Internal Waves in a Lagrangian Reference Frame

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ABSTRACT

Several recent studies of internal gravity waves have been expressed in a Lagrangian reference frame, motivated by the observation that in this frame the dispersion relation then excludes the Eulerian Doppler shifting term due to a background flow. Here the dispersion relation in a Lagrangian reference frame is explicitly derived for a background flow and background density that are slowly varying with respect to the waves, but are otherwise arbitrary functions of space and time. Two derivations are given, both yielding the same result. The first derivation involves a transformation of the dispersion relation from Eulerian to Lagrangian coordinates, while the second derivation involves a wave-packet analysis of the equations of motion directly in Lagrangian coordinates. The authors show that, although the Eulerian Doppler shifting term is removed from the dispersion relation by the transformation of the frequency when passing from an Eulerian to a Lagrangian reference frame, a dependence on the background shear is then introduced by the transformation of the wavenumber. This dependence on the background shear is the term that accounts for wave refraction in the Lagrangian frame, and its role has apparently not been fully appreciated in the aforementioned previous studies.

1. Introduction

There has been a recent revival of interest in formulating a theory for internal gravity waves in a Lagrangian frame of reference, rather than the usual Eulerian frame of reference. Following the work of Allen and Joseph (1989), Hines (2001), and Chunchuzov (2002) have argued that the nonlinear terms in the Lagrangian form of the equations are, in some cases, of less importance than the corresponding nonlinear terms in the Eulerian form of the equations. In effect, the implication is that simply using a coordinate transformation from the Eulerian frame to the Lagrangian frame can remove some of the effects of nonlinearity. This is an interesting, albeit provocative, suggestion. It is of course true that, in some special cases, a system of partial differential equations can be totally or partially explicitly linearized by a coordinate transformation, but this is not so in general for the transformation from Eulerian to Lagrangian coordinates.

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The above studies are concerned with an internal wave field in a background composed of other internal waves. In each of these studies, the Lagrangian internal wave dispersion relation is taken to have the same functional form as the well-known Eulerian internal wave dispersion relation, but without the Doppler shifting term [see, for instance, (2.3) of Allen and Joseph (1989) or (59) below]. Hence, such a Lagrangian dispersion relation loses a crucial dependence on the background flow and, in particular, cannot account for refraction by the background flow. Thus, in particular, Hines (2002) has suggested that under appropriate conditions (see section 5), there are consequent simplifications for the ray tracing of internal waves if the rays are expressed in Lagrangian coordinates.

These issues have motivated us to reexamine the structure of the internal wave dispersion relation in a Lagrangian reference frame vis-à-vis the corresponding internal wave dispersion relation in an Eulerian reference frame. Hence, in this paper, we explicitly and systematically derive the Lagrangian dispersion relation for small-amplitude internal waves in the presence of a general background flow of finite amplitude. For simplicity, we consider two-dimensional flow and ignore the effects of the earth's rotation. Two derivations are given, which lead to the same result. The first involves a transformation of the dispersion relation from Eulerian to Lagrangian coordinates (section 3a), and the second involves a wave-packet asymptotic analysis of the Lagrangian equations of motion (section 3b). A general, slowly varying, space- and time-dependent background is treated first. Then, for illustration, we specialize to a horizontal background flow that varies only in the vertical direction (section 4). The main conclusion from our analysis is that there is a complete equivalence between the Eulerian and Lagrangian wave dispersion relations and that internal wave rays that refract in an Eulerian frame will refract in a Lagrangian frame, and vice versa.

2. The Eulerian dispersion relation

The Eulerian coordinates are $\mathbf{x} = (x, z)$ with z vertical. We use an overbar to denote quantities associated with the background. For example, the background velocity $\overline{\mathbf{u}}(\mathbf{x}, t) = (\overline{u}, \overline{w})$ and the background density $\overline{\rho}(\mathbf{x}, t)$. The background flow is incompressible and satisfies the equations for conservation of density, but we allow for the possible presence of arbitrary body forces in the momentum equations to maintain the background flow.

Linearizing about this background, the equations for the associated perturbations u, w, ρ, and pressure p are

$$\overline{\rho}(u_t + \overline{u}u_x + \overline{w}u_z) + p_x + \cdots = 0, \qquad (1)$$

$$\overline{\rho}(w_t + \overline{u}w_x + \overline{w}w_z) + p_z + g\rho + \cdots = 0, \qquad (2)$$

$$\rho_t + \overline{u}\rho_x + \overline{w}\rho_z + u\overline{\rho}_x + w\overline{\rho}_z = 0, \quad \text{and} \qquad (3)$$

$$u_x + w_z = 0. \qquad (4)$$

Here, subscripts denote partial derivatives, and g is the acceleration due to gravity. In the momentum equations $\{\cdots\}$ denote terms that can be neglected at leading order since they involve derivatives of $\overline{u}$, $\overline{w}$. More formally, we could introduce the slow variables $X = \epsilon x$, $Z = \epsilon z$, $T = \epsilon t$, where ϵ is a small parameter characterizing the slow variation of the background with respect to the wave field, so that $\overline{u} = \overline{u}(X, Z, T)$, etc. The derivatives are then $O(\epsilon)$ with respect to the wave frequency. But it is important here to note that $g\overline{\rho}_x/\overline{\rho}$ and $g\overline{\rho}_z/\overline{\rho}$ are $O(1)$ quantities with respect to the square of the wave frequency. The reason for this is that for these internal waves the wave frequency scales with $(g/H)^{1/2}$, where H is a scale height for the density stratification. To avoid excessive notation, we will not formally introduce these slow variables into the derivation, although their presence is always understood.

Introducing the perturbation streamfunction

$$u = -\psi_z, \qquad w = \psi_x, \qquad (5)$$

and eliminating the pressure from the momentum equations gives, at leading order,

$$\overline{\rho}[(\nabla^2\psi)_t + \overline{u}(\nabla^2\psi)_x + \overline{w}(\nabla^2\psi)_z] + g\rho_x + \cdots = 0, \qquad (6)$$

$$\rho_t + \overline{u}\rho_x + \overline{w}\rho_z - \psi_z\overline{\rho}_x + \psi_x\overline{\rho}_z = 0. \qquad (7)$$

We let

$$\psi, \rho \propto a(\mathbf{x}, t)e^{i\theta(\mathbf{x},t)} + \text{c.c.}, \qquad (8)$$

where a is the amplitude, θ is the phase for a wave packet, and c.c. denotes complex conjugate. More formally, $a = a\,(X, Z, T)$ is slowly varying on the same scale as the background, whereas the phase $\theta = \theta(X, Z, T)/\epsilon$ is rapidly varying. The Eulerian wavenumber and frequency are then defined by

$$\mathbf{k} = (k, m) = \boldsymbol{\nabla}\theta, \qquad (9)$$

$$\omega = -\theta_t. \qquad (10)$$

Because the phase is rapidly varying with respect to the background, at leading order the derivatives of the field variables are taken with respect to the phase only. Substituting (8) into (6)–(7) then leads to the Eulerian dispersion relation

$$\hat{\omega}^2 = (k^2N^2 - kmM^2)/(k^2 + m^2), \qquad (11)$$

with

$$\hat{\omega} = \omega - k\overline{u} - m\overline{w}. \qquad (12)$$

Here $N^2 = -g\overline{\rho}_z/\overline{\rho}$ and $M^2 = -g\overline{\rho}_x/\overline{\rho}$. Often the horizontal density gradient is ignored; that is, $M = 0$.

An equivalent representation for the Eulerian dispersion relation, and one that is more convenient for our derivation of the Lagrangian dispersion relation in section 3a, is obtained by using the fact that the background density $\overline{\rho}$ satisfies the equation for the conservation of density. This can be solved to give $\overline{\rho} = \rho_0(z - \overline{\zeta})$. Here $\rho_0(z)$ is the undisturbed density field (in the absence of any flow) and $\overline{\zeta}$ is the vertical displacement induced by the background flow, which satisfies the equation:

$$\overline{\zeta}_t + \overline{u}\overline{\zeta}_x + \overline{w}\overline{\zeta}_z = \overline{w}. \qquad (13)$$

We then use the relations

$$\overline{\rho}_z = \rho_0'(1 - \overline{\zeta}_z), \qquad \overline{\rho}_x = \rho_0'(-\overline{\zeta}_x), \qquad (14)$$

where the prime indicates differentiation of ρ_0 with respect to its argument $(z - \overline{\zeta})$. Consequently, the Eulerian dispersion relation can be rewritten as

$$\hat{\omega}^2 = N^2k\hat{k}/(k^2 + m^2). \qquad (15)$$

Here

$$\hat{k} = k + m\overline{\zeta}_x - k\overline{\zeta}_z, \qquad (16)$$

$$N^2 = -g\rho_0'/\rho_0. \qquad (17)$$

Note that $N = N(z - \overline{\zeta})$ and so varies on the same temporal and spatial scales as the background flow.

The Eulerian ray equations (Lighthill 1978) are readily obtained from the Eulerian dispersion relation $\omega = \omega(\mathbf{k}, \mathbf{x}, t)$ in (15) and are given by

$$\frac{d\mathbf{x}}{dt} = \nabla_k \omega, \qquad \frac{d\mathbf{k}}{dt} = -\nabla \omega. \tag{18}$$

Here d/dt is the rate of change following the ray at the local group velocity $\nabla_{\mathbf{k}} \omega = (\omega_k, \omega_m)$. Along an Eulerian ray, we also have $d\omega/dt = \omega_t$.

3. The Lagrangian dispersion relation

In Lagrangian coordinates $\mathbf{x}' = (x', z')$, the equations of motion are (e.g., Lamb 1932)

$$\rho_0(x_{tt}x_{x'} + z_{tt}z_{x'} + gz_{x'}) + p_{x'} = 0, \tag{19}$$

$$\rho_0(x_{tt}x_{z'} + z_{tt}z_{z'} + gz_{z'}) + p_{z'} = 0, \tag{20}$$

$$\rho = \rho_0(z')\left[\frac{\partial(x, z)}{\partial(x', z')}\right] = 1. \tag{21}$$

To derive the Lagrangian dispersion relation, we can apply an analogous wave-packet analysis to these Lagrangian equations, or we can transform the Eulerian dispersion relation to the Lagrangian frame of reference. We give both derivations, starting with the latter.

a. *Derivation by transformation of the Eulerian dispersion relation*

This derivation is based on the mapping between Eulerian and Lagrangian coordinates, whose Jacobian is given in (21). The Lagrangian momentum equations (19)–(20) are not explicitly used in this derivation.

We introduce the particle displacements ξ, ζ, $\overline{\xi}$, $\overline{\zeta}$, such that

$$x = x' + \xi(x', z', t) + \overline{\xi}(x', z', t), \quad \text{and} \tag{22}$$

$$z = z' + \zeta(x', z', t) + \overline{\zeta}(x', z', t). \tag{23}$$

It is sufficient here to consider only the displacements $\overline{\xi}$, $\overline{\zeta}$ due to the background flow. To retain the perturbation displacements ξ, ζ would invalidate the linearization procedure as applied in both the Eulerian and Lagrangian frames. In particular, their retention would lead to their occurrence in the Lagrangian dispersion relation, clearly inconsistent with a linear theory. For the rest of this subsection, we therefore neglect ξ, ζ in (22)–(23).

For the background flow, the particle displacement fields are related to the Eulerian background velocity field through the equations

$$\overline{\xi}_t = \overline{u}(x' + \overline{\xi}, z' + \overline{\zeta}, t), \quad \text{and} \tag{24}$$

$$\overline{\zeta}_t = \overline{w}(x' + \overline{\xi}, z' + \overline{\zeta}, t), \tag{25}$$

with $\overline{\xi} = 0$, $\overline{\zeta} = 0$ at $t = 0$. Formally, the background displacements depend on the slow variables $X' = \epsilon x'$, $Z' = \epsilon z'$, $T = \epsilon t$, which are the Lagrangian counterparts of the Eulerian slow variables $X = \epsilon x$, $Z = \epsilon z$, $T = \epsilon t$ defined in section 2. Note that the displacements are formally of $O(1/\epsilon)$, although of course their first derivatives are $O(1)$. As in the Eulerian case, we will not explicitly use these slow variables here, but their presence is always understood.

It will be useful for the present derivation to construct the matrix B, defined by

$$\mathsf{B} = \begin{pmatrix} x_{x'} & x_{z'} \\ z_{x'} & z_{z'} \end{pmatrix} = \begin{pmatrix} 1 + \overline{\xi}_{x'} & \overline{\xi}_{z'} \\ \overline{\zeta}_{x'} & 1 + \overline{\zeta}_{z'} \end{pmatrix}. \tag{26}$$

Incompressibility of the background flow implies that $\det\mathsf{B} = 1$.

To transform the Eulerian dispersion relation into Lagrangian coordinates, we must transform not only k, m, ω but also the Eulerian derivatives $\overline{\zeta}_x$, $\overline{\zeta}_z$ that occur in the definition of $\hat{k}$ in (16). Starting with this latter transformation we first note that

$$\begin{pmatrix} x_{x'} & x_{z'} \\ z_{x'} & z_{z'} \end{pmatrix}\begin{pmatrix} x'_x & x'_z \\ z'_x & z'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{27}$$

Thus,

$$\begin{pmatrix} x'_x & x'_z \\ z'_x & z'_z \end{pmatrix} = \mathsf{B}^{-1} = \begin{pmatrix} z_{z'} & -x_{z'} \\ -z_{x'} & x_{x'} \end{pmatrix} \tag{28}$$

so that

$$x'_x = z_{z'}, \quad z'_z = x_{x'}, \quad x'_z = -x_{z'}, \quad z'_x = -z_{x'}. \tag{29}$$

These four equations, on using (22)–(23) for the background flow only, imply that

$$-\overline{\xi}_x = \overline{\zeta}_{z'}, \quad -\overline{\zeta}_z = \overline{\xi}_{x'}, \quad \overline{\xi}_z = \overline{\xi}_{z'},$$
$$\overline{\zeta}_x = \overline{\zeta}_{x'}. \tag{30}$$

Hence $\hat{k}$ in (16) becomes

$$\hat{k} = k + m\overline{\zeta}_{x'} + k\overline{\xi}_{x'}. \tag{31}$$

Next we transform the Eulerian wavenumber and frequency $\mathbf{k}$, ω. The Lagrangian wavenumber and frequency will be denoted by $\mathbf{k}'$, ω'. We start with the phase θ in (8), which transforms from Eulerian to Lagrangian coordinates according to

$$\theta(\mathbf{x}, t) = \theta[\mathbf{x}(\mathbf{x}', t), t] = \theta'(\mathbf{x}', t). \tag{32}$$

Defining $\nabla' = (\partial_{x'}, \partial_{z'})$, we have

$$\mathbf{k}' = \nabla'\theta' \qquad \omega' = -\theta'_t. \tag{33}$$

Using $\theta'_{x'} = \theta_x x_{x'} + \theta_z z_{x'}$ and a similar expression for $\theta'_{z'}$, we arrive at the wavenumber transformation

$$k' = kx_{x'} + mz_{x'}, \tag{34}$$

$$m' = kx_{z'} + mz_{z'}. \tag{35}$$

Equivalently,

$$\mathbf{k}' = \mathsf{B}^{\mathrm{T}}\mathbf{k}, \tag{36}$$

where superscript T indicates the transpose. Using the second expression for B in (26), we have

$$k' = k + k\overline{\xi}_{x'} + m\overline{\zeta}_{x'}, \tag{37}$$

$$m' = m + k\overline{\xi}_{z'} + m\overline{\zeta}_{z'}. \tag{38}$$

From (31) and (37) we now see that $k' = \hat{k}$; that is, $\hat{k}$ is in fact the Lagrangian horizontal wavenumber.

The inverse relations are

$$k = k' + k'\overline{\zeta}_{z'} - m'\overline{\zeta}_{x'}, \tag{39}$$

$$m = m' - k'\overline{\xi}_{z'} + m'\overline{\xi}_{x'}. \tag{40}$$

For the frequency,

$$\omega' = -\theta_t - \theta_x x_t - \theta_z z_t, \tag{41}$$

$$= \omega - k\overline{u} - m\overline{w}, \tag{42}$$

$$= \hat{\omega}. \tag{43}$$

Thus the Eulerian dispersion relation (15) can be written as

$$\hat{\omega}^2 = N^2 kk'/(k^2 + m^2). \tag{44}$$

Substituting for $\hat{\omega}$, k, m from (39)–(43) gives the Lagrangian dispersion relation

$$\omega'^2 = \frac{N^2 k'(k' + k'\overline{\zeta}_{z'} - m'\overline{\zeta}_{x'})}{(k' + k'\overline{\zeta}_{z'} - m'\overline{\zeta}_{x'})^2 + (m' - k'\overline{\xi}_{z'} + m'\overline{\xi}_{x'})^2}. \tag{45}$$

The derivatives of $\overline{\xi}$, $\overline{\zeta}$ appearing here are, by (24)–(25), functions of the background shear.

The Lagrangian ray equations are the counterpart of the Eulerian ray equations (18). That is, given the Lagrangian dispersion relation $\omega' = \omega'(\mathbf{k}', \mathbf{x}', t)$ in (45), the Lagrangian ray equations are

$$\frac{d\mathbf{x}'}{dt} = \nabla_{\mathbf{k}'}\omega', \qquad \frac{d\mathbf{k}'}{dt} = -\nabla'\omega'. \tag{46}$$

Along a Lagrangian ray, $d\omega'/dt = \omega'_t$. Using the transformations (22)–(23), for the background flow alone, along with (36) and (42)–(43), it can now be shown that there is a complete equivalence between these Lagrangian ray equations (46) and the Eulerian ray equations (18). The demonstration is straightforward, but note the relation

$$\mathbf{B}_t = \begin{pmatrix} \overline{u}_x & \overline{u}_z \\ \overline{w}_x & \overline{w}_z \end{pmatrix}\mathbf{B} \tag{47}$$

obtained by differentiation of (24)–(25) with respect to x', z'.

b. Derivation by an asymptotic analysis of the Lagrangian equations

This derivation first requires a linearization of the Lagrangian equations (19)–(21) about the background flow, using the background and perturbation displacements defined in (22)–(23). The result is

$$\rho_0[\xi_{tt}(1 + \overline{\xi}_{x'}) + \zeta_{tt}\overline{\zeta}_{x'} + g\zeta_{x'}] + p_{x'} = 0, \tag{48}$$

$$\rho_0[\xi_{tt}\overline{\xi}_{z'} + \zeta_{tt}(1 + \overline{\zeta}_{z'}) + g\zeta_{z'}] + p_{z'} = 0, \tag{49}$$

$$\xi_{x'}(1 + \overline{\zeta}_{z'}) + \zeta_{z'}(1 + \overline{\xi}_{x'}) - \xi_{z'}\overline{\zeta}_{x'} - \zeta_{x'}\overline{\xi}_{z'} = 0. \tag{50}$$

Eliminating the pressure from (48)–(49) gives, to leading order,

$$(\xi_{x'} - \zeta_{z'})_{tt} + \zeta_{x'tt}\overline{\zeta}_{z'} + \xi_{x'tt}\overline{\zeta}_{z'} - \xi_{ttz'}\overline{\xi}_{x'} - \zeta_{ttz'}\overline{\zeta}_{x'} + N^2\zeta_{x'} = 0. \tag{51}$$

We seek solutions of the form

$$\xi,\ \zeta \propto a'(\mathbf{x}', t)e^{i\theta'(\mathbf{x}',t)} + \text{c.c.}, \tag{52}$$

with $\mathbf{k}' = \nabla'\theta'$ and $\omega' = -\theta'_t$ as in (33). Substitution into (50)–(51) gives

$$\omega'^2(m\xi - k\zeta) + N^2 k'\zeta = 0, \tag{53}$$

$$m\zeta + k\xi = 0, \tag{54}$$

where we have used the wavenumber relation (36) to write the result in a compact form. Eliminating ξ or ζ from these equations, we obtain

$$\omega'^2 = N^2 kk'/(k^2 + m^2), \tag{55}$$

which agrees with (43) and (44).

4. A steady horizontal background flow

We now apply the general Lagrangian dispersion relation just obtained to the special case in which the background flow $\overline{u}$ is horizontal and varies with depth only. The background displacements are then given by

$$\overline{\xi} = \overline{u}(z')t, \qquad \overline{\zeta} = 0. \tag{56}$$

From (34)–(35), the relation between the Eulerian and Lagrangian wavenumbers is then

$$k = k', \qquad m = m' - k'\overline{u}_{z'}t. \tag{57}$$

The Lagrangian dispersion relation (45) reduces to

$$\omega'^2 = k'^2 N^2/[k'^2 + (m' - k'\overline{u}_{z'}t)^2]. \tag{58}$$

The background shear $\overline{u}_{z'}$ is small compared with ω' since, when expressed in terms of the slow variables, it is $\epsilon\overline{u}_{Z'}$. This does not imply, however, that we can neglect $k'\overline{u}_{z'}t$ in comparison with m' in (58). Indeed, in terms of the slow variables this term is $k'\overline{u}_{Z'}T$ and is $O(1)$ with respect to the small parameter ϵ. Alternatively we could estimate $k'\overline{u}_{z'}t$ as $k'\overline{u}/c_g$. Here c_g is the vertical component of group velocity (which for this problem has the same value in both the Eulerian and Lagrangian frames). We have used the estimates that along a ray path $\overline{u}_{z'}z' \sim \overline{u}$ and $z'/t \sim c_g$. Thus, $k'\overline{u}_{z'}t$ is an $O(1)$ quantity and of the same order as m'.

It is clear from the study of Hartman (1975) that the term $k'\overline{u}_{z'}t$ in (58) can be important for the description of rays in the Lagrangian frame. He considers the particular case of constant N and constant background shear $\overline{u}_{z'}$. Then m', as well as k', are constant along a Lagrangian ray, but ω' is not constant. This follows from the Lagrangian dispersion relation (58), which depends explicitly on t but not on z' when $\overline{u}_{z'}$ is constant. For this particular case, the Lagrangian rays are straight lines in (x', z'), but the group velocity changes along the ray due to the change in ω'. The Lagrangian ray reflects from the same turning point height (where $\omega' \to N$) and asymptotes toward the same critical layer height (where $\omega' \to 0$) as for the corresponding Eulerian ray.

Note that the coordinates used by Hartman (1975) are Lagrangian in the sense that we have defined here. Hartman's coordinates are $x' = x - \overline{u}(z)t$ and $z' = z$, that is, the position that moves with the velocity of the background flow. As the notation indicates, these are the same as our Lagrangian coordinates $x' = x - \overline{\xi}$ and $z' = z - \overline{\zeta}$ when our coordinates are specialized to this case using (56).

5. Discussion

The main result of this paper is the Lagrangian dispersion relation (45) for internal gravity waves. This differs from the form used, for instance, by Allen and Joseph (1989), which in present notation is (apart from the Coriolis term and ignoring the third dimension)

$$\omega'^2 = N^2 k'^2/(k'^2 + m'^2). \tag{59}$$

Unlike this form, which is appropriate for a uniform background without any shear, our Lagrangian dispersion relation (45) depends on the background straining field. That is, it contains terms involving the derivatives of the particle displacements $\overline{\xi}$ and $\overline{\zeta}$ associated with the background flow. These terms are functions of the background shear through (24)–(25), and their occurrence in the Lagrangian dispersion relation then accounts for wave refraction in the Lagrangian frame.

Allen and Joseph (1989) used Lagrangian coordinates to describe a spectrum of internal waves, with each wave component of the Lagrangian spectrum satisfying (59). When this Lagrangian spectrum is transformed to the Eulerian frame, the resulting high-wavenumber components of the Eulerian spectrum were found to be nonwavelike. That is, these components did not satisfy the Eulerian dispersion relation (15). Our work suggests a possible factor contributing to this result, in that Allen and Joseph's Lagrangian dispersion relation (59) and the Eulerian dispersion relation (15) are not equivalent to each other under the Lagrangian-to-Eulerian coordinate mapping except in the limit of no background shear. For these high wavenumbers, this limit is not appropriate because the high wavenumbers are subjected to strong background shears resulting from the longer waves. A comprehensive discussion of this issue of the high-wavenumber Eulerian spectrum can be found in the recent review by Fritts and Alexander (2003).

Hines (2002) examines another issue. He considers a ''test wave'' in a background of waves and asks the question: when can the interaction of the test wave with the background be ignored? Hines then finds that the criteria for neglecting the interaction with the background depends on the test-wave wavenumber in different ways according to whether one uses an Eulerian or a Lagrangian coordinate system (see Fig. 1 of Hines 2002).

In the Lagrangian frame, the Hines criteria for negligible interaction with the background involve his quantities $S_{1,1}$, $S_{1,3}$, $S_{3,1}$, $S_{3,3}$, which correspond respectively to our quantities $\overline{\xi}_{x'}$, $\overline{\xi}_{z'}$, $\overline{\zeta}_{x'}$, $\overline{\zeta}_{z'}$. The conditions on $S_{1,1}$, $S_{1,3}$ in Hines' (3.17), (3.22), correspond here to $|\overline{\xi}_{x'}| \ll 1$ and $|\overline{\xi}_{z'}| \ll |m'/k'|$. The conditions on $S_{3,1}$, $S_{3,3}$ in Hines' (3.21), (3.18), correspond here to $|\overline{\zeta}_{x'}| \ll |k'/m'|$ and $|\overline{\zeta}_{z'}| \ll 1$. Under these conditions, the background terms in our Lagrangian dispersion relation (45) become small, so in this respect our work is consistent with that of Hines (2002).

In an Eulerian frame of reference, the corresponding conditions under which the background flow can be ignored are readily obtained from (15). Thus, we now require the conditions under which $\omega \approx \hat{\omega}$, or equivalently $|\mathbf{k} \cdot \overline{\mathbf{u}}| \ll \hat{\omega}$, and under which $\hat{k} \approx k$. On using the definitions (16) and (15), these can be expressed in the form $|\overline{\mathbf{u}}| \ll N|k|/|\mathbf{k}|^2$, $|\overline{\zeta}_x| \ll |k/m|$ and $|\overline{\zeta}_z| \ll 1$. Again, these conditions are consistent with those obtained by Hines (2002) for the neglect of the effects of the background in an Eulerian reference frame. However, we point out here that the criterion $|\mathbf{k} \cdot \mathbf{u}| \ll \hat{\omega}$ essentially estimates advection by the background flow, rather than refraction, per se. To take account of refraction, one needs to estimate the variation of $\mathbf{k} \cdot \mathbf{u}$ along a ray. When this is done, the criterion is replaced by $|\delta\mathbf{u}| \ll N/|\mathbf{k}|$, where $\delta\mathbf{u}$ is a measure of the spatial variability in $\mathbf{u}$. Further, we also need to ensure that $N(z - \overline{\zeta})$ does not vary significantly with $\overline{\zeta}$. This can be achieved by requiring either that $|\overline{\zeta}_x| \ll 1$ and $|\overline{\zeta}_z| \ll 1$, or that N itself is approximately constant.

Hines (2002) argues that there is an advantage in using a Lagrangian frame when, as judged by the above criteria, the interaction between the Lagrangian ray and the background is weak (see appendix C of Hines 2002). But note that the above criteria are local and wavenumber dependent. They can be used to describe the conditions under which the refraction is *locally* minimal, either in an Eulerian frame on the one hand or in a Lagrangian frame on the other hand. However, the presence of variations in the background, no matter how weak, will cause some refraction and, since the criteria for minimal refraction are wavenumber dependent, it is not clear whether the refraction can necessarily remain minimal along a ray over the time scales of interest. Indeed Hines (2002, p. 26) notes that such conditions for minimal refraction may be breached locally and need to be continually confirmed. On the other hand, our present formulation based on the full Lagrangian dispersion relation (45) is not subject to any such restriction. Numerical ray tracing based on the full Lagrangian dispersion relation (45) might help to resolve this question, and to test the criteria derived by Hines (2002) and described above.

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