# NOTES AND CORRESPONDENCE

## Similarity Functions for Turbulence in Neutral Air Above Swell

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### 1. Introduction

To avoid the problem of closure arising in the theory of turbulence, in the past few decades several semi-empirical theories and similarity models have been proposed, that have been quite successful in describing the flow over solid surfaces. It appears, however, that to date no such model has yet been developed for turbulent flow over a free-water surface disturbed by waves. Nevertheless, it has been suggested by Kitaygorodskiy (1969) that, beside the usual parameters needed to describe the flow over a solid surface, two additional ones must be considered; namely $c$, the phase velocity of the dominant wave and $\lambda$, the corresponding wavelength. Thus, he reasoned that in the near-water air layer a state of developed shear turbulence exists in which, on the basis of dimensional arguments, the characteristics of the turbulence, such as the mean wind $U=U(z)$ or the variance of the longitudinal velocity fluctuations $\sigma_u^2=\overline{u^2}$, are given by

$$\frac{dU}{dz}=(u_*/kz)\phi_0, \quad \text{for } h_s \ll z < z_c, \tag{1}$$

$$\sigma_u^2=u_*^2\phi_1, \quad \text{for } h_s \ll z < z_c, \tag{2}$$

where $z$ is the elevation, $u_*=(\tau_0/\rho)^{\frac{1}{2}}$ the friction velocity, $\tau_0$ the surface shear stress, $\rho$ the density of the air, $k$ von Kármán's constant, $h_s$ the effective height of the roughness of the water surface, and $z_c$ Miles' critical height, which is defined as the height at which the mean wind speed equals the phase velocity, namely $U(z_c)=c$. Kitaygorodskiy (1969) hypothesized that, in general, $\phi_0=\phi_0(z/\lambda, c/u_*)$ and $\phi_1=\phi_1(z/\lambda, c/u_*)$ are dimensionless functions, but he did not specify their form or nature.

One difficulty with this hypothesis is that it is not always clear how the dominant wave should be defined. The waves on a free-water surface are usually quite random in both space and time, covering a wide spectrum of wavelengths. Especially when the waves are growing under the influence of the wind, the interaction among the various wavelengths of the developing spectrum is a complex matter (e.g., Phillips, 1966), so that it is rather difficult to characterize the wave effect on the turbulence in terms of a single wavelength and phase velocity. This difficulty does not exist under conditions of a relatively weak wind over well-developed slowly decaying smooth waves, or swell, and the characterization of the wavefield by a single $c$ and $\lambda$ is more meaningful.

Accordingly, it is the purpose of this note to present on the basis of simple assumptions and dimensional considerations a plausible mathematical expression for $\phi_0$, $\phi_1$, etc., over relatively low-frequency waves, that is, for large values of $(c/u_*)$. The constants arising in the analysis are obtained from available experimental data.

### 2. Turbulent energy equation over a wavy water surface

The equation for the energy balance of the turbulence is a useful tool to analyze the mechanics of turbulent flow and it has been the starting point for several similarity models to describe flow in the lower atmosphere (e.g., Monin and Yaglom, 1971). Under steady neutral conditions over a homogeneous uniform surface it can be written as

$$\overline{wu}\frac{\partial U}{\partial z}+\frac{\partial}{\partial z}\left(\frac{\overline{wq^2}}{2}+\frac{\overline{wp}}{\rho}\right)=\epsilon, \tag{3}$$

where $u$, $v$, $w$ are the components of the velocity fluctuations parallel to the $x$-, $y$-, $z$-axes, respectively; $U$ is the mean wind speed in its direction along the $x$-axis; $\overline{q^2}/2=(\overline{u^2}+\overline{v^2}+\overline{w^2})/2$ is the turbulent kinetic energy per unit mass; $p$ is the pressure fluctuation; and $\rho$ is the density of the air. The first term in (3) represents turbulent energy production since it expresses the transformation of mechanical energy from the mean motion through the Reynolds stresses. The last term is essentially the work done per unit mass and per unit time by the viscous shear stresses of the turbulence. This dissipation is usually assumed to be independent of the viscosity and expressable in terms of the variances of the velocity fluctuations, or of the cross correlation,

$\epsilon=(-\overline{uw})^{\frac{3}{2}}/L_1$, where $L_1$ is a length scale characteristic of the turbulence which may be taken proportional to the elevation $z$.

Consider now that the underlying water surface is disturbed by waves with a dominant phase velocity $c$. Because of the difference in the scales of the wave-induced turbulence and the shear- and buoyancy-induced turbulence, it is convenient to express the fluctuations in the air in terms of a random or turbulent component superimposed on a dominant wave-induced component, as

$$\left.\begin{aligned} u&=u_w+u_t \\ v&=v_w+v_t \\ &\text{etc.} \end{aligned}\right\}. \qquad (4)$$

The dominant wave is characterized by a single frequency so that any long-term second-order cross correlations of $t$- with $w$-subscripted quantities are zero; the same is true for third-order cross correlations of a $t$-subscripted variable with two $w$-subscripted variables resulting from the second term on the left in (3). However, as a result of the interaction between air turbulence and wave-induced motion (e.g., Chang and Cheng, 1972), this is not necessarily the case for the third-order cross correlations of a $w$-subscripted variable with two $t$-subscripted variables. Thus, Eq. (3) may be written as

$$\overline{uw}\frac{dU}{dz}+\frac{d}{dz}\left(\frac{\overline{w_tq_t^2}}{2}+\frac{\overline{w_tp_t}}{\rho}\right)+\frac{d}{dz}\left(\frac{\overline{w_wq_w^2}}{2}+\frac{\overline{w_wp_w}}{\rho}\right)$$
$$+\frac{d}{dz}\left(\frac{\overline{w_wq_t^2}}{2}+\overline{u_wu_tw_t}+\overline{v_wv_tw_t}+\overline{w_ww_t^2}\right)$$
$$-\frac{(-\overline{uw})^{\frac{3}{2}}}{c_1z}=0, \qquad (5)$$

where $c_1$ is a constant, and where $\overline{uw}=\overline{u_tw_t}+\overline{u_ww_w}$ is the total shear stress.

## 3. Similarity assumptions

Over a solid surface the vertical flux of turbulent energy, represented by the second term in (3), is often considered to be zero. Accordingly, let it be assumed that over waves this only holds for the "random" turbulence, but not for the wave-induced turbulence. In other words, the turbulent energy produced by the turbulent Reynolds stresses is being dissipated locally without any convective transport; on the other hand, the wave-related energy is not only being dissipated locally, but it is also diffused convectively upward, since it originates from the motion of the swell. This first assumption allows putting the second term in (5) equal to zero.

For well-developed deep water waves $c$ and $\lambda$ are strongly interdependent. Moreover, it has been found (e.g., Phillips, 1966; Kitaygorodskiy, 1969) that even over a wavy surface the Reynolds shear stress is approximately constant in the vertical, or

$$\overline{uw}=-u_*^2. \qquad (6)$$

Consequently, Kitaygorodskiy's $\phi$-functions may probably be assumed to be independent of $\lambda$ and $z$, and it is acceptable to simplify them as

$$\phi_0=\phi_0(c/u_*), \quad \text{for} \quad h_s\ll z, \qquad (7)$$

and in a similar way $\phi_1$, $\phi_2$, etc.

A third assumption stems from the observation that the relative speed of the air with respect to that of the waves is probably the main factor governing the magnitudes of the wave-induced fluctuations. Thus

$$\left.\begin{aligned} u_w&=f_1(c-U,x,y,z,t) \\ v_w&=f_2(c-U,x,y,z,t) \\ &\text{etc.} \end{aligned}\right\}, \qquad (9)$$

where the $f$'s are unknown functions that may however be approximated by a McLaurin series as

$$f(c-U,x,y,z,t)$$
$$=f(0,x,y,z,t)+(c-U)f'(0,x,y,z,t)$$
$$+\frac{(c-U)^2}{2!}f''(0,x,y,z,t)\cdots,$$

where the primes denote differentiation with respect to $(c-U)$. Because the wave-induced fluctuations probably become negligible when the air moves as fast as the waves, $f(0,x,y,z,t)$ may be neglected. To a first approximation, terms in second and higher order derivatives may also be neglected.

Finally, there is a vast amount of experimental evidence (e.g., Roll, 1965; Ruggles, 1970) showing that under neutral conditions the mean wind velocity profile over a wavy water surface is for all practical purposes logarithmic. Bearing also in mind that the Reynolds stresses are approximately constant in the vertical, one sees that the $z$-dependence in any of the terms in (5) should be given either by $(dU/dz)$ or by $(u_*/z)$.

With these four assumptions, dimensional considerations suggest now the substitutions

$$\frac{d}{dz}\left(\frac{\overline{w_wq_w^2}}{2}+\frac{\overline{w_wp_w}}{\rho}\right)=c_2(c-U_r)^2\frac{dU}{dz}, \qquad (10)$$

$$\frac{d}{dz}\left(\frac{\overline{w_wq_t^2}}{2}+\overline{u_wu_tw_t}+\overline{v_wv_tw_t}+\overline{w_ww_t^2}\right)=c_3u_{*t}^2\frac{dU}{dz}$$
$$=[c_3u_*-c_4(c-U_r)^2]\frac{dU}{dz}, \qquad (11)$$

where $c_2$, $c_3$ and $c_4$ are constants and $U_r$ is a reference or characteristic wind to be specified below. Thus, Eq. (5) reduces to

$$c_5 u_*^2 \frac{dU}{dz} - c_6(c-U_r)^2 \frac{u_*}{z} + \frac{u_*^3}{c_1 z} = 0, \qquad (12)$$

where $c_5$ and $c_6$ are constants. Comparison with (1) yields

$$\frac{dU}{dz} = \frac{u_*}{kz}\left[1+\beta\left(\frac{c}{u_*}-\alpha\right)^2\right], \quad \text{for } (c/u_*)>\alpha, \qquad (13)$$

or

$$\phi_0 = 1+\beta\left(\frac{c}{u_*}-\alpha\right)^2, \qquad \text{for } (c/u_*)>\alpha, \qquad (14)$$

where $\alpha=(C_{Dr})^{-\frac{1}{2}}$, in which $C_{Dr}=(u_*/U_r)^2$ is the drag coefficient corresponding to $U_r$, and $\beta$ is a constant to be determined experimentally. Eq. (14) is the main result of this note, providing the functional form of Kitaygorodskiy's function used in Eq. (1).

## 4. Discussion

The constant $\alpha$ is related to the drag coefficient $C_{Dr}$, corresponding to $U_r$. The drag coefficient over water disturbed by waves has been the subject of much experimental work. Reviews of recently obtained values of $C_D$ (e.g., Stewart, 1967; Toba and Kunishi, 1970; Pond, 1971; DeLeonibus, 1971) show that there is very little evidence of wind speed dependence except at high wind speeds; it appears to range between 0.0010 and 0.0015. Although $C_D$ probably varies with $c/u_*$ (Volkov, 1970), herein $C_{Dr}$ is merely a parameter which assumes the value of $C_D$ when $c=U_r$ or when $(c/u_*)=\alpha$. If the above-mentioned values of $C_D$ represent "average" wave conditions, i.e., for $(c/u_*)=\alpha$, it follows that $\alpha=(C_{Dr})^{-\frac{1}{2}}=(U_r/u_*)$ lies in the range between approximately 25 and 32. Hence let it be assumed that $\alpha=29$. This also shows that the present characteristic velocity $U_r$ is related to $U_c$, the convection velocity of the large-scale atmospheric pressure fluctuations at sea, which Phillips (1966) found to be of the order of $25u_*$.

The only other parameter necessary in the formulation of $\phi_0$ is $\beta$. Unlike $\alpha$ however, $\beta$ cannot be determined from heuristic arguments but must be obtained from wind profile measurements. Volkov (1969, 1970) has presented and analyzed the results of experiments carried out on the Western Mediterranean which consisted mainly of measurements of the sea swell and of various characteristics of the turbulence at 2 m and the wind profile below 3.5 m above the water surface under slightly unstable conditions. Two relationships are of interest. The first one showed $u_*$ vs $u_*^p$, where $u_*^p$ is the apparent friction velocity obtained from the wind profile, namely

$$\frac{dU}{dz} = (u_*^p/kz), \qquad (15)$$

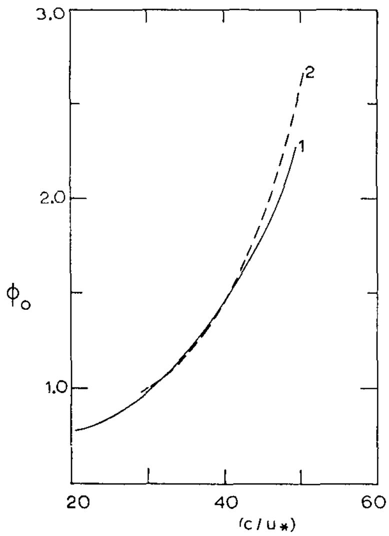


FIG. 1. Dependence of the shear function $\phi_0=(kz/u_*)dU/dz$ on the parameter $c/u_*$. Curve 1 represents experimental data published by Volkov (1969, 1970) and curve 2 is based on Eq. (14).

and where $u_*=(-\overline{uw})^{\frac{1}{2}}$ is the actual friction velocity. The second relationship shows $\tau_0$ vs $c/u_*$. The combination of these two curves allows immediately the construction of $\phi_0=\phi_0(c/u_*)$. This empirical function is shown in Fig. 1. It can be seen that it is in agreement with the present theoretical result of (14) with $\alpha=29$. The value of $\beta$ thus obtained is 0.00600.

So far the analysis has centered on the shear function $\phi_0=\phi_0(c/u_*)$. As stated with Eq. (7) all other turbulence statistics may be described by analogous functions. In fact, using the argument leading to (10) and (11), one obtains for the longitudinal fluctuations

$$\overline{u_w u_w} = c_7(c-U_r)^2, \qquad (16)$$

where $c_6$ is a constant. This gives, then, since $\sigma_u^2=\overline{u_t^2}+\overline{u_w^2}$ from Eq. (2),

$$\phi_1 = \left[a_1+b_1\left(\frac{c}{u_*}-\alpha\right)^2\right], \quad \text{for } \frac{c}{u_*}>\alpha, \qquad (17)$$

where $a_1$ and $b_1$ are constants. Similarly, for the variance of the vertical velocity fluctuations, one may write

$$\sigma_w^2 = u_*^2\phi_3, \quad \text{for} \quad h_s \ll z, \qquad (18)$$

in which the subscript $w$ refers to the $z$-velocity component, and in which again $\phi_3=\phi_3(c/u_*)$ is of the form

$$\phi_3 = \left[a_3+b_3\left(\frac{c}{u_*}-\alpha\right)^2\right], \quad \text{for } \frac{c}{u_*}>\alpha, \qquad (19)$$

where $a_3$ and $b_3$ are constants. Measurements of $\sigma_u$ and $\sigma_w$ have been published by Volkov (1969) and DeLeonibus (1971), which are in general agreement with these equations. With $\alpha=29$, Volkov's data appear to follow approximately $a_1=12.0$, $b_1=0.0600$, $a_3=3.0$, $b_3=0.0109$. DeLeonibus' neutral data follow approximately $a_3=2.5$ and $b_3=0.00495$. Actually, DeLeonibus (1971) also performed a linear least-squares regression fit of all data and obtained (in the present notation) $\phi_3=\{1.68+0.019[(c/u_*)-29]\}^2$, which in view of the small value of $b_3$ of 0.019 is very close to the form of (19). Even over solid surfaces there is a wide scatter in the experimental results of $(\sigma_w/u_*^2)$ and $(\sigma_u/u_*)$ (e.g. Monin and Yaglom, 1971), so that also here differences between the various sets of available experimental results are to be expected. Obviously, more work will be necessary to come to a consensus.

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