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Inertial coupling of fluids with large density contrast

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Abstract

The coupling of two fluids of large density contrast is considered, and it is shown that the presence of a discontinuity of Lagrangian velocity across the interface between the two fluids, due to gravity waves, leads to an inertially coupled drag law of the form $\tau_s = K \mid \rho_1^{1/2}\hat{u}_1 - \rho_2^{1/2}\hat{u}_2 \mid (\rho_1^{1/2}\hat{u}_1 - \rho_2^{1/2}\hat{u}_2)$, where ρ_1 and ρ_2 are the densities of the fluids, $\hat{u}_1$ and $\hat{u}_2$ are the tangential fluid velocities relative to the interfacial non-wave induced velocity, K is a drag coefficient and τ_s is the interfacial shearing stress.

1. Introduction

The concept of viscosity in fluids as originally developed by Newton has been applied universally in fluid dynamics [1]. There is one important situation in which its application gives misleading results. This is the representation of the shearing stress between two fluids of large density contrast, e.g. air and water.

2. Viscous coupling

Consider the matching along a linear boundary segment between two fluids. The application of the viscous hypothesis leads to two interfacial boundary conditions,

$$u_0 = u_1 \mid_{n=0} = u_2 \mid_{n=0}, \tag{1}$$

$$\tau_s = \eta_1 \frac{\partial u_1}{\partial n}\bigg|_{n=0} = \eta_2 \frac{\partial u_2}{\partial n}\bigg|_{n=0}, \tag{2}$$

where n is normal to the interface ($n = 0$), and u_i and η_i are respectively the tangential component of velocity, and Newtonian viscosity for the two fluids, u_0 is the velocity of the interface, and τ_s is the interfacial shearing stress.

In this formulation the tangential velocity is continuous, but the velocity shear is discontinuous. In particular if η_1 and η_2 differ greatly, the discontinuity in shear is strong, and for a gas and liquid it would be due mainly to the difference in density between the two fluids.

On integrating over a small normal distance (n) in each fluid, over which τ_s is assumed to be a constant and eliminating the interfacial velocity, we obtain

$$\tau_s = \frac{\eta_1 \eta_2}{\eta_1 + \eta_2} \frac{u_1 - u_2}{n}, \tag{3}$$

where $u_1 = u_1(n)$ and $u_2 = u_2(-n)$ and

$$u_0 = \frac{u_1 \eta_1 + u_2 \eta_1}{\eta_1 + \eta_2}. \tag{4}$$

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Eq. (3) has the property that the shearing stress is proportional to the difference in velocity between the two fluids.

In the limit of $\eta_1/\eta_2 \to \infty$ in which $\boldsymbol{u}_1 \to \boldsymbol{u}_0$, it reduces to the relation for a single fluid adjacent to a solid surface moving at a velocity $\boldsymbol{u}_0$. Similar expressions apply if $\eta_2/\eta_1 \to \infty$.

3. Inertial coupling

For turbulent flow in the constant stress layer adjacent to a solid surface, the shearing stress is represented by a drag law [2], such that in fluid (1)

$$\boldsymbol{\tau}_s = \rho_1 K_1 \mid \boldsymbol{u}_1 - \boldsymbol{u}_0 \mid (\boldsymbol{u}_1 - \boldsymbol{u}_0), \quad \boldsymbol{u}_2 - \boldsymbol{u}_0 = \mathbf{0}, \tag{5}$$

and in fluid (2)

$$\boldsymbol{\tau}_s = \rho_2 K_2 \mid \boldsymbol{u}_0 - \boldsymbol{u}_2 \mid (\boldsymbol{u}_0 - \boldsymbol{u}_2), \quad \boldsymbol{u}_1 - \boldsymbol{u}_0 = \mathbf{0}, \tag{6}$$

in which ρ_1 and ρ_2 are respectively the density of each fluid, and K_i is a drag coefficient, applicable to the velocity $(\boldsymbol{u}_i)$. For similarity flows adjacent to an identical solid surface the drag coefficients would be expected to be independent of the fluid density, i.e. $K(\mid n \mid) = K_1(n) = K_2(-n)$.

Eqs. (5) and (6) are analogous to the Newtonian relations for viscous shear relative to a *solid* surface, which is moving at a velocity $(\boldsymbol{u}_0)$. There is however an important difference at the interface between two fluids. Here the motion is augmented by interfacial wave induced velocities which are *not* continuous across the interface.

Let us suppose that the interfacial Lagrangian velocities may be partitioned into two parts: a non-wave induced velocity $(\boldsymbol{u}_0)$ which is common to both fluids, and a wave induced velocity, $\boldsymbol{u}_L$ in fluid 1, and $A\boldsymbol{u}_L$ in fluid 2, where A is a constant, so that adjacent to the interface, (5) and (6) are replaced by

$$\boldsymbol{\tau}_s = \rho_1 K_L \mid \boldsymbol{u}_1 - \boldsymbol{u}_0 - \boldsymbol{u}_L \mid (\boldsymbol{u}_1 - \boldsymbol{u}_0 - \boldsymbol{u}_L), \quad \boldsymbol{u}_L \neq 0, \tag{7}$$

$$\boldsymbol{\tau}_s = \rho_2 K_L \mid \boldsymbol{u}_0 + A\boldsymbol{u}_L - \boldsymbol{u}_2 \mid (\boldsymbol{u}_0 + A\boldsymbol{u}_L - \boldsymbol{u}_2), \quad A \neq 0, \tag{8}$$

where $K_L(\mid n \mid)$ is a drag coefficient applicable to the velocities $(\boldsymbol{u}_i)$ adjacent to the interface, and seek the condition that (7) and (8) jointly reduce to (5) when $\boldsymbol{u}_2 \to \boldsymbol{u}_0$, and to (6) when $\boldsymbol{u}_1 \to \boldsymbol{u}_0$. On eliminating $\boldsymbol{u}_L$, we find that

$$A = \epsilon, \tag{9}$$

where $\epsilon = (\rho_1/\rho_2)^{1/2}$, which yields

$$\boldsymbol{\tau}_s = \rho_1 K \left| \boldsymbol{u}_1 - \boldsymbol{u}_0 - \frac{1}{\epsilon}(\boldsymbol{u}_2 - \boldsymbol{u}_0) \right| \times \left(\boldsymbol{u}_1 - \boldsymbol{u}_0 - \frac{1}{\epsilon}(\boldsymbol{u}_2 - \boldsymbol{u}_0) \right), \tag{10}$$

in which $K = \frac{1}{4} K_L$ and

$$\epsilon \boldsymbol{u}_1 + \boldsymbol{u}_2 = 2\epsilon \boldsymbol{u}_L + (1 + \epsilon) \boldsymbol{u}_0. \tag{11}$$

Eq. (10) is the relation for inertial coupling of the two fluids, in distinction to (3), which represents viscous coupling. Three important properties are:

(i) $\boldsymbol{u}_L$ is arbitrary, hence (10) would apply generally provided that the wave induced motion exists, which it does in any inertial system. The strength of the wave-induced motion may however determine K.

(ii) The interfacial shearing stress $(\boldsymbol{\tau}_s)$ does not depend on the difference between the velocities in the two fluids as in viscous coupling, but on the difference between two inertially weighted vectors which tend to velocities as the densities become equal. For fluids of large density difference, e.g. liquids and gases, this difference is very significant.

(iii) Eq. (10) indicates that a zero shearing stress balance $(\boldsymbol{\tau}_s = 0)$ is obtained for

$$(\boldsymbol{u}_2 - \boldsymbol{u}_0) = \epsilon(\boldsymbol{u}_1 - \boldsymbol{u}_0), \tag{12}$$

such that close to the interface the kinetic energy of the two fluids relative $\boldsymbol{u}_0$ is equally partitioned.

From (7) and (8), respectively, we also obtain

$$\boldsymbol{u}_1 = \boldsymbol{u}_L + \boldsymbol{u}_0, \tag{13}$$

$$\boldsymbol{u}_2 = \epsilon \boldsymbol{u}_L + \boldsymbol{u}_0, \tag{14}$$

such that $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are constant velocities (within the constant stress layer). For $\mid \boldsymbol{\tau}_s \mid \neq 0$, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are functions of n.

4. Discussion

Inertial coupling finds its full expression in the interaction of large systems of greatly differing density, in which at a given position the two fluid velocities ($\boldsymbol{u}_1$ and $\boldsymbol{u}_2$) which generate the interfacial wavefield are determined by non-local processes.

A topical example, which is the subject of intense study at present, is climate [3] in which the interaction of the ocean and atmosphere is mediated by the wind wave field, and momentum passes from the atmosphere to the ocean on the scale of the synoptic weather system, and from the ocean to the atmosphere on the scale of mesoscale oceanic eddies. The interfacial wave-induced velocities in this situation are set by the particle velocities of the gravity waves in the water, and the phase velocities of the gravity waves in the air [4,5].

References

[1] H. Lamb, Hydrodynamics, 6th Ed. (Dover, New York, 1932).
[2] J.O. Hinze, Turbulence, 2nd Ed. (McGraw-Hill, New York, 1975).
[3] IPCC (Intergovernmental Panel on Climate Change) Climate Change, 1992, Supp. Rep. to the IPCC Scientific Assessment (Cambridge Univ. Press, Cambridge, 1992).
[4] J.A.T. Bye, J. Mar. Res. 46 (1988) 467.
[5] J.A.T. Bye, J. Mar. Res., submitted (1994).