the calculations were based entirely on the values of $M_1(N)$ given in Appendix 1, using equation (16) and suitable finite-difference formulae. The latter method could, of course, be used for other values of ϵ if required. The curves shown approach their limiting values relatively slowly, so that for values of N generally used (say between 100 and 1000), values differing considerably from the limits will apply.

The process described above could be extended to give the parameters for the mth highest wave, but with increasing complication. In some cases the following alternative method of reducing the sampling errors may be worth considering. A record containing N waves may be divided into n subgroups, and the average of the maximum waves in all these subgroups may be taken. The standard error of this estimate will be reduced from that of the overall maximum in the ratio

$$n^{-\frac{1}{2}}\frac{D(N/n)}{D(N)} \sim \left[n\left(1-\frac{\ln n}{\ln N}\right)\right]^{-\frac{1}{2}}$$

for large N. For $n = 2$, the reduction in sampling error is numerically about the same as that of the second highest wave.

4. The effect of correlation

The theory has so far been developed entirely on the assumption that the N wave crests in the sample are mutually uncorrelated. In practice this condition could be met by ignoring all but every nth wave, where n may be 3 or 4, say, but this process would be both tedious and wasteful of available data. However, if each wave is highly correlated to the adjacent waves, the effective number of independent waves is clearly reduced to some extent, and we shall now consider the magnitude of this reduction. Watson (1954) has shown that if the members of a population are unbounded and m-dependent (that is correlation is negligible only between waves separated in order by more than m), and the ratio of the probability of any two waves exceeding c to the probability that one wave exceeds c tends to zero as c tends to infinity, then the distribution of extremes tends to the same value for increasing N as in the case of independence. We shall here extend Watson's analysis to give a higher approximation for the moderately large values of N in which we are mainly interested.

Suppose the ordered set of crest heights are $x_1, x_2, x_3, \ldots, x_N$. Let

$$Q(x_{R_1}, x_{R_2}, \ldots, x_{R_r}; x) \quad (r \leqslant N)$$

denote the probability that the r quoted waves shall exceed x, regardless of the value of the remaining $N-r$ waves, and $\Sigma Q_r(x)$ denote the sum of such probabilities for all the $\binom{N}{r}$ combinations possible. Then it can be shown that the probability $p_N(x)$ that all the waves (and therefore the maximum wave) shall be less than x can be expressed in general by

$$p_N(x) = 1-\Sigma Q_1(x)+\Sigma Q_2(x)-\Sigma Q_3(x)+\ldots+(-1)^N\Sigma Q_N(x). \tag{23}$$

Let us first apply (23) to the simplest case, that of independence. Here the probability of any r waves exceeding x is $q_1^r(x)$, and so

$$\Sigma Q_r(x) = N(N-1)(N-2)\ldots(N-R+1)(q_1^r/r!).$$

Equation (23) then gives immediately

$$p_N(x) = [1-q_1(x)]^N,$$

as in equation (1).

Following Watson (1954), we define a sequence $c_N(\xi)$ such that for any given positive number $\xi \leqslant 1$:

$$q_1(c_N) = \xi/N.$$

Clearly $c_N \to \infty$ as $N \to \infty$, so that if we put x equal to c_N we obtain in the limit the well-known result for independence

$$p_N(c_N) \sim \mathrm{e}^{-\xi}(1-\xi^2/2N) \sim \mathrm{e}^{-\xi}.$$

Next consider the waves to be 1-dependent, that is, to have correlation only between adjacent members. If $q_r'(x)$ represents the probability of r *consecutive* waves all exceeding x, the components of (23) can be expressed in the form

$$\begin{aligned}
\Sigma Q_1(x) &= Nq_1(x),\\
\Sigma Q_2(x) &= (N-1)(N-2)(q_1^2/2!)+(N-1)q_2',\\
\Sigma Q_3(x) &= (N-2)(N-3)(N-4)(q_1^3/3!)+(N-2)(N-3)q_1q_2'+(N-2)q_3',\\
\Sigma Q_4(x) &= (N-3)(N-4)(N-5)(N-6)(q_1^4/4!)+(N-3)(N-4)(N-5)[(q_1^2q_2'/2!]\\
&\qquad +(N-3)(N-4)[q_1q_3'+(q_2')^2/2!]+(N-3)q_4',
\end{aligned}$$

etc.

With $x = c_N$ and $q_1 = \xi/N$, (23) then consists of a number of terms independent of N, and a remainder of order N^{-1}. The former terms may be divided into a group involving ξ only, namely

$$1-\xi+\xi^2/2!-\ldots+(-1)^N\xi^N/N! \sim \mathrm{e}^{-\xi},$$

and another finite group of terms involving q_2'/q_1, starting as follows

$$(q_2'/q_1)(\xi-\xi^2+\xi/2!-\ldots)+1/2!\,(q_2'/q_1)^2(\xi^2-\xi^3+\xi^4/2!-\ldots)$$
$$+1/3!\,(q_2'/q_1)^3(\xi^3-\xi^4+\xi^5/2!-\ldots)+\ldots$$

and tending asymptotically to

$$\mathrm{e}^{-\xi}(\mathrm{e}^{(q_2'/q_1)\xi}-1).$$

On collecting similarly the terms in q_3'/q_1, etc., we obtain finally as the asymptotic form for $p_N(c_N)$, correct to order N^{-1},

$$p_N(c_N) \sim \mathrm{e}^{-\xi}\{1+[\exp(+\xi q_2'/q_1)-1]+[\exp(-\xi q_3'/q_1)-1]+[\exp(+\xi q_4'/q_1)-1]+\ldots\}.$$

Since $q_n'/q_1 = (q_2'/q_1)^{n-1}$, it follows that if $q_2'/q_1 \ll 1$, then q_3'/q_1 and higher terms can be neglected, and the first approximation to $p_N(c_N)$ for 1-dependence is

$$\begin{aligned}
p_N(c_N) &\sim \exp\{-\xi[1-(q_2'/q_1)]\}\\
&= \exp\{-Nq_1[1-(q_2'/q_1)]\}.
\end{aligned}$$

The general result for m-dependence can be obtained in a similar way. If $q_2^{(r)}$ $(r \leqslant m)$ is the joint probability that two waves separated in order by r exceed x, and Σq_2 denotes $\sum_{r=1}^{m} q_2^{(r)}$ and Σq_s denotes a similar sum covering all combinations of groups of size s, no two members of which are separated in order by more than m, then it can be shown that

$$p_N(c_N) \sim e^{-\xi}\{1+[\exp(+\xi\Sigma q_2/q_1)-1]+[\exp(-\xi\Sigma q_3/q_1)-1]+\ldots\}. \tag{24}$$

Watson (1954) used the condition that

$$\lim_{c_N\to\infty} \max_{r\leqslant m} (q_2^{(r)}/q_1) = 0, \tag{25}$$

from which it follows that

$$\lim_{c_N\to\infty} \Sigma q_s/q_1 = 0 \quad (s \geqslant 2).$$

He deduced that

$$\lim_{c_N\to\infty} p_N(c_N) = e^{-\xi},$$

so that for large N the distribution tends to the same form as for independence. Condition (25) will certainly hold for the type of distribution with which we are dealing, but in order to assess the effect of correlation for moderate values of N we shall retain a few more terms in (24). Assume for convenience that quotients of higher order than q_3/q_1 can be neglected. Then (24) can be written

$$\begin{aligned} p_N(c_N) &\sim \exp\{-\xi[1-(\Sigma q_2/q_1)]\}\{1-\xi(\Sigma q_3/q_1)\exp[-\xi(\Sigma q_2/q_1)]\} \\ &\doteqdot \exp[-\xi\{1-(\Sigma q_2/q_1)+(\Sigma q_3/q_1)\exp[-\xi(\Sigma q_2/q_1)]\}]. \end{aligned} \tag{26}$$

If we now replace c_N by x, and ξ by $Nq_1(x)$, (26) can be interpreted as meaning that the distribution of the highest wave for m-dependence is to a first approximation equivalent to that for independence with N multiplied by

$$1-\Sigma q_2/q_1+(\Sigma q_3/q_1)\,e^{-N\Sigma q_2} = 1-\alpha, \tag{27}$$

say, where for the approximation to be valid α must of course be fairly small.

In order to assess relative magnitudes of α it is necessary to obtain an expression for $\Sigma q_r/q_1$ in terms of the energy spectrum of $f(t)$. An exact expression would be very difficult to obtain. However, when the spectrum is narrow, which is the case when correlation becomes important, the following treatment gives a good approximation.

For a narrow spectrum we have

$$q_1(x) \doteqdot e^{-\frac{1}{2}x^2},$$

and, given a wave crest x_1, the next wave crest, x_2, occurs approximately after an interval for which the autocorrelogram of $f(t)$ has its first maximum, ρ_1, say, with a mean value $\rho_1 x_1$ and a nearly normal distribution of variance $(1-\rho_1^2)$. Hence, the joint probability that both x_1 and x_2 shall exceed x is given by

$$\begin{aligned} q_2'(x) &\doteqdot \int_x^\infty dx_1\, x_1\, e^{-\frac{1}{2}x^2} \int_x^\infty [2\pi(1-\rho_1^2)]^{-\frac{1}{2}} \exp[-\tfrac{1}{2}(x_2-\rho_1 x_1)^2/(1-\rho_1^2)]\, dx_2 \\ &= [(1+\rho_1)/\sqrt{(2\pi)}]\, e^{-\frac{1}{2}x^2} \int_{\mu_1 x}^\infty e^{-\frac{1}{2}t^2}\, dt, \end{aligned}$$

where

$$\mu_1 = [(1-\rho_1)/(1+\rho_1)]^{\frac{1}{2}}.$$

Therefore

$$q_2'/q_1 \doteqdot \tfrac{1}{2}(1+\rho_1)\,\mathrm{erfc}\,(\mu_1 x/\sqrt{2}), \tag{28}$$

where

$$\mathrm{erfc}\, u = 2/\sqrt{\pi} \int_u^\infty e^{-\frac{1}{2}t^2}\, dt$$

provided x is large.

Figure 5 shows curves of the function given by (28) for various values of ρ_1. It will be seen that all curves tend to zero with increasing x, the more rapidly the smaller the value of ρ_1, as we should expect. In fact, for values of x greater than about 4, q_2'/q_1 has appreciable magnitude only for values of ρ_1 in the range $0{\cdot}8 < \rho_1 < 1$.

For q_2''/q_1 we replace ρ_1 by ρ_2 and μ_1 by μ_2 in (28), where ρ_2 is the second maximum of the autocorrelogram, and so on. To the same order of accuracy,

$$q_3'/q_1 = (q_2'/q_1)^2$$

and similar expressions hold for other combinations of three waves.

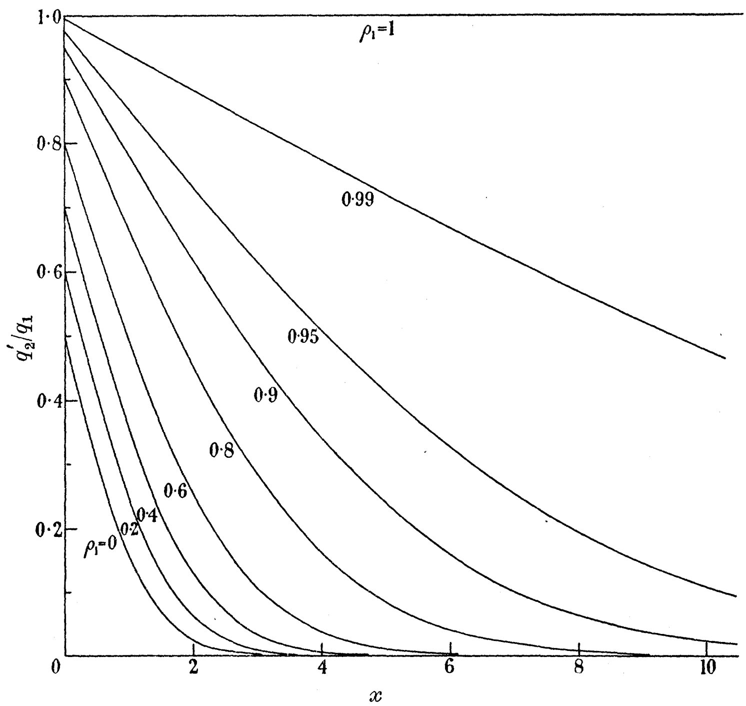


FIGURE 5. Curves of q_1'/q_1 (equation (28)).

Substitution of expressions as above in (26) gives an approximation to $p_N(x)$ in terms of the autocorrelogram, and it would be possible in theory to integrate the result to obtain the effect of correlation on the moments of the distribution. However, the integrations would prove very laborious, and it is convenient again to approximate, this time using (27) with a suitable constant value of x, namely

$$x = (2 \ln N)^{\frac{1}{2}}, \quad \text{or} \quad \xi = 1.$$

This is close to the mean value of x in the case of independence, and even closer to the mode of the distribution (Longuet-Higgins 1952). Further, since the distribution is confined to a relatively narrow region about the mode, the approximation should be reasonably close. This then gives for the effective value of N

$$N(1-\alpha) \doteqdot N[1-\beta(N)+\gamma(N)\,\mathrm{e}^{-\beta(N)}], \tag{29}$$

where
$$\beta(N) = \sum_{r=1}^{m} \tfrac{1}{2}(1+\rho_r)\,\mathrm{erfc}\,\{[(1-\rho_r)/(1+\rho_r)]\ln N\}^{\frac{1}{2}}$$
$$= \sum_{r=1}^{m} \beta_r(N)$$
and
$$\gamma(N) = \sum_{r=1}^{m}\sum_{s=1}^{m} \beta_r(N)\,\beta_s(N).$$

(a) *Application to second and third maxima*

The type of argument considered above can be applied to the distribution of the second and third highest waves for m-dependence, by replacing N by $N(1-\alpha)$ in the results of § 3, on condition that the first and second maxima are independent,

TABLE 2. SAMPLING ERRORS AND OTHER PARAMETERS FOR ENERGY SPECTRA OF VARIOUS WIDTHS

	$\delta=0{\cdot}1,\ \epsilon=0{\cdot}12$			$\delta=0{\cdot}2,\ \epsilon=0{\cdot}22$			$\delta=0{\cdot}3,\ \epsilon=0{\cdot}32$			$\delta=0{\cdot}4,\ \epsilon=0{\cdot}40$		
N	10^2	10^3	10^4	10^2	10^3	10^4	10^2	10^3	10^4	10^2	10^3	10^4
α	0·63	0·48	0·38	0·19	0·13	0·09	0·06	0·03	0·01	0·02	0·00	0·00
Δ	0·15	0·09	0·07	0·13	0·09	0·07	0·12	0·08	0·06	0·12	0·08	0·06
Δ_s	0·11	0·04	0·01	0·08	0·03	0·01	0·07	0·02	0·01	0·06	0·02	0·01

	$\delta=0{\cdot}6,\ \epsilon=0{\cdot}53$			$\delta=0{\cdot}8,\ \epsilon=0{\cdot}61$			$\delta=1{\cdot}0,\ \epsilon=0{\cdot}67$		
N	10^2	10^3	10^4	10^2	10^3	10^4	10^2	10^3	10^4
α	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
Δ	0·125	0·085	0·065	0·126	0·086	0·065	0·127	0·087	0·066
Δ_s	0·051	0·016	0·005	0·047	0·015	0·005	0·044	0·014	0·004

or if the third is considered, that it should be independent of both the first and the second. This proviso means that the second highest wave must be counted as the greatest of all waves barring the highest itself *and* m waves on either side of it (assuming of course that $m \ll N$). Without this proviso the problem is much more difficult, and further, if the correlation is high there is a strong tendency for the true second highest wave to be adjacent to the highest, and nearly equal to it; their distributions would therefore be nearly identical, and there would be no advantage gained by way of reduction in sampling error.

(b) *Numerical example*

Values of α given by equation (29) were calculated for a series of spectra, each spectrum consisting of a uniform band of energy centred on a mean angular frequency $\bar{\sigma}$, and extending between $\bar{\sigma}(1+\delta)$ and $\bar{\sigma}(1-\delta)$. The results, for $\delta = 0{\cdot}1, 0{\cdot}2, 0{\cdot}3, 0{\cdot}4, 0{\cdot}6, 0{\cdot}8, 1{\cdot}0$ and $N = 100$, 1000 and 10 000, are shown in table 2.

The autocorrelogram for this form of spectrum is
$$\frac{\sin(\delta\bar{\sigma}\tau)}{\delta\bar{\sigma}\tau}\cos\bar{\sigma}\tau,$$

whose maxima (except for $\tau = 0$) were used as successive values of ρ_r. The corresponding value of ϵ is

$$\epsilon = \delta[(60+4\delta^2)/(45+90\delta^2+9\delta^4)]^{\frac{1}{2}}.$$

The relatively large values of α obtained for the narrowest spectrum, $\delta = 0{\cdot}1$, are probably not very accurate, owing to the omission of terms of order q_4/q_1 in (27); accuracy will again be impaired for δ greater than about 0·4, since some of the assumptions used in deriving (29) do not hold for wide spectra. However, it is clear that for the type of spectrum considered, and for δ greater than about 0·4, α rapidly becomes so small as to have almost negligible effect on calculations.

Table 2 also offers comparison between the relative standard errors in estimating $m_0^{\frac{1}{2}}$ from the highest wave, and from direct measurement of the r.m.s. value of $f(t)$ over the time taken by N waves. The former was in general calculated by interpolation of values of $\Delta = D/M_1$ from Appendix 1 with $\epsilon = 0$ using an effective value of N equal to

$$N(1-\epsilon^2)^{\frac{1}{2}}(1-\alpha).$$

However, when α was negligible Appendix 1 was used with the true values of N and ϵ.

We have denoted by Δ_s the sampling error, relative to the mean, of the standard deviation of $f(t)$ evaluated over N waves. This was calculated from equation (3·9–9) of Rice (1945), which in our notation can be written

$$\left[\frac{\text{s.e. of (s.d.)}^2}{m_0}\right]^2 = (2\delta\bar{\sigma})^{-2}\int_{\bar{\sigma}(1-\delta)}^{\bar{\sigma}(1+\delta)} d\sigma_1 \int_{\bar{\sigma}(1-\delta)}^{\bar{\sigma}(1+\delta)} d\sigma_2$$

$$\times\left\{\left[\frac{\sin\frac{1}{2}(\sigma_1-\sigma_2)\,T}{\frac{1}{2}(\sigma_1-\sigma_2)\,T}\right]^2+\left[\frac{\sin\frac{1}{2}(\sigma_1=\sigma_2)\,T}{\frac{1}{2}(\sigma_1+\sigma_2)\,T}\right]^2\right\},$$

where T is the duration of the record, taken as

$$T = N(2\pi/\bar{\sigma})\,[(15+5\delta^2)/(15+30\delta^2+3\delta^4)]^{\frac{1}{2}},$$

which is the expected value for N waves. The double integral can be evaluated to give a bulky expression involving sine and cosine integrals, which for the values of N concerned reduces to

$$\left[\frac{\text{s.e. of (s.d.)}^2}{m_0}\right]^2 = \frac{2\pi}{2\delta\bar{\sigma}T}+O(\delta\bar{\sigma}T)^{-2}$$

and further calculation shows that the sampling error required is given by

$$\Delta_s = \frac{\text{s.e. of s.d.}}{m_0^{\frac{1}{2}}} \doteqdot \frac{1}{2}\left[\frac{2\pi}{2\delta\bar{\sigma}T}\right]^{\frac{1}{2}}.$$

An alternative method is to take the mean of all the maxima of $f(t)$, for which the sampling error would be roughly equal to Δ_s. Comparison of Δ with Δ_s in table 2 shows that for $N = 100$ there is little advantage gained in estimating $m_0^{\frac{1}{2}}$ by the laborious direct method, for Δ is only slightly greater than Δ_s, except for large δ, when both Δ and Δ_s are very small anyway. It is also seen that the two sampling errors tend to equality as δ decreases; this is because both tend to the error of an estimate based on a single wave crest as the spectral width tends to zero for constant N. For the larger values of N the relative differences increase, because Δ_s decreases like $N^{-\frac{1}{2}}$, while Δ decreases like $(\ln N)^{-1}$. The slight increase in both sampling errors for large values of δ is caused by a decrease in T for fixed N as ϵ increases.

5. A practical application

The theory developed in the previous sections was applied to a practical instance of a very long continuous record of 10 000 waves, taken in the Bay of Biscay on the R.R.S. *Discovery II* on 21 and 22 May 1955. Conditions were specially chosen to be as nearly as possible stationary. The record had previously been analyzed by Mr M. J. Tucker for the variation of mean square wave height in successive 10 min samples (Tucker 1957). Simultaneously with the record of wave height, $f(t)$, an electrical device recorded 'mean rectified wave height', or effectively, the mean value of $|f(t)|$ over a few minutes. On multiplication by a calibration factor and

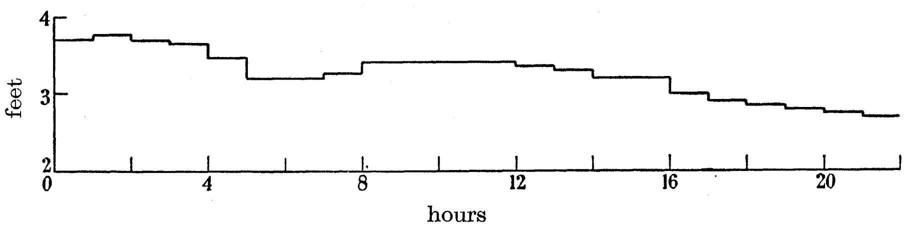
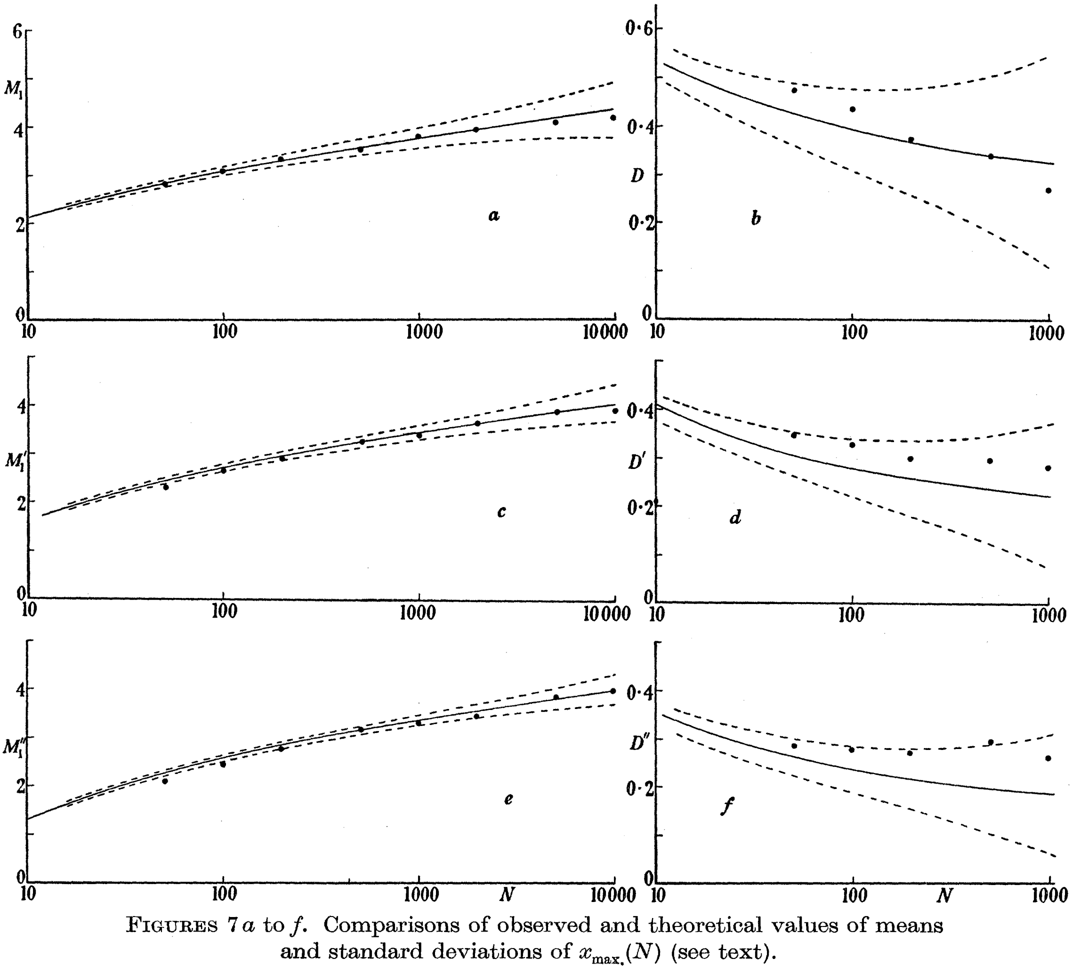


Figure 6. Estimates of $m_0^{\frac{1}{2}}$ over 24 h continuous wave record.

using the Gaussian property of $f(t)$, average values of this quantity taken over 3 h give reliable estimates of $m_0^{\frac{1}{2}}$. Successive estimates of $m_0^{\frac{1}{2}}$ so obtained are represented in figure 6, which covers the 22 h period containing the 10 000 waves analyzed. The constant value shown over each hourly period is the average over the period consisting of that hour and the hours preceding and following it, so that the averages actually extended over 24 h altogether. It is clear that, in spite of a slow irregular decrease from 3·7 to 2·7 ft., any short section of $f(t)$ lasting about an hour or less can be regarded as stationary with the value of $m_0^{\frac{1}{2}}$ read off figure 6 at the instant corresponding to the mid-point of the section.

Values of ϵ were estimated at several stages of the record by counting zero crossings and turning points of $f(t)$; the ratio of the numbers of these two events has the expectation $(1-\epsilon^2)^{\frac{1}{2}}$. It was found that ϵ varied little from an average value of 0·52 throughout the record. Autocorrelograms measured by an analogue computor at various stages gave practically negligible values of $\rho_1, \rho_2, \ldots$, although it was clear from visual inspection that there was some correlation between at least consecutive wave crests. The failure of the autocorrelogram to show up this correlation is understandable from the wide variation in time intervals between successive waves associated with the wide spectrum. To obtain a more realistic estimate of ρ_1, the correlation between successive wave crests was calculated numerically at various stages, and these gave estimates of ρ_1 of about 0·5. However, the corresponding values of α ranged only from 0·08 for $N = 50$ to 0·01 for $N = 10\,000$, and so the error in taking ρ_1 from the autocorrelogram and thus ignoring correlation altogether would not be at all large.

The waves were divided into 200 consecutive groups of 50 waves each, and the maximum wave crest in each of the 200 groups was measured and divided by the appropriate value of $m_0^{\frac{1}{2}}$ from figure 6 to give a sample value of $x_{\text{max.}}(50)$ (the notation is obvious). Sample values of $x'_{\text{max.}}(50)$ and $x''_{\text{max.}}(50)$, the second and third highest maxima, were obtained similarly, taking care that none of these three maxima was



FIGURES 7 *a* to *f*. Comparisons of observed and theoretical values of means and standard deviations of $x_{\text{max.}}(N)$ (see text).

consecutive (m being taken as 1). Then the 200 sets were combined in consecutive pairs, giving 100 sample values of $x_{\text{max.}}(100)$, $x'_{\text{max.}}(100)$ and $x''_{\text{max.}}(100)$, and the process was continued to give 50 samples of first, second and third maxima for 200 waves, 20 for 500, 10 for 1000, 5 for 2000, 2 for 5000, and one value of each maximum for 10 000 wave crests. From these measurements, means and standard deviations were computed, and the results are plotted against curves derived from theory in figures 7 *a* to *f*.

In figures 7 *a* and *b* the solid curves were obtained by interpolation from the values of M_1 and D, respectively, in Appendix 1, with $\epsilon = 0{\cdot}52$ and $N(1-\alpha)$ in place of N.

In figures 7 *c* to *f*, the solid curves were obtained from the curves of M_1'/M_1, D'/D, M_1''/M_1, D''/D, shown in figure 5, with $N(1-\epsilon^2)^{\frac{1}{2}}(1-\alpha)$ in place of N, and the values of M_1 and D as derived above. The broken curves represent confidence limits for the $(10\,000/N)$ sample values from which the estimates were derived, and in fact differ from the solid curves by twice the standard sampling error, thus corresponding roughly to 5 % confidence limits. In figures 7 *a*, *c* and *e* the sampling errors are $D/\surd n$, $D'/\surd n$ and $D''/\surd n$, respectively, where n is the sample size, $10\,000/N$. For the standard deviations (figures 7 *b*, *d* and *f*), the sampling variance is theoretically

$$(m_4-m_2^2)/(4m_2\surd n) \quad \text{or} \quad (\beta_2-1)\,m_2/4\surd n$$

(see Kendall 1945, p. 224). The exact value of the fourth moment about the mean not being readily available, we have here used the limiting value $\beta_2 = 5\cdot4$, deduced from the asymptotic formulae for large N in § 2. With this figure, the sampling errors are simply $(1\cdot1)^{\frac{1}{2}}$ times those used in figures 7 *a*, *c* and *e*. No comparisons were made beyond $N = 1000$ in figures 7 *b*, *d* and *f*, since the confidence limits there became too wide: for example, for $N = 2000$, the number of sample maxima was only 5, from which it is clearly impossible to make a valuable estimate of standard deviation.

The comparisons of mean values in figures 7 *a*, *c* and *e* are all quite good, particularly for the larger values of N. The estimates tend to fall slightly below the lower confidence limits for M_1' and M_1'' for $N = 50$ and 100, probably because the effect of correlation has not been fully allowed for in the theory of the second and third maxima. However, the error in these cases is numerically quite small. The comparisons of standard deviations with D, D' and D'' are again mostly within the confidence limits, but there is a tendency to be above the theoretical curve rather than below it. This again may be partly due to inadequate allowance for correlation, but the random errors in measurement of x, which are certainly of the order of $0\cdot1$, must also be responsible. Again, the differences are numerically quite small compared with the mean values, M_1, etc.

6. Concluding remarks

From the analysis and computations presented above, expectancies and probability distributions of the highest three maxima of a record may be calculated, given m_0 and the degree of correlation. Conversely, from measurements of these maxima one may rapidly estimate m_0. The whole theory is based ultimately on the assumption that the function $f(t)$ is a stationary random function of time which can be represented by the linear sum of an infinite number of sinusoids in random phase (Rice 1945; Cartwright & Longuet-Higgins 1956). Though this assumption is known to be justified in the main for sea waves and similar variables, it cannot be completely accurate; one may justifiably suspect that non-linearities might become important for the largest waves considered in a theory of extreme values. But the satisfactory results of measurements shown in figures 7 *a* to *f* confirm that the assumed representation still holds good well into the tail of the probability distribution. Nevertheless, errors will always be likely to arise from non-stationarity of the wave system, which for sea waves will limit the number of waves to which the theory can confidently be applied to about 1000 at most. For very long periods of time, such as when one considers the maximum wave height over the course of years, the more general theory as described by Gumbel (1954) is appropriate.

The treatment of correlation in § 4 can hardly be called precise, since it was found necessary to make several simplifying assumptions. Luckily it appears that unless N is very small, or the spectrum particularly narrow, the effect of correlation will be slight, so that a certain amount of inaccuracy in estimating the effect will be unimportant. When the effect does become important, the basic assumptions, most of which involve the narrowness of the spectrum, are more justifiable, and so accuracy can be expected to increase with the magnitude of the effect. In any case, under most conditions, errors will not be great if correlation is ignored altogether, and so the tables in Appendix 1 can frequently be used with confidence without any additional correction.

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Appendix 1. Tables of $M_1(N, \epsilon)$, $M_2(N, \epsilon)$ and $D(N, \epsilon)$ for $N = 2^m$

$\epsilon = 0$

m	M_1	M_2	D
0	1·253 314	2·000 000	0·6551
1	1·620 401	3·000 000	0·6118
2	1·963 643	4·166 667	0·5575
3	2·275 764	5·435 714	0·5066
4	2·558 673	6·761 458	0·4633
5	2·816 765	8·116 990	0·4276
6	3·054 383	9·487 782	0·3982
7	3·275 156	10·866 294	0·3737
8	3·481 954	12·248 690	0·3531
9	3·677 015	13·633 033	0·3356
10	3·862 086	15·018 351	0·3204
11	4·038 545	16·404 158	0·3071
12	4·207 489	17·790 208	0·2954
13	4·369 806	19·176 380	0·2849
14	4·526 225	20·562 613	0·2755
15	4·677 349	21·948 877	0·2670

$\epsilon = 0.1$

m	M_1	M_2	D
0	1·247 032	1·990 000	0·6595
1	1·616 701	2·989 950	0·6134
2	1·960 850	4·156 616	0·5583
3	2·273 441	5·425 664	0·5071
4	2·556 644	6·751 408	0·4637
5	2·814 940	8·106 940	0·4278
6	3·052 710	9·477 732	0·3984
7	3·273 602	10·856 244	0·3739
8	3·480 496	12·238 640	0·3533
9	3·675 637	13·622 983	0·3357
10	3·860 776	15·008 301	0·3205
11	4·037 293	16·394 107	0·3072
12	4·206 289	17·780 158	0·2955
13	4·368 651	19·166 330	0·2850
14	4·525 111	20·552 563	0·2756
15	4·676 272	21·938 827	0·2670

$\epsilon = 0.2$

m	M_1	M_2	D
0	1·227 992	1·960 000	0·6723
1	1·605 255	2·959 200	0·6183
2	1·952 263	4·125 845	0·5608
3	2·266 312	5·394 892	0·5087
4	2·550 421	6·720 636	0·4647
5	2·809 345	8·076 168	0·4287
6	3·047 583	9·446 960	0·3990
7	3·268 839	10·825 472	0·3744
8	3·476 029	12·207 868	0·3537
9	3·671 415	13·592 211	0·3360
10	3·856 763	14·977 529	0·3208
11	4·033 459	16·363 336	0·3075
12	4·202 612	17·749 386	0·2957
13	4·365 114	19·135 558	0·2852
14	4·521 697	20·521 792	0·2758
15	4·672 970	21·908 055	0·2672

$\epsilon = 0.3$

m	M_1	M_2	D
0	1·195 586	1·910 000	0·6932
1	1·585 080	2·905 942	0·6273
2	1·937 204	4·072 362	0·5653
3	2·253 853	5·341 404	0·5114
4	2·539 562	6·667 147	0·4667
5	2·799 591	8·022 680	0·4301
6	3·038 648	9·393 471	0·4001
7	3·260 544	10·771 984	0·3753
8	3·468 250	12·154 379	0·3544
9	3·664 065	13·538 722	0·3367
10	3·849 776	14·924 041	0·3214
11	4·026 786	16·309 847	0·3080
12	4·196 213	17·695 897	0·2961
13	4·358 957	19·082 070	0·2856
14	4·515 757	20·468 303	0·2761
15	4·667 225	21·854 567	0·2675

$\epsilon = 0.4$

m	M_1	M_2	D
0	1·148 681	1·840 000	0·7215
1	1·554 558	2·827 117	0·6407
2	1·914 377	3·992 408	0·5723
3	2·235 048	5·261 365	0·5157
4	2·523 212	6·587 105	0·4696
5	2·784 924	7·942 637	0·4322
6	3·025 225	9·313 428	0·4018
7	3·248 088	10·691 941	0·3766
8	3·456 574	12·074 337	0·3556
9	3·653 037	13·458 680	0·3376
10	3·839 296	14·843 998	0·3222
11	4·016 779	16·229 804	0·3087
12	4·186 619	17·615 854	0·2968
13	4·349 728	19·002 027	0·2862
14	4·506 854	20·388 260	0·2766
15	4·658 615	21·774 524	0·2680

$\epsilon = 0{\cdot}5$

m	M_1	M_2	D
0	1·085 402	1·750 000	0·7562
1	1·511 284	2·718 246	0·6590
2	1·881 614	3·879 733	0·5825
3	2·208 107	5·148 113	0·5219
4	2·499 856	6·473 784	0·4738
5	2·764 013	7·829 309	0·4354
6	3·006 111	9·200 100	0·4042
7	3·230 367	10·578 612	0·3786
8	3·439 974	11·961 008	0·3572
9	3·637 364	13·345 351	0·3390
10	3·824 408	14·730 669	0·3234
11	4·002 567	16·116 476	0·3097
12	4·172 997	17·502 526	0·2977
13	4·336 627	18·888 698	0·2870
14	4·494 218	20·274 931	0·2774
15	4·646 397	21·661 195	0·2687

$\epsilon = 0{\cdot}6$

m	M_1	M_2	D
0	1·002 651	1·640 000	0·7967
1	1·451 656	2·572 952	0·6824
2	1·835 400	3·724 396	0·5964
3	2·169 914	4·990 254	0·5308
4	2·466 783	6·315 344	0·4799
5	2·734 460	7·670 741	0·4399
6	2·979 142	9·041 503	0·4077
7	3·205 393	10·420 009	0·3814
8	3·416 600	11·802 403	0·3595
9	3·615 311	13·186 746	0·3410
10	3·803 472	14·572 064	0·3251
11	3·982 590	15·957 870	0·3112
12	4·153 855	17·343 921	0·2990
13	4·318 224	18·730 093	0·2882
14	4·476 472	20·116 326	0·2784
15	4·629 242	21·502 590	0·2696

$\epsilon = 0{\cdot}7$

m	M_1	M_2	D
0	0·895 045	1·510 000	0·8420
1	1·369 824	2·381 722	0·7108
2	1·769 851	3·510 320	0·6148
3	2·114 917	4·767 903	0·5432
4	2·418 920	6·090 019	0·4887
5	2·691 664	7·444 336	0·4464
6	2·940 119	8·814 698	0·4128
7	3·169 296	10·193 051	0·3855
8	3·382 852	11·575 386	0·3629
9	3·583 500	12·959 705	0·3439
10	3·773 293	14·345 014	0·3275
11	3·953 811	15·730 816	0·3134
12	4·126 296	17·116 865	0·3009
13	4·291 739	18·503 036	0·2899
14	4·450 943	19·889 269	0·2800
15	4·604 572	21·275 533	0·2710

$\epsilon = 0{\cdot}8$

m	M_1	M_2	D
0	0·751 989	1·360 000	0·8914
1	1·254 781	2·128 375	0·7442
2	1·673 847	3·208 839	0·6380
3	2·032 224	4·443 457	0·5599
4	2·345 850	5·754 267	0·5013
5	2·625 795	7·102 829	0·4561
6	2·879 815	8·470 166	0·4205
7	3·113 418	9·846 885	0·3918
8	3·330 581	11·228 315	0·3682
9	3·534 232	12·612 122	0·3483
10	3·726 569	13·997 136	0·3314
11	3·909 277	15·382 766	0·3167
12	4·083 671	16·768 713	0·3039
13	4·250 795	18·154 824	0·2925
14	4·411 497	19·541 021	0·2823
15	4·566 468	20·927 262	0·2732

$\epsilon = 0{\cdot}9$

m	M_1	M_2	D
0	0·546 307	1·190 000	0·9442
1	1·079 002	1·776 430	0·7824
2	1·519 750	2·753 913	0·6665
3	1·894 363	3·927 271	0·5819
4	2·220 539	5·199 902	0·5188
5	2·510 463	6·523 682	0·4704
6	2·772 616	7·874 375	0·4324
7	3·012 985	9·239 598	0·4019
8	3·235 876	10·612 903	0·3768
9	3·444 446	11·990 861	0·3559
10	3·641 061	13·371 598	0·3380
11	3·827 527	14·754 061	0·3226
12	4·005 253	16·137 636	0·3092
13	4·175 354	17·521 955	0·2973
14	4·338 734	18·906 786	0·2867
15	4·496 130	20·291 981	0·2771

$\epsilon = 1{\cdot}0$

m	M_1	M_2	D
0	0·000 000	1·000 000	1·0000
1	0·564 190	1·000 000	0·8256
2	1·029 375	1·551 329	0·7012
3	1·423 600	2·399 535	0·6106
4	1·765 991	3·413 736	0·5431
5	2·069 669	4·525 147	0·4915
6	2·343 733	5·696 573	0·4511
7	2·594 597	6·907 180	0·4186
8	2·826 863	8·144 806	0·3920
9	3·043 903	9·402 028	0·3697
10	3·248 240	10·674 093	0·3508
11	3·441 799	11·957 814	0·3344
12	3·626 082	13·250 957	0·3201
13	3·802 279	14·551 896	0·3075
14	3·971 351	15·859 406	0·2963
15	4·134 083	17·172 536	0·2862

Appendix 2. Asymptotic formulae for the moments $M_r(N)$ of the probability distribution $q_N(x, 0)$

From § 2, equations (3) and (4), we have for $\epsilon = 0$

$$M_r(N) = \int_0^\infty x^r \, \mathrm{d}[(1-\mathrm{e}^{-\frac{1}{2}x^2})^N].$$

With the substitution $x^2 = 2(y+\theta)$, $\theta = \ln N$, this becomes

$$M_r(N) = 2^{\frac{1}{2}r} \int_{-\theta}^\infty (y+\theta)^{\frac{1}{2}r} \, \mathrm{d}[(1-\mathrm{e}^{-y}/N)^N]. \tag{A 1}$$

We shall now prove two lemmas.

Lemma 1. *When* $r \geqslant 0$

$$\int_{-\theta}^\infty (y+\theta)^{\frac{1}{2}r} \, \mathrm{d}[(1-\mathrm{e}^{-y}/N)^N] = \int_{-\theta}^\infty (y+\theta)^{\frac{1}{2}r} \, \mathrm{d}[\exp(-\mathrm{e}^{-y})] + R(r, N),$$

where $R(r, N)$ *is* $O(N^{-1}(\ln N)^{\frac{1}{2}r})$.

Now

$$R(r, N) = \int_0^N (\ln N/u)^{\frac{1}{2}r} g'(u) \, \mathrm{d}u \tag{A 2}$$

$$= \tfrac{1}{2}r \int_0^N (\ln N/u)^{\frac{1}{2}r-1} g(u) \, \mathrm{d}u/u \quad (r \geqslant 0), \tag{A 3}$$

where

$$g(u) = \mathrm{e}^{-u} - (1-u/N)^N, \quad g'(u) = -\mathrm{e}^{-u} + (1-u/N)^{N-1}.$$

In the range $0 \leqslant u \leqslant N$, $g(u)$ is zero when

$$\mathrm{e}^{-u/N} = 1 - u/N,$$

which equation is only satisfied by $u = 0$. Therefore $g(u)$ does not change sign, and since $g(N) = \mathrm{e}^{-N} > 0$, it follows that $g(u) \geqslant 0$ throughout the range of integration.

From (A 3), therefore, $R(r, N) \geqslant 0$ if $r \geqslant 0$, since the integrand is never negative.

On the other hand, $g'(u)$ is zero when

$$\mathrm{e}^{-u/(N-1)} = 1 - u/N, \tag{A 4}$$

and it can be shown that this equation is satisfied for $u = 0$ and just one other value of u, say $u = k(N)$, where $0 < k(N) < N$. Since $g'(N) = -\mathrm{e}^{-N} < 0$, it follows that $g'(u) \leqslant 0$ for $k(N) \leqslant u \leqslant N$, and $g'(u) \geqslant 0$ for $0 \leqslant u \leqslant k(N)$. By expanding (A 4) in series, it can be seen that for large N

$$k(N) \sim 2(1-1/N).$$

In fact $k(N) < 2$ for all N, because

$$\mathrm{e}^{-2/(N-1)} - 1 + 2/N = \frac{2(N-3)}{3N(N-1)^3} + \left[\frac{2^4}{(N-1)^4 4!} - \frac{2^5}{(N-1)^5 5!}\right] + \left[\frac{2^6}{(N-1)^6 6!} - \frac{2^7}{(N-1)^7 7!}\right] + \ldots > 0 \quad \text{if} \quad N \geqslant 3,$$

so that $g'(2) < 0$ for all $N \geqslant 3$ (and, it is easily checked, for $N = 1$ and 2 also).

By Rolle's theorem, $g'(u)$ has at least one positive maximum for $u = k_1(N)$, say, where $0 \leqslant k_1(N) \leqslant k(N) < 2$, and

$$e^{-k_1} = (1-1/N)(1-k_1/N)^{N-2},$$

so that $0 < g'(k_1) = 1/N(1-k_1)(1-k_1/N)^{N-2} \leqslant 1/N$ (incidentally proving also that $k_1(N) \leqslant 1$).

Using (A 2), we can now state that

$$R(r,N) \leqslant 1/N \int_0^2 (\ln N/u)^{\frac{1}{2}r}\, du.$$

On integrating by parts, and putting $\theta_1 = \ln \tfrac{1}{2}N$, the last integral is shown equal to

$$2[\theta_1^{\frac{1}{2}r} + \tfrac{1}{2}r\theta_1^{\frac{1}{2}r-1} + \tfrac{1}{2}r(\tfrac{1}{2}r-1)\,\theta_1^{\frac{1}{2}r-2} + \ldots + (\tfrac{1}{2}r)!]$$

if r is even, or

$$2[\theta_1^{\frac{1}{2}r} + \tfrac{1}{2}r\theta_1^{\frac{1}{2}r-1} + \ldots + \tfrac{1}{2}r(\tfrac{1}{2}r-1)\ldots\tfrac{3}{2}.\tfrac{1}{2}R_1],$$

where
$$R_1 = \int_0^2 (\ln N/u)^{-\frac{1}{2}}\, du < 2\theta_1^{-\frac{1}{2}},$$

if r is odd.

Thus
$$0 \leqslant R(r,N) \leqslant 2/N(\ln \tfrac{1}{2}N)^{\frac{1}{2}r}\{1 + O[(\ln N)^{-1}]\},$$

which proves the lemma.

LEMMA 2. *Let* $I(r,N)$ *denote* $\int_{-\theta}^{\infty} (y+\theta)^{\frac{1}{2}r}\, d[\exp(-e^{-y})]$. *Then for any* $r \geqslant 0$ *and large* N

$$I(r,N) \sim \theta^{\frac{1}{2}r}[1 + \tfrac{1}{2}r\theta^{-1}A_1 + \tfrac{1}{2}!\,\tfrac{1}{2}r(\tfrac{1}{2}r-1)\,\theta^{-2}A_2 + \ldots],$$

where $A_s = \int_{-\infty}^{\infty} y^s\, d[\exp(-e^{-y})]$.

We first observe that

$$I(r,N) = \int_{-\theta}^{\theta} (y+\theta)^{\frac{1}{2}r}\, d[\exp(-e^{-y})] + I'(r,N),$$

where
$$\begin{aligned} I'(r,N) &= \int_{\theta}^{\infty} (y+\theta)^{\frac{1}{2}r}\, d[\exp(-e^{-y})] \\ &= N\int_{N^2}^{\infty} (\ln v)^{\frac{1}{2}r} \exp(-N/v)\, v^{-2}\, dv \\ &< N\int_{N^2}^{\infty} (ln\, v)^{\frac{1}{2}r}\, v^{-2}\, dv = O[(2\ln N)^{\frac{1}{2}r} N^{-1}], \end{aligned}$$

as can be verified by integration by parts. Therefore, with any value of r, for large N,

$$\begin{aligned} I(r,N) &\sim \int_{-\theta}^{\theta} (y+\theta)^{\frac{1}{2}r}\, d[\exp(-e^{-y})] \\ &\sim \theta^{\frac{1}{2}r}\int_{-\theta}^{\theta} [1 + \tfrac{1}{2}r\theta^{-1}y + \ldots + 1/s!\,\tfrac{1}{2}r(\tfrac{1}{2}r-1)\ldots(\tfrac{1}{2}r-s+1)\,\theta^{-s}y^s]\, d[\exp(-e^{-y})], \end{aligned} \tag{A 5}$$

where s can be chosen sufficiently large to make the last approximation as close as desired.

Now, for any $p \geqslant 0$,

$$\int_{\theta}^{\infty} y^p \,\mathrm{d}[\exp(-\mathrm{e}^{-y})] = \int_N^{\infty} (\ln v)^p \,\mathrm{e}^{-1/v} v^{-2} \mathrm{d}v < \int_N^{\infty} (\ln v)^p v^{-2} \mathrm{d}v = O[(\ln N)^p N^{-1}]$$

and
$$\int_{-\infty}^{-\theta} |y^p| \,\mathrm{d}[\exp(-\mathrm{e}^{-y})] = \int_N^{\infty} (\ln u)^p \,\mathrm{e}^{-u} \mathrm{d}u = O[(\ln N)^p \,\mathrm{e}^{-N}].$$

Therefore the range of integration in (A 5) can be extended to $(-\infty, \infty)$ with errors of any required degree of smallness, for all N greater than a certain number, and

$$I(r, N) \sim \theta^{\frac{1}{2}r}[1 + \tfrac{1}{2}r\theta^{-1}A_1 + \ldots + 1/s!\,\tfrac{1}{2}r(\tfrac{1}{2}r-1)\ldots(\tfrac{1}{2}r-s+1)\,\theta^{-s}A_s].$$

From (A 1), and the results of the above two lemmas, we can write the moments $M_r(N)$ in the following form

$$M_r(N) = (2\theta)^{\frac{1}{2}r}[1 + \tfrac{1}{2}r\theta^{-1}A_1 + \tfrac{1}{2}r(\tfrac{1}{2}r-1)\,\theta^{-2}A_2 + \ldots] + O[(\ln N)^{\frac{1}{2}r} N^{-1}].$$

It remains to evaluate the integrals A_s.

We have
$$\begin{aligned} A_s &= \int_{-\infty}^{\infty} y^s \,\mathrm{d}[\exp(-\mathrm{e}^{-y})] \\ &= \left\{\frac{\mathrm{d}^s}{\mathrm{d}t^s}\int_{-\infty}^{\infty} \mathrm{e}^{yt}\,\mathrm{d}[\exp(-\mathrm{e}^{-y})]\right\}_{t=0} \\ &= \left[\frac{\mathrm{d}^s}{\mathrm{d}t^s}(-t)!\right]_{t=0} = (-1)^s \frac{\mathrm{d}^s}{\mathrm{d}t^s}[\Gamma(t)]_{t=1}. \end{aligned}$$

The derivatives of the Γ-function may be expressed in terms of the derivatives of $\psi(z) = (\mathrm{d}/\mathrm{d}z)\ln\Gamma(z)$ by means of Leibnitz's theorem, as follows

$$\left.\begin{aligned} \Gamma'(z) &= \Gamma(z)\,\psi(z), \\ \Gamma''(z) &= \Gamma'\psi + \Gamma\psi', \\ \Gamma'''(z) &= \Gamma''\psi + 2\Gamma'\psi' + \Gamma\psi'', \\ \Gamma^{\mathrm{IV}}(z) &= \Gamma'''\psi + 3\Gamma''\psi' + 3\Gamma'\psi'' + \Gamma\psi''', \quad \text{etc.} \end{aligned}\right\} \tag{A6}$$

Given $\Gamma(z)$, the first s of these equations can be solved by a simple process to give $\Gamma^{(s)}(z)$ in terms of ψ, ψ', ψ'', ..., $\psi^{(s-1)}$, and it can be shown (Whittaker & Watson 1952, p. 241) that

$$\left.\begin{aligned} \psi(1) &= -\gamma = -0{\cdot}577\,216\ldots, \\ \psi'(1) &= S_2 = \tfrac{1}{6}\pi^2 = 1{\cdot}644\,934\ldots, \\ \psi''(1) &= -2S_3 = -2{\cdot}404\,114\ldots, \\ \psi'''(1) &= 6S_4 = \tfrac{1}{15}\pi^4 = 6{\cdot}493\,939\ldots, \\ \psi^{(n)}(1) &= (-1)^n n!\,S_{n+1}, \quad \text{where} \quad S_n = \sum_1^{\infty} t^{-n}. \end{aligned}\right\} \tag{A 7}$$

With $\Gamma(1) = 1$, (A 6) and (A 7) give the numerical values for A_s quoted in § 2.