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On the speed and profile of steep solitary waves

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Previous estimates of the speed of solitary waves in shallow water unexpectedly showed that the speed and energy were greatest for waves of less than the maximum possible height. These calculations were based on Padé approximants. In the present paper we present some quite independent calculations based on an integral equation for the wave profile (Byatt-Smith 1970), now modified so that the wave speed appears as a dependent variable. There is remarkably good agreement with the previous method. In particular the existence of a maximum speed and energy are verified.

The method also yields a more accurate profile for the free surface of steep solitary waves. As the wave amplitude increases, it is found that the point of intersection of neighbouring profiles moves up towards the crest. Hence the highest wave lies mostly *beneath* its neighbours, which helps to explain why its speed is less.

Tables are given not only of the wave speed but also of the maximum surface slope as a function of wave amplitude. In no case does the slope exceed 30°, but for still higher waves this possibility is not excluded.

1. INTRODUCTION

Despite many experimental and theoretical investigations since the time of Scott Russel (1845) and Rayleigh (1876) the exact form of solitary waves on water of uniform depth has remained an interesting and unsolved problem. In a recent study, Longuet-Higgins & Fenton (1974) made extensive numerical calculations which yielded the unexpected result that the speed of a solitary wave in water of undisturbed depth h does not increase monotonically with the wave amplitude a, but instead reaches a maximum at a fairly high value of a/h and then diminishes. The highest wave is therefore neither the fastest, nor the most energetic. Similar results have been found by Longuet-Higgins (1975) for waves on deep water and by Cokelet (1975) for all steady, irrotational waves in water of any uniform depth.

These conclusions were however reached by the extensive use of Padé approximants, a device for summing power series beyond their ordinary radius of convergence. Well known in other branches of physics, this technique was recently introduced into the study of gravity waves by Schwartz (1974). Essentially the idea is very simple: to approximate the infinite sum of a power series not by its partial sum, which is a polynomial expression, but more generally by the ratio of two polynomials, having the same partial power-series expansion. This has the effect of distorting the circle of convergence so that it extends well beyond the nearest singularity in the complex plane.

While there is little reason to doubt the validity of this technique as applied to gravity waves, it cannot yet be said to have been justified rigorously. Moreover, though the speed and the energy of solitary waves were both found to converge satisfactorily for all waves up to the highest, the same was not true of the surface profiles. Hence there is some interest in confirming and extending these results by an independent method of calculation.

A quite different approach to the calculation of solitary waves was proposed by Byatt-Smith (1970). In this approach, the form of the wave profile is given directly as the solution to a singular integral equation. In general the integral equation may be solved numerically. The method is particularly successful for waves of moderately large amplitude, though it fails for the very highest waves, where the sharp curvature at the wave crest makes numerical integration increasingly less accurate. Nevertheless, Byatt-Smith was able to calculate approximately the speed and profile of solitary waves up to non-dimensional speeds ($F = c/\surd(gh)$) equal to about 1.293. Beyond this point the method did not work.

Now the above value of F is very close to the maximum value $F = 1.294$ found by Longuet-Higgins & Fenton (1974). These authors suggested that the reason for the failure of the integral-equation technique was that where F is an almost stationary function of the dimensionless amplitude a/h the wave profile, with F as independent parameter, cannot be accurately determined. Indeed, in the range where F diminished with amplitude the solution was not unique, but two-valued. Moreover the second solution might be difficult to obtain.

In the present paper this difficulty is overcome by recasting the solution of the integral equation in terms of a new parameter ω which, unlike F, is monotonic throughout the whole range of wave heights. ω is in fact the same parameter introduced by Longuet-Higgins & Fenton (1974), being defined by

$$\omega = 1 - q^2/gh. \tag{1.1}$$

Here q denotes the particle-velocity at the wave crest, in the frame of reference travelling with the wave speed (so that the motion appears steady). Generally ω lies between 0 and 1. The value $\omega = 1$ corresponds to the highest wave. An application of Bernoulli's equation gives

$$\omega = 2a/h - (F^2 - 1). \tag{1.2}$$

In this paper ω is taken as the independent parameter, and a/h (and hence F) as a dependent parameter. This then allows practically the whole range of solitary waves to be explored, without the necessity for F to be monotonic.

Using a digital computer with a large core-store, it has been found possible to compute the wave speeds and profiles with convincing accuracy as far as $\omega = 0.96$. The computations show a maximum in the computed value of F at about $\omega = 0.917$, corresponding to $F = 1.294$, in good agreement with the different calculations based on Padé approximants.

The present paper therefore provides a welcome confirmation of the previous results of Longuet-Higgins & Fenton (1974). Unlike the previous paper it also yields the form of the wave profile at high amplitudes. Moreover, the form of these profiles provide confirmation of a speculation by Longuet-Higgins & Fenton (1974) as to the cause of the maximum value of F; namely that the profiles of the highest waves intersect the more rounded profiles of lower waves, at points not far from the wave crest. Hence the highest waves actually lie beneath the not-so-high waves over most of the profile (see § 5 below).

2. The integral equation

Following Byatt-Smith (1970), let us consider an irrotational, solitary wave of amplitude a propagating with velocity $-c$ in water of undisturbed depth h. Viewed by an observer moving with the phase-velocity, the motion becomes a steady stream, which at large distances has a uniform horizontal velocity equal to c. It will be convenient to choose units of length and time so that

$$h = c = 1 \quad \text{and hence} \quad g = 1/F^2. \tag{2.1}$$

In the moving frame of reference, we take rectangular axes (x, y) with the x-axis horizontal and the origin in the mean surface level. ϕ is the velocity potential and $y = \eta$ the surface elevation. Byatt-Smith (1970) showed that $\eta(\phi)$ must satisfy the integral equation

$$1 + \eta(\phi) + (1/\pi)\int_{-\infty}^{\infty} S(\phi') \ln \tanh \tfrac{1}{4}\pi|\phi - \phi'| \,\mathrm{d}\phi' = 0, \tag{2.2}$$

where $$S(\phi) = (\partial x/\partial \phi)_{y=\eta} = [F^2/(F^2 - 2\eta) - (\mathrm{d}\eta/\mathrm{d}\phi)^2]^{\frac{1}{2}}. \tag{2.3}$$

If we take $\phi = 0$ at the crest of the wave, then η is an even function of ϕ. Hence (2.2) may be written

$$1 + \eta(\phi) + (1/\pi)\int_0^{\infty} [S(\phi + \phi') + S(|\phi - \phi'|)] \ln \tanh (\tfrac{1}{4}\pi\phi') \,\mathrm{d}\phi' = 0. \tag{2.4}$$

The integrand has a logarithmic singularity at $\phi' = 0$, which can however be reduced by using the identity

$$(1/\pi)\int_0^{\infty} \ln \tanh (\tfrac{1}{4}\pi\phi') \,\mathrm{d}\phi' = -\tfrac{1}{2}.$$

Thus we have

$$1 + \eta(\phi) - S(\phi) + (1/\pi)\int_0^{\infty} [S(\phi + \phi') + S(|\phi - \phi'|) - 2S(\phi)] \ln \tanh (\tfrac{1}{4}\pi\phi') \,\mathrm{d}\phi' = 0, \tag{2.5}$$

and as $\phi' \to 0$ the integrand is now like $\phi'^2 \ln \phi'$, which is small.

To solve equation (2.5) numerically, we aim to evaluate the surface elevation η at the $N+1$ points

$$\phi = \phi_j = j\Delta\phi, \quad j = 0, 1, 2, \ldots, N, \tag{2.6}$$

where N is some large (even) integer and $\Delta\phi$ is a small step-length. For brevity, write

$$\eta(\phi_j) = \eta_j, \quad S(\phi_j) = S_j, \quad R(\phi_j) = R_j,$$

where $R(\phi)$ denotes the left-hand side of equation (2.5). Then given some fixed value of the parameter ω, and trial values of the η_j, we approximate S_j and R_j, ($j = 0, 1, 2, \ldots, N$). The vanishing of the $N+1$ residuals R_j, with equation (1.2), then gives us $N+2$ nonlinear equations for determining the η_j and F by successive approximations.

The values of η_j are defined in the first place up to and including $j = N$. For the evaluation of the integral in (2.5) we need values of η_j beyond ϕ_N. By Stokes's result on the asymptotic behaviour of η as $x \to \infty$, we may approximate these by

$$\eta_{N+l} = \eta_N \, \mathrm{e}^{-\alpha l \Delta\phi}, \tag{2.7}$$

where α is the smallest positive root of the equation

$$\frac{\tan\alpha}{\alpha} = F^2. \tag{2.8}$$

The values of S_j may be calculated from (2.3). We can then evaluate the integral in (2.5) by Simpson's rule (setting the integrand $G(\phi, \phi')$ equal to 0 at $\phi' = 0$). The range of integration must be truncated at some suitable large value of ϕ', which we take to be ϕ_N (equation (2.6)). Thus we set

$$\int_0^\infty G(\phi, \phi')\, \mathrm{d}\phi' \doteqdot \tfrac{1}{3}\Delta\phi \sum_{n=1}^{\frac{1}{2}N} (G_{2n-2} + 4G_{2n-1} + G_{2n}), \tag{2.9}$$

where G_j denotes $G(\phi, \phi'_j)$.

An initial approximation to the η_j is provided either by the small-amplitude theory of solitary waves, or else by previously computed solutions corresponding to neighbouring values of ω or ϕ_N. At each approximation, the matrix $\partial R_j/\partial\eta_i$ was calculated, and a new approximation $(\eta_j + \mathrm{d}\eta_j)$ was obtained by solving the linear equations

$$(\partial R_j/\partial\eta_i)\, \mathrm{d}\eta_i = -R_j. \tag{2.10}$$

The process was repeated until the total absolute 'error', $\sum_j |R_j|$, was less than some assigned bound, usually 10^{-5}. This normally required four or five iterations. Representative runs were carried out in both single and double precision. There were no significant differences between the corresponding results, showing that rounding-errors were negligible.

After calculating the η_j, the horizontal coordinates x_j were calculated from

$$x_j = \int_0^{\phi_j} (\partial x/\partial\phi)\, \mathrm{d}\phi = \int_0^{\phi_j} S\, \mathrm{d}\phi. \tag{2.11}$$

Also calculated were the values of the kinetic and potential energies, and the 'circulation' C. In terms of the present units (see equation (2.1)) these are given by

$$\left.\begin{aligned} T &= \tfrac{1}{2}F^2\int_{-\infty}^{\infty} \eta(S-1)\,\mathrm{d}\phi, \\ V &= \frac{1}{2}\int_{-\infty}^{\infty} \eta^2 S\,\mathrm{d}\phi, \\ C &= F\int_{-\infty}^{\infty} (S-1)\,\mathrm{d}\phi, \end{aligned}\right\} \tag{2.12}$$

respectively. The integrals were evaluated by Simpson's rule.

3. Discussion of errors

Solutions were easily obtained up to and including $\omega = 0.98$ with at most five iterations. However, at the higher values of ω the profile tended to develop an instability in the form of a saw-tooth, or ripple, whose period was of the same order as the step-length. This instability was almost certainly numerical and not physical, being associated with the sharp curvature, and its derivatives, which occur near the wave crest as ω approaches 1. These instabilities could always be eliminated by reducing the step-length $\Delta\phi$ at constant ϕ_N until practical limitations prevented any further increase in the magnitude of N.

There appear to be four main sources of error in the computation. The most important arises from the truncation of the integral in equation (2.5). The part neglected is of order

$$\int_{\phi_N}^{\infty} \ln\tanh\left(\tfrac{1}{4}\pi\phi'\right)\mathrm{d}\phi'.$$

For large ϕ_N, the integral is order $\exp\left(-\tfrac{1}{2}\pi\phi_N\right)$. Hence we expect an error

$$\epsilon_1 \propto \mathrm{e}^{-1.57\phi_N} \tag{3.1}$$

approximately.

A second error also arises from the finite values of ϕ_N. For in approximating η_j by an expression proportional to $\exp(-\alpha\phi_j)$ (see equation (2.7)) we neglect some terms asymptotically proportional to $\exp(-2\alpha\phi_j)$. Hence† we expect that

$$\epsilon_2 \propto \mathrm{e}^{-2\alpha\phi_N}. \tag{3.2}$$

In fact $2\alpha > \tfrac{1}{2}\pi$ whenever

$$\alpha > \tfrac{1}{4}\pi, \quad \text{i.e.} \quad \omega > 0.27$$

so that for the larger values of ω, ϵ_2 will decay more rapidly with ϕ_N than will ϵ_1.

† Witting (1974) has suggested that the asymptotic form should generally contain terms like $\exp(-n\alpha_m\phi)$ where α_m is the mth positive root of equation (2.8). However we have $\alpha_2 > \pi > 2\alpha_1$, so our conclusion still applies.

We expect that both ϵ_1 and ϵ_2 may be made negligible by increasing ϕ_N sufficiently, or by extrapolating to $\phi_N = \infty$ for fixed $\Delta\phi$. There then remain errors due to the finite size of $\Delta\phi$.

The first of these arises from the finite-difference approximation to the gradient $\mathrm{d}\eta/\mathrm{d}\phi$ in the expression for S (see equation (2.3)). The formula used was

$$(\mathrm{d}\eta/\mathrm{d}\phi)_j^2 \doteqdot [(\eta_{j+1}-\eta_j)^2+(\eta_j-\eta_{j-1})^2]/2(\Delta\phi)^2$$

which contains errors of order $(\Delta\phi)^2$. So we expect

$$\epsilon_3 \sim A(\Delta\phi)^2, \tag{3.3}$$

where $A(\omega)$ is some constant (at fixed ω). On the other hand the contribution of $(\mathrm{d}\eta/\mathrm{d}\phi)^2$ to the value of S is relatively small except possibly near the wave crest, and hence we expect that A will not generally be large.

The use of Simpson's rule for integrating a function f with a continuous fourth derivative $f^{(\mathrm{iv})}$ gives rise to errors of order $(\Delta\phi)^4 f^{(\mathrm{iv})}$. However, the integrand $G(\phi, \phi')$ has a singularity of order $\phi'^2 \ln \phi'$ at one end of the range ($\phi' = 0$). Hence we may on the contrary expect errors of order

$$\epsilon_4 \sim B(\Delta\phi)^3 \ln \Delta\phi + C(\Delta\phi)^3. \tag{3.4}$$

Since such errors will occur in each of the R_j, whereas the magnitude of $(\mathrm{d}\eta/\mathrm{d}\phi)^2$ is important only near the wave crest, the coefficients B and C will possibly be large compared to the coefficient A in (3.3).

Both of the errors ϵ_3 and ϵ_4 will diminish rapidly as $\Delta\phi$ decreases, and it may be possible to extrapolate to the limit $\Delta\phi = 0$. We emphasize, however, that in all extrapolations with respect to ϕ_N or $\Delta\phi$ the parameter ω is to be kept constant, that is, we consider only one particular wave profile at a time. Since the behaviour of the solitary wave profile as ω approaches its maximum is not yet well understood, all extrapolations with respect to ω will be avoided.

4. Results for $\omega = 0.96$

The largest value of ω for which reliable results were obtained was 0.96. Table 1 shows the calculated values of F, for different values of ϕ_N and $\Delta\phi$. Under ΔF are listed the first-differences, and under r the ratio of successive values of ΔF.

Evidently, for each value of $\Delta\phi$, r is practically constant, indicating an exponential rate of decreases of the error ϵ; thus

$$\epsilon \propto \mathrm{e}^{-\mu\phi_N}; \quad r \doteqdot \mathrm{e}^{-\mu\Delta\phi_N}, \tag{4.1}$$

where μ is given by

$$\mu = -\frac{\ln r}{\Delta\phi_N}. \tag{4.2}$$

The values of μ in table 1 show that μ is almost independent of both ϕ_N and $\Delta\phi$. In fact, for large values of ϕ_N we have $\mu \doteqdot 1.6$. This is in good agreement with the value $\frac{1}{2}\pi$ suggested in § 3.

This exponential rate of decay enables the extrapolation to $\phi_N = \infty$ to be carried out very simply by means of the formula

$$F_\infty = F_n + \frac{r_n}{1-r_n}(\Delta F)_n \tag{4.3}$$

(where the suffix n denotes the entry in table 1 which corresponds to the highest value of ϕ_{max}) for each value of $\Delta\phi$. In this way we obtain the four values of F_∞ shown in table 2.

In the third and fourth columns of table 2 are shown the first-differences $\Delta' F_\infty$ and the ratios r' of successive values of $\Delta' F_\infty$. Table 3 shows the values of r' that would be expected if the error in F_∞ were proportional to $\Delta\phi$, $(\Delta\phi)^2$, $(\Delta\phi)^3 \ln(\Delta\phi)$ or $(\Delta\phi)^3$, respectively. Evidently r' is too small to be proportional to $\Delta\phi$. Guided by the analysis of § 3 we assume that the error $(F - F_\infty)$ is given by an expression of the form

$$A(\Delta\phi)^2 + B(\Delta\phi)^3 \ln(\Delta\phi) + C(\Delta\phi)^3, \tag{4.4}$$

TABLE 1. CALCULATED VALUES OF F, WHEN $\omega = 0.96$

(a) $\Delta\phi = 0.03$

ϕ_{max}	F	ΔF	r	μ
3.6	1.28225			
		0.00257		
4.2	1.28482		0.41	1.49
		0.00105		
4.8	1.28587		0.40	1.53
		0.00042		
5.4	1.28629		0.38	1.6
		0.00016		
6.0	1.28645			

(b) $\Delta\phi = 0.02$

ϕ_{max}	F	ΔF	r	μ
3.6	1.28750			
		0.00132		
4.0	1.28882		0.55	1.52
		0.00072		
4.4	1.28954		0.54	1.53
		0.00039		
4.8	1.28993		0.54	1.55
		0.00021		
5.2	1.29014		0.52	1.62
		0.00011		
5.6	1.29025		0.5	1.5
		0.00006		
6.0	1.29031			

(c) $\Delta\phi = 0.015$

ϕ_{max}	F	ΔF	r	μ
3.6	1.28920			
		0.00139		
4.2	1.29059		0.40	1.52
		0.00056		
4.8	1.29115		0.39	1.56
		0.00022		
5.4	1.29137		0.39	1.5
		0.00009		
6.0	1.29146			

(d) $\Delta\phi = 0.01$

ϕ_{max}	F	ΔF	r	μ
3.2	1.28885			
		0.00152		
3.6	1.29037		0.57	1.42
		0.00086		
4.0	1.29123			

TABLE 2. VALUES OF F EXTRAPOLATED TO $\phi_{\text{max}} = \infty$

$\Delta\phi$	F_∞	$\Delta' F_\infty$	r'
0.030	1.28655		
		0.00382	
0.020	1.29037		0.30
		0.00115	
0.015	1.29152		0.62
		0.00071	
0.010	1.29223		

TABLE 3. VALUES OF r' CORRESPONDING TO VARIOUS ERROR LAWS

$\Delta\phi$	error $\propto \Delta\phi$	$(\Delta\phi)^2$	$(\Delta\phi)^3 \ln(\Delta\phi)$	$(\Delta\phi)^3$
0.030, 0.020, 0.015	0.500	0.350	0.320	0.243
0.020, 0.015, 0.010	1.000	0.714	0.561	0.514

where A, B, C are constants to be determined. From table 2 we then have the following simultaneous equations for A, B, C:

$$\left.\begin{aligned} 50.0A-6.338B+1.900C &= 382, \\ 17.5A-1.712B+0.462C &= 115, \\ 12.5A-0.957B+0.237C &= 71, \end{aligned}\right\} \tag{4.5}$$

(each side has been multiplied by 10^5). Solving these, we find

$$A = 1.709, \quad B = -75.94, \quad C = -97.33. \tag{4.6}$$

The final value of F is then found from table 2 and equation (4.5) with $\Delta\phi = 0.01$, that is to say

$$F = 1.29223 + 0.00042 = 1.2926 \tag{4.7}$$

to four decimal places. This is to be compared with the value $F = 1.2919$ found by Longuet-Higgins & Fenton (1974). Both values appear to be significantly below the maximum value 1.2941 found previously at $\omega \doteqdot 0.913$ (see figure 1).

The value of A given by equation (4.6) is considerably smaller than that of B or C, as was to be expected from the discussion in § 3.

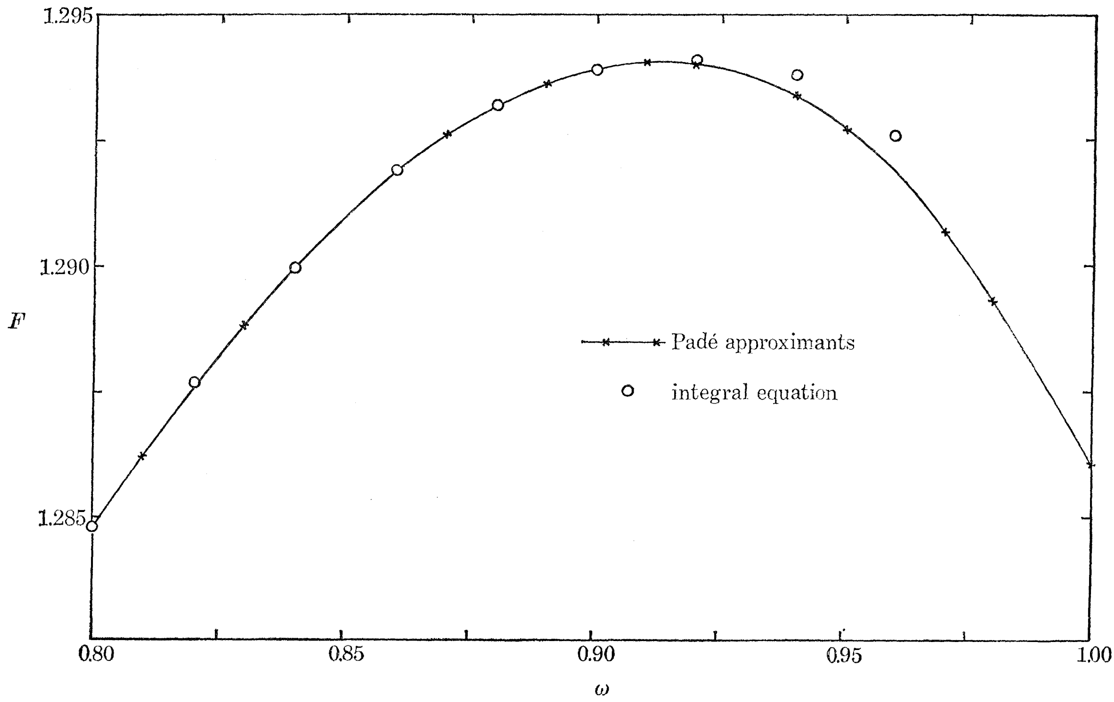

FIGURE 1. The dimensionless speed F of very steep solitary waves ($0.80 \leqslant \omega \leqslant 1.0$). The full curve represents the values obtained from Padé approximants; plotted points correspond to the integral equation.

5. F as a function of ω

For all values of ω, up to and including 0.96, it was found that the first-differences, for constant $\Delta\phi$ and increasing ϕ_N, behaved exponentially; hence the extrapolation to $\phi_N = \infty$ could be carried out quickly and accurately. These extrapolated values, for $\omega = 0.90$, 0.92 and 0.94, are shown in table 4. In the case $\omega = 0.94$ it makes no significant difference whether the extrapolation to $\Delta\phi = 0$ is carried out by means of the expression (4.5) or by the first term only (representing an error proportional to $(\Delta\phi)^2$). Accordingly we have adopted the simpler method. Similar considerations apply all the more to the lower values of ω.

Table 4. Values of F_∞ when $\omega = 0.94$, 0.92 and 0.90

	$\omega = 0.94$		$\omega = 0.92$		$\omega = 0.90$	
$\Delta\phi$	F_∞	ΔF_∞	F_∞	ΔF_∞	F_∞	ΔF_∞
0.030	1.29220		1.29379		1.29393	
		0.00092		0.00020		0.00000
0.020	1.29312		1.29399		1.29393	
		0.00030		0.00009		0.00001
0.015	1.29342		1.29408		1.29394	
		0.00023				
0.010	1.29365					

Table 5. Calculated values of F, T, V and C when $0.8 \leqslant \omega \leqslant 0.96$

	F		T		V		C	
ω	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)
0.80	1.2848	1.2848	0.533	0.533	0.441	0.441	1.783	1.782
0.82	1.2877	1.2876	0.541	0.540	0.446	0.446	1.780	1.779
0.84	1.2900	1.2899	0.547	0.546	0.450	0.450	1.775	1.774
0.86	1.2919	1.2919	0.550	0.550	0.452	0.452	1.769	1.771
0.88	1.2932	1.2932	0.552	0.552	0.453	0.452	1.759	1.758
0.90	1.2939	1.2939	0.553	0.551	0.453	0.451	1.753	1.746
0.92	1.2941	1.2940	0.551	0.548	0.451	0.448	1.743	1.732
0.94	1.2938	1.2934	0.547	0.542	0.448	0.443	1.733	1.716
0.96	1.2926	1.2919	0.542	0.533	0.443	0.435	1.723	1.697

The final values of F, for ω in the range $0.80 \leqslant \omega \leqslant 0.96$, are given in table 5, and are shown in figure 1 (plotted points) compared with the values derived from Padé approximants (solid curve). The agreement is striking, especially since the vertical scale in figure 1 corresponds to only about 1% of the total variation of F.

Also in table 5 are shown the calculated values of the kinetic energy T, the potential energy V and the circulation C (as defined by Longuet-Higgins 1974). These are listed in the first column (*a*) under each heading. In the second column (*b*) the values calculated by the Padé approximant method (Longuet-Higgins & Fenton 1974) are shown for comparison. The values of T, V and C converged less rapidly than those for F, by an order of magnitude,† and are given to three decimal

† This is to be expected, since a further integration is necessary for their evaluation.

places only. Nevertheless, there is still good agreement between the two methods of calculation. In particular, the existence of a maximum in each of T, V and C is verified.

For values of ω less than 0.80, the values of T, V and C as calculated by the two different methods were in agreement to four significant figures, and the values of F were in agreement to five significant figures. For these the reader is referred to table 5 of Longuet-Higgins & Fenton (1974).

6. Surface profiles

One advantage of the present method of computation is that the profile of the free surface may be calculated with greater accuracy. The advantage is most apparent for steep waves, when ω approaches its maximum value 1.

Figure 2 shows a succession of wave profiles, from $\omega = 0.3$ up to $\omega = 0.9$. Evidently the height of the crest increases monotonically with ω. This is in accordance with the Rayleigh–Boussinesq theory for waves of small amplitude in which we have

$$y = \tfrac{1}{3}\alpha^2 \operatorname{sech}^2(\tfrac{1}{2}\alpha x) \tag{6.1}$$

and

$$F^2 = 1 + \tfrac{1}{3}\alpha^2. \tag{6.2}$$

Therefore

$$\omega = 2a/h - (F^2 - 1) = \tfrac{1}{3}\alpha^2 \tag{6.3}$$

and the wave height is approximately equal to ω.

As ω increases, so at first the horizontal scale of the wave, which is proportional to α^{-1} or $\omega^{-\frac{1}{2}}$, tends to diminish. The surface of each wave therefore lies *above* its predecessor (corresponding to a lower value of ω) at the crest, but lies *below* its predecessor in the wave 'tails'. It follows that each wave profile must intersect its neighbour in at least one intermediate point on either side of the wave crest.

The point of intersection of each profile with its neighbour lies by definition on the *envelope* of the surface profiles. The equation for this is found by differentiation with respect to the parameter ω or α. From equation (6.1) we easily find that a point on the envelope must satisfy

$$(\tfrac{1}{2}\alpha x)\tanh(\tfrac{1}{2}\alpha x) = 1 \tag{6.4}$$

so

$$\tfrac{1}{2}\alpha x = 1.1997\ldots = G, \tag{6.5}$$

say. Hence as α increases so the point of contact with the wave envelope moves in towards the wave crest. In fact we have

$$\left.\begin{aligned} x &= 2G/\alpha = 1.3853\omega^{-\frac{1}{2}}, \\ y &= \frac{G^2-1}{3G^2}\alpha^2 = 0.3052\omega, \end{aligned}\right\} \tag{6.6}$$

so that for small values of ω the wave envelope has the asymptote

$$x^2 y = \tfrac{4}{3}(G^2 - 1) = 0.5856. \tag{6.7}$$

This is shown by the broken curve in figure 4.

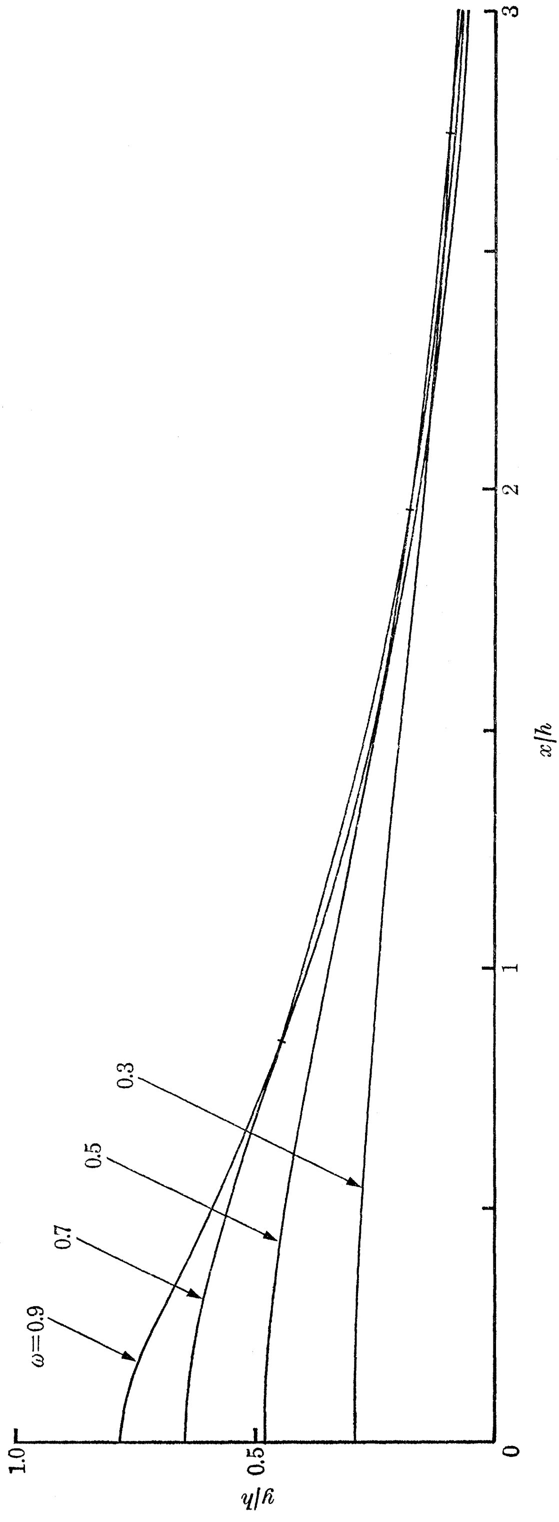

FIGURE 2. Surface profiles of solitary waves of different amplitudes, in water of constant depth.

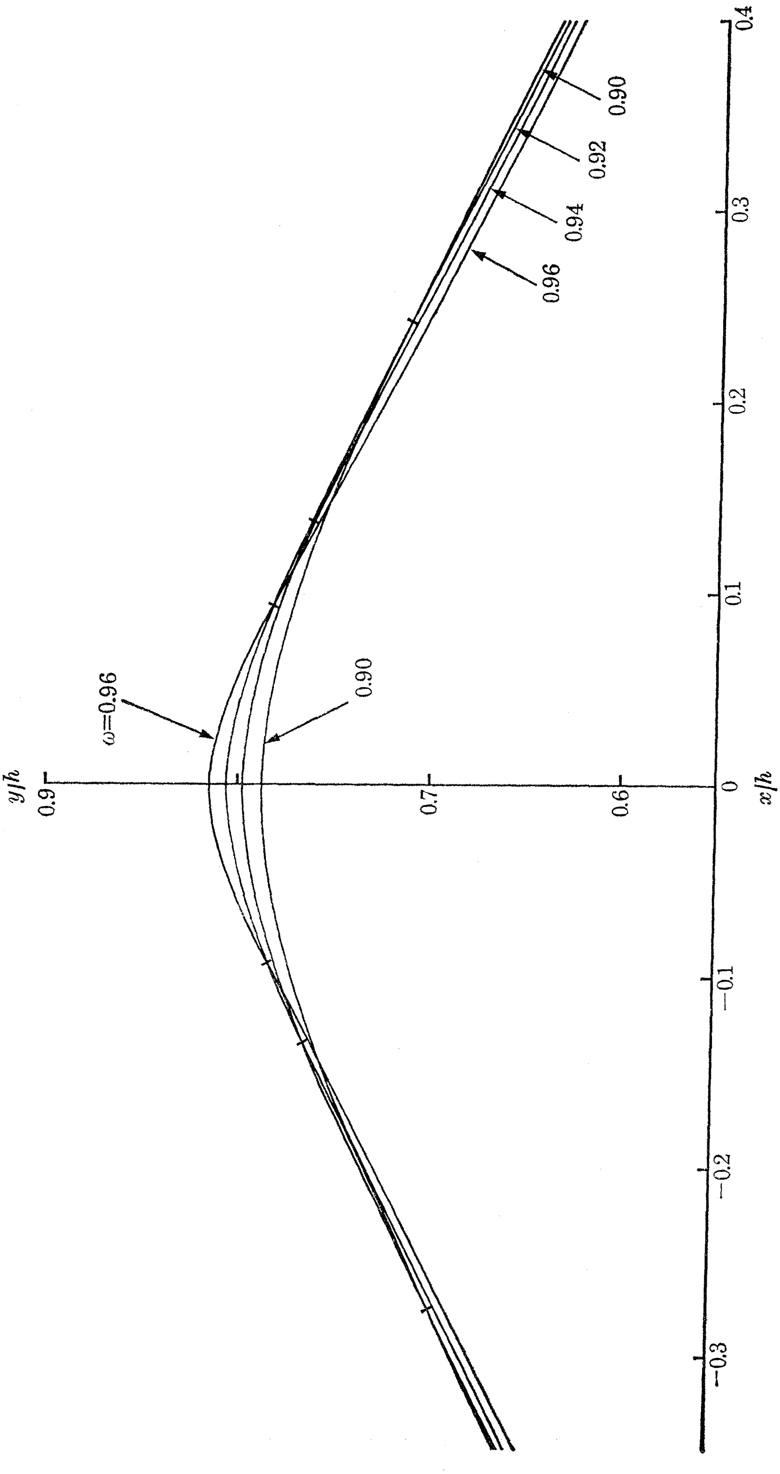

FIGURE 3. Enlargement of the surface profiles near the wave crests, for steep waves ($\omega = 0.90, 0.92, 0.94$ and 0.96).

Figure 3 shows an enlargement of the profiles in the neighbourhood of the wave crest, at $\omega = 0.90$, 0.92, 0.94 and 0.96 respectively. It is apparent that as $\omega \to 1$ so the points of intersection continue to move up towards the crest. The upper envelope of the wave profiles, as deduced from the present calculations, is shown in figure 4 (full curve). The points of contact at different values of ω are indicated. The broken

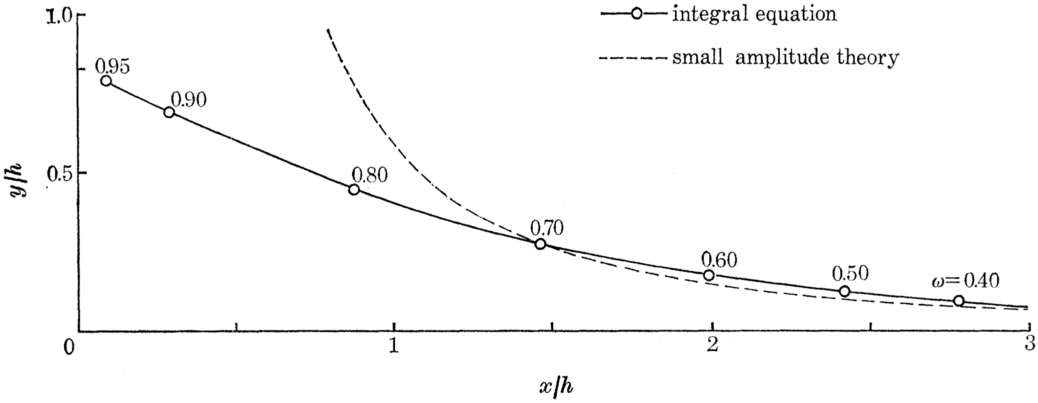

FIGURE 4. The upper envelope of the profiles of solitary waves, in water of constant depth.

curve corresponds to the asymptote given by equation (6.7) which is valid only for small values of ω.

This behaviour provides an explanation for the decreases in the wave speed F at high values of ω. For, over a large part of the wave, the profile of the highest waves lies *below* that of the not-so-high waves, implying that the total mass, or volume, of the wave actually diminishes as ω approaches 1. The potential and kinetic energies, which are integrals over the volume of the wave, likewise attain a maximum, and then diminish. Moreover the *average* surface elevation

$$\bar{\eta} = \int_{-\infty}^{\infty} \eta^2 \, \mathrm{d}x \bigg/ \int_{-\infty}^{\infty} \eta \, \mathrm{d}x$$

evidently decreases as ω increases, for the very highest waves. But $\bar{\eta}$ is related to the dimensionless speed F by the *exact* relation

$$\bar{\eta} = \tfrac{2}{3}(F^2 - 1)$$

which holds for solitary waves of any amplitude (see Longuet-Higgins & Fenton 1974, § 6). Hence for the highest waves F tends to decrease also. Thus the unexpected behaviour of T, V and F can be seen to be reasonable after all.

It will be noticed that because of the final decrease in F, the 'tails' of the wave profile, which are known to decay like $\exp(-\alpha|x|)$ (see equation (2.8)) must begin to spread out slightly again. This implies that there must be a second point of intersection far out in the tail, on each side of the wave crest. The lower branch of the envelope is too far out to be shown in figure 4.

A question of some interest is the behaviour of the maximum surface slope, as a function of the parameter ω. In order to prove the existence of waves of finite amplitude, Krasovskii (1961) has introduced the assumption that the maximum slope does not exceed 30° – the angle in Stokes's limiting flow. A similar assumption was made by Keady & Pritchard (1974). On the other hand, certain recent computations (Sasaki & Murakami 1973) have suggested that for steep waves the slope may indeed exceed 30°. In table 6 below we give the values of the maximum height and slope, for given ω, as accurately as can be determined from the present calculations (see also figure 5). Clearly the angle increases monotonically with ω. In no case do we find that it exceeds 30°. However the limiting value cannot be determined from the present method without extrapolating numerically with regard to ω, which in our view would be unjustified.

Table 6. Maximum inclination of the free surface

ω	θ_{max}/deg	ω	θ_{max}/deg
0.1	1.15	0.82	22.37
0.2	3.15	0.84	23.15
0.3	5.59	0.86	23.99
0.4	8.35	0.88	24.86
0.5	11.34	0.90	25.6
0.6	14.54	0.92	26.5
0.7	17.95	0.94	27
0.8	21.60	0.96	28

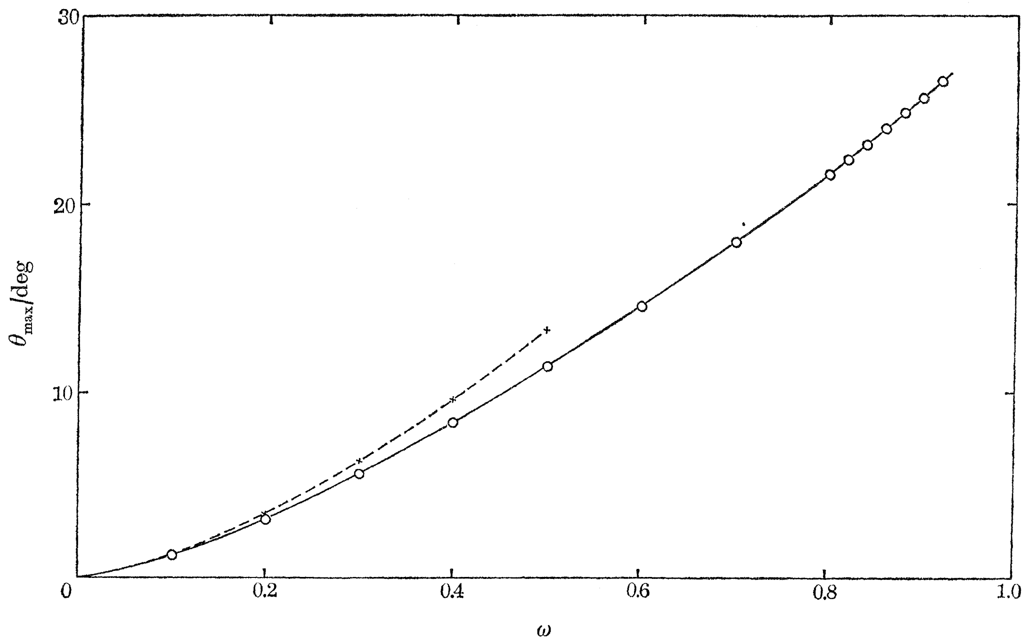

Figure 5. The maximum gradient dy/dx of the free surface, as a function of the parameter ω. ——, Integral equation; – – – –, small amplitude theory.

7. Conclusions

The parameters of steep solitary waves have been calculated by a method which is independent of the use of Padé approximants. We have also avoided all extrapolation with regard to the parameter ω. The existence of maxima in the dimensionless speed, energy and circulation has been verified numerically. A suggested reason for this is also confirmed, namely that a solitary wave touches its wave envelope from below; as the wave steepness increases, so the point of contact with the wave envelope moves up towards the crest. Thus the steepest waves lie mostly beneath the less steep waves, and it is not surprising that the total mass and energy of the steepest waves are less, hence also their speed.

Throughout the computed range of ω, the maximum wave steepness increases monotonically with the wave height. In none of our computations does it exceed 30°. However, the accurate determination of the limiting value of $\theta_{\max}$ as $\omega \to 1$ must await a study of the asymptotic form of the surface profile in the neighbourhood of a sharply rounded crest.

Because of the large core-store required for these calculations, they were carried out mainly on the Rutherford Laboratory's I.B.M. 370/195, through the Atlas Laboratory of the Science Research Council, over a data-link to the Institute of Oceanographic Sciences at Wormley, Surrey. We are indebted to the staff of these laboratories, and in particular to Mr W. T. J. Slade of I.O.S. Wormley, for their valuable cooperation.

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