A Quasi-Eulerian, Quasi-Lagrangian View of Surface-Wave-Induced Flow in the Ocean

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ABSTRACT

In this study the influence of surface waves on the mean flow in an ocean of arbitrary depth is examined. The wave-induced forcing on the mean flow is obtained by integrating the Eulerian equations for mass and momentum balance from the bottom to an undulating material surface within the water column. By using the mean position of the material surface as the vertical coordinate, the authors obtain the depth dependence of the mean flow and the wave-induced forcing. Substitution of the vertical coordinate makes the model Lagrangian in the vertical direction. The model is Eulerian in the horizontal direction, allowing one to model the effects of a spatially nonuniform wave field or varying depth in a straightforward way.

1. Introduction

That surface waves give rise to a net mass transport, or wave drift, has been known for over 150 years (Stokes 1847). Nevertheless, there is still an ongoing discussion on how the mass transport, and the associated momentum flux, should be incorporated into ocean models of various complexities. Several recent papers have provided a variety of descriptions (Ardhuin et al. 2004; Lane et al. 2007; McWilliams and Restrepo 1999; McWilliams et al. 2004; Mellor 2003; Weber 2003; Weber et al. 2006).

Wave drift is essentially a phenomenon of Lagrangian nature, that is, concerning the motion of individual fluid particles (e.g., Jenkins 1986; Pollard 1970; Weber 2003). The Eulerian description provides the spatial distribution of fluid motion, and information on individual fluid particles is not readily available. For example, the Stokes drift is the mean drift inherent in surface waves (Stokes 1847), which in the Lagrangian description is a current that decays uniformly with depth on the scale of the wave motion. The Stokes drift is a Lagrangian quantity in the sense that it cannot be represented as a current that decays uniformly with depth in an Eulerian frame of reference. In the Eulerian description the mean wave motion in the interior of the fluid becomes zero and the mass transport is therefore confined to the region above the wave troughs (Phillips 1977). This "deficit" of the Eulerian description implies that the motion of, for example, biologically active compounds is not captured in an adequate way. In addition to the Stokes drift, frictional forces acting at the boundaries or pressure gradient forces due to mass accumulation induce drift currents that are equal in the Eulerian and Lagrangian descriptions (Longuet-Higgins 1953). In contrast to the Stokes drift, these (often termed Eulerian) drift currents can be modeled equally well using an Eulerian or a Lagrangian description of the motion.

If the particle motion is important, a Lagrangian description of the motion is preferable: This is also the case if the problem involves moving boundaries (i.e., the sea surface in our case) and exchange of momentum at these boundaries. Difficulties with a direct Lagrangian approach arise when the problem involves lateral boundaries or varying bottom topography. Both the lateral boundary conditions and the topographic effects are easier to formulate in the Eulerian description. Some analytical models make use of curvilinear coordinates that fit the waves or use a hybrid coordinate system (e.g., Andrews and McIntyre 1978; Longuet-

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Higgins 1953; Mellor 2003). Since models of wave drift often use coordinate systems other than Cartesian Eulerian coordinates, they can be quite complex compared to theoretical models for other oceanographic phenomena. One unfortunate consequence is thus that many studies tend to be more oriented toward mathematical details than toward oceanographic applications.

In this paper we derive equations for the mean horizontal velocities and mass transport induced by the waves correct to second order in the wave amplitude. Because any mean quantity of surface waves is of second order, our discussion is limited to the lowest-order wave drift problem; for instance, we neglect fourth-order advection terms in the mean momentum equation. Still we argue that the shortcomings of our model are justified by its simple derivation and the simple physical interpretation of each forcing term in the mean momentum equation. We would also like to point to the inconsistency in keeping terms of third order and higher in any model for wave drift if the derivation of the forcing terms is based on linear wave solutions. Second-order periodic solutions (and possibly higher orders, depending on the desired accuracy) have to be taken into account in such higher-order drift models.

The influence of surface waves on barotropic flows were outlined in the early 1960s by Longuet-Higgins and Stewart, leading to a formulation where the waves were associated with a certain radiation stress (Longuet-Higgins and Stewart 1960, 1961, 1962, 1964; Whitham 1962). The idea is to integrate the governing equations from the bottom to the undulating sea surface; by taking an average over the wave period, the influence of the waves on the mean flow can be described. The theory has proven to be very useful for many applications and has been extensively verified (Longuet-Higgins 1970, 2005; Nielsen 1992; Phillips 1977; Thornton and Guza 1986). Although the concept is powerful, it does not provide insight into the depth distribution of the mass transport or the wave-induced forcing of the mean flow. Thus our question is: Can we use the strategy of Longuet-Higgins and Stewart to obtain a continuous description of the drift velocities and the wave-induced forcing?

In this study we follow the procedure of Longuet-Higgins and Stewart closely, although we consider the integration from the bottom to an undulating material surface within the water column. Accordingly, in the limit when the material surface approaches the sea surface we—by definition—recapture the results of Longuet-Higgins and Stewart. Furthermore, we argue that the mean horizontal velocity and the wave-induced forcing at any given depth is identical to the vertical derivative of the derived integral quantities (i.e., Lamb 1932, section 250). We use the position of the material surface within the water column as the vertical coordinate, which makes our model Lagrangian in the vertical direction. In the horizontal direction we retain the Eulerian description, which enables us to include effects of varying depth and spatial changes in the wave properties in a straightforward way. Our results can thus be used to develop a fully Eulerian numerical model.

The outline of the paper is as follows: the main results are derived in section 2. In section 2a we formulate the problem mathematically, deriving the vertically integrated momentum equation. We then use the linear wave solutions to evaluate the wave-induced forcing of the mean flow in section 2b. The depth dependence of this forcing is examined in section 2c. In section 3 we discuss possible ways of extending the validity of the model to include frictional forces. In section 3a we rederive the classical depth-integrated transport equation from our continuous drift equations. We briefly discuss the use of our results in numerical modeling in section 3b, while section 4 contains a short summary and concluding remarks.

2. Mathematical formulation

We consider an ocean where the bottom is located at $z = -d$. The surface is located at $z = \eta = h + \tilde{\eta}$, where $\tilde{\eta}$ denotes fast surface oscillations as described by oceanic surface waves, and h is the mean height of the sea surface (i.e., averaged over a wave cycle). The mean depth of the ocean in this geometrical setting becomes $H = h + d$. We will also consider a material surface located at $z = \zeta = Z + \tilde{\zeta}$, where Z is the mean vertical position of the surface and $\tilde{\zeta}$ denotes the oscillation of the material surface due to surface waves (see Fig. 1). It should be noted that $\tilde{\zeta}$ is related to $\tilde{\eta}$ and varies with the value of Z.

a. Vertically integrated mass and momentum equations

We consider two-dimensional motion and take the horizontal axis to be aligned in the direction of the waves and the vertical axis to be positive upward (see Fig. 1). We neglect Coriolis forces and assume that the density ρ is constant. Our main aim is to investigate the effects of a sloping bottom; thus, we consider irrotational motion and neglect frictional forces. The horizontal and vertical momentum equations become

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial z}(\rho uw) = 0, \qquad (1a)$$

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial z}(\rho w^2 + p + \rho gz) + \frac{\partial}{\partial x}(\rho uw) = 0, \qquad (1b)$$

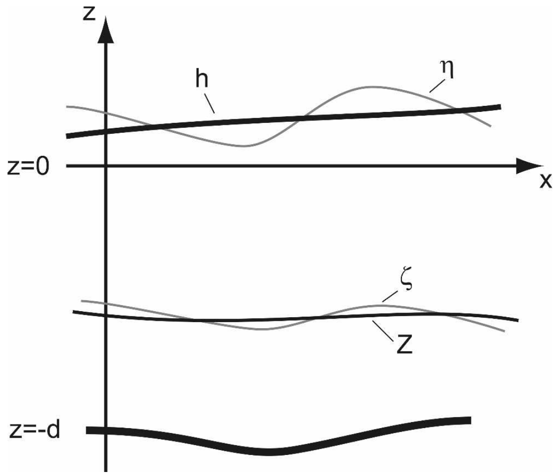

FIG. 1. A sketch of the geometry considered here. The bottom is located at $z = -d$; the sea surface is located at $z = \eta = h + \tilde{\eta}$, where h is the mean position of the sea surface and $\tilde{\eta}$ denotes the undulating part of the sea surface; further, we consider a material surface located at $z = \zeta = Z + \tilde{\zeta}$, where Z is the mean position of the material surface and $\tilde{\zeta}$ denotes the undulating movements of the material surface.

where (u, w) are the horizontal and vertical velocities, p is the pressure, and g is the acceleration due to gravity. In (1) we have used the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{2}$$

At the material surface the following kinematic condition holds:

$$w = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x}, \quad z = \zeta. \tag{3}$$

In particular, Eq. (3) is used as the kinematic boundary condition at the bottom $z = -d$, and at the surface $z = \eta$. The dynamic boundary condition at the surface $z = \eta$ is

$$-p = \sigma, \tag{4}$$

where σ is the vertical atmospheric stress. The local wave solutions to the linearized versions of (1)–(4) are

$$\tilde{u} = a\omega\frac{\cosh[k(d+z)]}{\sinh(kH)}\cos\theta, \tag{5a}$$

$$\tilde{w} = a\omega\frac{\sinh[k(d+z)]}{\sinh(kH)}\sin\theta, \tag{5b}$$

$$\tilde{p} = \rho\frac{a\omega^2\cosh[k(d+z)]}{k\sinh(kH)}\cos\theta, \tag{5c}$$

$$\theta = kx - \omega t, \tag{5d}$$

where a is the amplitude of the waves, and θ is the phase. The wavenumber k and the wave frequency ω are connected through the dispersion relation, which to $O(a)$ reads

$$\omega^2 = gk\tanh(kH). \tag{6}$$

The phase velocity of the waves is $c = \omega/k$, and the group velocity c_g is

$$c_g = \frac{\partial \omega}{\partial k} = \frac{c}{2}\left[1 + \frac{2kH}{\sinh(2kH)}\right]. \tag{7}$$

We assume that the wave frequency ω is constant, but that a, k, and the mean depth $H = d + h$ change slowly in x and t. Furthermore, we will assume that a and k are known a priori, for instance, from ray tracing or model results. The surface elevation corresponding to the wave solutions (5) is

$$\tilde{\eta} = a\cos\theta. \tag{8}$$

The vertical coordinate ζ of a material surface consists of a fluctuating part $\tilde{\zeta}$ of $O(a)$ and a part Z that changes slowly in x and t on the same scale as mean wave properties of $O(a^2)$. In view of (3) we find that the fluctuating part $\tilde{\zeta}$ must be given by

$$\tilde{\zeta} = \int \tilde{w}\,dt\,. \tag{9}$$

Integrating (1a) vertically from the bottom to the material surface $z = \zeta$, using (3), we obtain (e.g., Phillips 1977)

$$\frac{\partial}{\partial t}\int_{-d}^{\zeta}\rho u\,dz = -\frac{\partial}{\partial x}\int_{-d}^{\zeta}(p + \rho u^2)\,dz + p(z=\zeta)\frac{\partial \zeta}{\partial x} + p(z=-d)\frac{\partial d}{\partial x}\,. \tag{10}$$

Equation (10) is a conservation equation for the horizontal momentum in a water column extending from the bottom to the undulating material surface ζ and relates the rate of change with time to the divergence of the horizontal momentum fluxes (Longuet-Higgins and Stewart 1960). The first term on the right-hand side is the gradient of the horizontal momentum flux due to pressure and bodily transport of horizontal momentum. The remaining terms describe momentum fluxes at the upper and lower boundaries.

Equation (10) is the starting point for deriving the radiation stresses and the mean flow equations in the sense of Longuet-Higgins and Stewart. Alternative representations of the wave-induced forcing terms, for instance, the Craik–Leibovich vortex force, are described elsewhere (e.g., Lane et al. 2007; McWilliams et al. 2004; McWilliams and Restrepo 1999).

b. Wave-induced forcing

We will examine the variation with depth of the wave-induced mean flow and the mean momentum fluxes using Eq. (10) as a starting point. Since both the energy and the mean drift in surface waves are of second order in the wave amplitude, we assume that the main features of the dynamics are captured if the analysis is correct to this order and neglect terms of third order and higher. An average over a wave cycle is denoted by angle brackets, and will be used to assess mean quantities.

For the present, the left-hand side of (10) is kept unchanged, and we focus on the forcing terms on the right-hand side. The bodily transport of momentum yields a contribution

$$\left\langle \int_{-d}^{\zeta} \rho u^2\, dz \right\rangle = \int_{-d}^{Z} \rho\langle \tilde{u}^2\rangle\, dz + O(a^3). \qquad (11)$$

The various forcing terms in (10) involving the pressure need careful examination. In the absence of waves the pressure is hydrostatic, but due to waves the pressure will contain fluctuating parts of both first and second order in the wave amplitude. Furthermore, we must allow for a horizontal pressure gradient depending on x and t caused by a sloping surface. For p we therefore write

$$p = -\rho g z + \tilde{p} + p' + p_0(x, t), \qquad (12)$$

where p' is the $O(a^2)$ contribution due to the wave motion and p_0, which is assumed to have a mean value of $O(a^2)$, is the deviation from the hydrostatic pressure due to changes in the mean surface elevation. The fluctuating part $\tilde{p}$ is given by (5c). The second-order contribution p' can be found by vertically integrating (1b) as in Phillips (1977). However, here we follow Longuet-Higgins and Stewart (1964) and take

$$\langle p'\rangle = -\rho\langle \tilde{w}^2\rangle, \qquad (13)$$

which implies that we neglect the mean vertical acceleration, as well as the Reynolds stress component $\partial\langle \tilde{u}\tilde{w}\rangle/\partial x$.

To find p_0 we invoke the dynamic boundary condition (4) and assume that $\langle\sigma\rangle = 0$ so that the mean pressure at the surface is zero. By substitution from (13) into (12) we obtain from (4):

$$\begin{aligned}\langle p_0\rangle &= \rho g h + \left\langle \rho\tilde{w}^2 - \frac{\partial\tilde{p}}{\partial z}\tilde{\eta}\right\rangle_{z=h} + O(a^3)\\ &= \rho g h + O(a^3). \end{aligned} \qquad (14)$$

To obtain the last equality we use (9), the linearized version of (1b) in combination with (12), and the fact that the waves are periodic in time. Equation (14) demonstrates that the only $O(a^2)$ contribution from the waves to the pressure in our formulation is given by (13).

All mean second-order quantities in (12) are now known in terms of the wave solutions, and we can evaluate the forcing terms involving pressure in (10), averaged over a wave cycle. For the hydrostatic part we have exactly

$$\begin{aligned}\left\langle \int_{-d}^{\zeta} -\rho g z\, dz \right\rangle &= -\frac{1}{2}\rho g(\langle\zeta^2\rangle - d^2),\\ &= -\frac{1}{2}\rho g(Z^2 + \langle\tilde{\zeta}^2\rangle - d^2). \end{aligned} \qquad (15)$$

For the fluctuating part of the pressure we obtain

$$\begin{aligned}\left\langle \int_{-d}^{\zeta} \tilde{p}\, dz \right\rangle &= \left\langle \int_{-d}^{Z} \tilde{p}\, dz \right\rangle + \langle\tilde{p}\tilde{\zeta}\rangle_{z=Z} + O(a^3),\\ &= \langle\tilde{p}\tilde{\zeta}\rangle_{z=Z} + O(a^3), \end{aligned} \qquad (16)$$

which shows that the mean momentum flux due to the fluctuating pressure is confined to the region between the mean level Z of the material surface and the material surface itself. For the remaining pressure terms of $O(a^2)$ we find that

$$\begin{aligned}\left\langle \int_{-d}^{\zeta} p' + p_0\, dz \right\rangle &= \int_{-d}^{Z} -\rho\langle\tilde{w}^2\rangle\, dz + (Z + d)\langle p_0\rangle\\ &\quad + O(a^3). \end{aligned} \qquad (17)$$

In addition to the expressions (11) and (15)–(17), we need to evaluate the last two parts on the right of (10) associated with the momentum fluxes at the upper and lower boundaries. At the upper boundary we need $\langle p\partial\zeta/\partial x\rangle_{z=\zeta}$ correct to $O(a^2)$. From (9) and (12) we obtain

$$\begin{aligned}\left\langle p\frac{\partial\zeta}{\partial x}\right\rangle_{z=\zeta} &= -\frac{1}{2}\rho g\frac{\partial}{\partial x}(Z^2) - \frac{1}{2}\rho g\frac{\partial}{\partial x}\langle\tilde{\zeta}^2\rangle\\ &\quad + \left\langle \tilde{p}\frac{\partial\tilde{\zeta}}{\partial x}\right\rangle_{z=Z} + O(a^3). \end{aligned} \qquad (18)$$

For irrotational motion $\tilde{p}$ vanishes at the bottom, and the second-order term $\langle p'\rangle$ is negligible provided $|\partial d/\partial x| \ll 1$, as we assume here. It follows that the last term on the right-hand side of (10) can be written

$$\left\langle p\frac{\partial d}{\partial x}\right\rangle_{z=-d} = \rho g H\frac{\partial d}{\partial x} + O(a^3). \qquad (19)$$

Taking the average over a wave cycle and using (11)–(19), we can now rewrite Eq. (10) in the form

$$\rho\frac{\partial}{\partial t}\left\langle\int_{-d}^{\zeta} u\,dz\right\rangle = -\left\langle\tilde{p}\frac{\partial\tilde{\zeta}}{\partial x}\right\rangle_{z=Z} - \frac{\partial}{\partial x}\left[\rho\int_{-d}^{Z}\langle\tilde{u}^2 - \tilde{w}^2\rangle\,dz - \langle\tilde{p}\tilde{\zeta}\rangle_{z=Z} + \rho g(Z+d)h\right] + \rho gh\frac{\partial d}{\partial x} + O(a^3). \tag{20}$$

In (20) the total velocity u, the mean surface elevation h, and the mean position of the material surface Z are all unknown. By integrating the continuity equation (2) from the bottom to the surface, using (3), we find that

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left\langle\int_{-d}^{\eta} u\,dz\right\rangle, \tag{21}$$

which enables us to connect the mean surface elevation to the total horizontal volume transport. An additional equation for Z is obtained in a similar way by integrating (2) from the bottom to the material surface ζ:

$$\frac{\partial Z}{\partial t} = -\frac{\partial}{\partial x}\left\langle\int_{-d}^{\zeta} u\,dz\right\rangle. \tag{22}$$

In the following section we will examine the depth dependence of the forcing terms in (20). Note that the procedure we have followed here can be used to construct a model for the mass transport and momentum fluxes in a layered system by replacing the bottom coordinate d with the coordinate of a material surface.

c. *Depth distribution of the wave-induced forcing*

The velocity and forcing at each vertical level is simply defined as the vertical derivative of the integrated quantity up to this level, an idea that goes back to Lamb (1932). The forcing terms in (20) have the mean position Z of a material surface as the vertical coordinate, and we will take the derivative with respect to Z of this equation.

In general Z is an unknown function of x and t. A change in Z is necessarily a result of a divergent mass flux [see Eq. (22)], so we may safely assume that

$$Z = z_0 + a^2 f(x, z_0, t), \tag{23}$$

where z_0 is the initial vertical position of the material surface (or a suitable reference position), and f is slowly varying on a time scale and horizontal scale comparable to any other second-order quantity. Equation (23) implies that we can, to the present order of approximation, interchange the order of differentiation with respect to x, t, and Z freely: Noting that z_0 is independent of x and t, we apply $\partial/\partial z_0$ to Eq. (20), and use

$$\frac{\partial}{\partial z_0} = [1 + O(a^2)]\frac{\partial}{\partial Z}.$$

Although we may consider the first-order horizontal velocity as known, we will not separate between the first- and the second-order horizontal velocity on the left-hand side of (20). Instead, we define

$$\overline{u} = \frac{\partial}{\partial Z}\left\langle\int_{-d}^{\zeta} u\,dz\right\rangle \tag{24}$$

so that $\overline{u}$ is the rate of change with depth of the total mean horizontal momentum up to level Z and can be viewed as the mean horizontal velocity at this level (Lamb 1932, section 250). Applying $\partial/\partial Z$ to Eq. (20) we obtain

$$\rho\frac{\partial\overline{u}}{\partial t} = \frac{\partial}{\partial Z}\left\langle\tilde{p}\frac{\partial\tilde{\zeta}}{\partial x}\right\rangle - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial Z}\langle\tilde{p}\tilde{\zeta}\rangle\right) - \rho\frac{\partial}{\partial x}\langle\tilde{u}^2 - \tilde{w}^2\rangle - \rho g\frac{\partial h}{\partial x}, \tag{25}$$

where all variables are evaluated at $z = Z$ (in the following all mean quantities are to be evaluated at $z = Z$ unless explicitly stated otherwise). The first term on the right-hand side is analogous to the rate of change with depth of a form stress (sometimes referred to as form drag) acting on the material surface. The second term is the divergence of the mean flux of momentum, per unit distance in the vertical direction, due to the fluctuating pressure acting on the vertical plane between Z and ζ. The third term is the divergence of (i) the bodily transport of horizontal momentum, and (ii) the mean flux of momentum due to the second order wave-induced pressure (both per unit distance in the vertical direction). The expression $\rho\langle\tilde{u}^2 - \tilde{w}^2\rangle$ is independent of depth when the motion is irrotational (Longuet-Higgins and Stewart 1964). The last term on the rhs of (25) is the pressure gradient arising from the mean surface slope.

Up to this point in the analysis we have made no explicit use of the wave solutions (5), and Eq. (25) is therefore the most general result in this paper.

d. *Waves without external forcing*

For irrotational waves in the absence of external forcing $\sigma = 0$, the amplitude a can only vary as a result of a change in the depth; hence $a = a(x)$. The Stokes drift velocity is given by (Stokes 1847)

$$u_S = a^2\omega k\frac{\cosh[2k(Z+d)]}{2\sinh^2(kH)} \tag{26}$$

and can be calculated from the wave solutions (5) and (9) by evaluating the expression

$$u_S = \frac{\partial}{\partial Z}\left\langle \int_{-d}^{\zeta} \tilde{u}\, dz \right\rangle \approx \frac{\partial}{\partial Z}\langle \tilde{u}\tilde{\zeta}\rangle. \tag{27}$$

We see from the right-hand side of (27) that the Stokes drift is confined to the region between the crests and the troughs of the wavy material surface ζ as expected in an Eulerian frame of reference (Phillips 1977). Our formulation is partly Lagrangian because we trace the vertical position of the material surface and hence obtain the Stokes drift in Lagrangian form, that is, continuously distributed with depth, when we differentiate with respect to Z. It would be more precise, however, to refer to (27) as the rate of change with depth of the total mean wave momentum: The total mean wave momentum can be written E/c, where the wave energy density E is

$$E = \frac{1}{2}\rho g a^2. \tag{28}$$

It follows from the dispersion relation (6), the definitions of the Stokes drift (26), and the wave energy density (28) that

$$\rho \int_{-d}^{h} u_S\, dZ = \frac{E}{c}, \tag{29}$$

confirming that the total mean wave momentum is represented by the integrated Stokes drift (Starr 1959).

We will now find expressions for the forcing terms in (25) by using the wave solutions (5) and (9). If the wave properties do not change along the propagation direction, the form stress term $\langle \tilde{p}\partial\tilde{\zeta}/\partial x\rangle$ is zero because the fluctuating pressure and the slope of the material surface will be exactly out of phase. Taking into account that a, H, k, and Z depend on x, we find that

$$\frac{\partial}{\partial Z}\left\langle \tilde{p}\frac{\partial\tilde{\zeta}}{\partial x}\right\rangle = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial Z}\langle \tilde{p}\tilde{\zeta}\rangle\right) + \frac{1}{2k}\left\{\rho c u_S + \left[\frac{2}{H}\left(\frac{c_g}{c} - \frac{1}{2}\right)E\right]\right\}\frac{\partial k}{\partial x}. \tag{30}$$

The expression

$$\frac{1}{H}\left(\frac{c_g}{c} - \frac{1}{2}\right)E = \frac{\rho a^2 \omega^2}{4\sinh^2(kH)} \tag{31}$$

becomes zero in the deep-water limit when $c_g \to c/2$. The first part on the right-hand side of (30) equals minus half the second term on the right-hand side of (25), and from the wave solutions and (26) we find that

$$-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial Z}\langle \tilde{p}\tilde{\zeta}\rangle\right) = -\rho\frac{\partial}{\partial x}(c u_S). \tag{32}$$

The term $\langle \tilde{p}\tilde{\zeta}\rangle$ is the mean flux of momentum due to the fluctuating pressure (5c). Just as for the Stokes drift velocity (26), this flux of momentum is confined to the region between the crests and the troughs of the material surface, and by taking the derivative with respect to Z we obtain a continuous expression for the rate of change with depth of this flux.

For the momentum flux due to the bodily transport of momentum we find

$$\rho\frac{\partial}{\partial x}\langle \tilde{u}^2\rangle = \frac{\rho}{2}\frac{\partial}{\partial x}(c u_S) + \frac{\partial}{\partial x}\left[\frac{1}{H}\left(\frac{c_g}{c} - \frac{1}{2}\right)E\right], \tag{33}$$

and for the momentum flux due to the second-order pressure p' we find

$$\rho\frac{\partial}{\partial x}\langle \tilde{w}^2\rangle = \frac{\rho}{2}\frac{\partial}{\partial x}(c u_S) - \frac{\partial}{\partial x}\left[\frac{1}{H}\left(\frac{c_g}{c} - \frac{1}{2}\right)E\right]. \tag{34}$$

We see from (33) and (34) that the difference $\rho\langle \tilde{u}^2 - \tilde{w}^2\rangle$ is independent of Z, as mentioned previously.

We have now found explicit expressions for all forcing terms in the mean momentum equation (25), specific to the case of nondissipative, nonforced waves. Using (30)–(34), we obtain from (25):

$$\rho\frac{\partial \overline{u}}{\partial t} = -\frac{\rho}{2}\frac{\partial}{\partial x}(c u_S) - 2\frac{\partial}{\partial x}\left[\frac{1}{H}\left(\frac{c_g}{c} - \frac{1}{2}\right)E\right] + \frac{1}{2k}\left\{\rho c u_S + \left[\frac{2}{H}\left(\frac{c_g}{c} - \frac{1}{2}\right)E\right]\right\}\frac{\partial k}{\partial x} - \rho g\frac{\partial h}{\partial x}. \tag{35}$$

Using the definition of the mean horizontal velocity $\overline{u}$, we can write the integrated continuity equation in (21) and (22) as

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left\langle \int_{-d}^{\zeta} u\, dz\right\rangle = -\frac{\partial}{\partial x}\int_{-d}^{h} \overline{u}(Z)\, dZ \equiv -\frac{\partial \overline{U}}{\partial x}, \tag{36a}$$

$$\frac{\partial Z}{\partial t} = -\frac{\partial}{\partial x}\left\langle \int_{-d}^{\eta} u\, dz\right\rangle = -\frac{\partial}{\partial x}\int_{-d}^{Z} \overline{u}(Z')\, dZ'. \tag{36b}$$

Here $\overline{U}$ is the vertically integrated volume transport. Equations (36a, b) connect the mean surface elevation h and the mean position of the material surface Z to the mean drift velocity $\overline{u}$. Together with (35) they form a closed set of equations for the mean surface elevation, the mean horizontal momentum, and the mass transport. To solve Eqs. (35) and (36) we need initial conditions and one boundary condition in x for $\overline{u}$, h, and Z. Since we have neglected atmospheric forcing and the frictional boundary layers at the surface and the bottom, we do not need any boundary conditions in the vertical direction.

3. Discussion

It is clear from (35) that the horizontal fluxes of momentum (not to be confused with the gradients of these fluxes) are either (i) independent of depth or (ii) vertically distributed in the same way as the Stokes drift velocity (26). This fact does not imply that all depth-dependent forcing terms in (35) are directly proportional to the Stokes drift. Since Z in general is a function of x we will find terms proportional to

$$\frac{\partial u_S}{\partial Z}\frac{dZ}{dx}$$

on the rhs of (35), which obviously do not have the same Z dependence as u_S. The definition of Z in (23) is formally equal to the first few terms in an asymptotic series for the vertical position of a fluid particle, with the periodic wave component averaged out and with z_0 as the Lagrangian vertical coordinate (Pierson 1962). A similar definition could be made for the horizontal position of a fluid particle say, X, and because the material surface consists of the same particles, X will increase along lines of constant Z. Since the deviation in Z from the initial value z_0 is assumed small we could, to the present order of approximation, replace $\partial/\partial x$ with $\partial/\partial X$ in the mean momentum equation. In such a frame of reference $dZ/dX = 0$, and the depth-dependent forcing terms become proportional to the Stokes drift at all times, as expected if we use a direct Lagrangian approach (Weber et al. 2007). Since we have neglected dissipative forces, there are no mechanisms that can redistribute the momentum from the waves (and hence the Stokes drift) to a mean drift current with a depth dependence other than the waves and the mean wave momentum.

On the other hand, in real fluids mean momentum diffuses into the interior from the frictional boundary layers at the surface and the bottom. This diffusion of momentum creates a Couette-type flow that comes in addition to the Stokes drift (Longuet-Higgins 1953). To the present order of approximation the drift velocity can be split into different parts reflecting the underlying physical mechanisms that govern the flow. Thus, for the Lagrangian drift velocity we may write

$$\overline{u} = u_S + \overline{u}_E + \overline{u}_{\mathrm{bl}}, \tag{37}$$

where $\overline{u}_{\mathrm{bl}}$ is the solution for the steady streaming in the boundary layers (e.g., Jenkins 1986). It is necessary to apply either a Lagrangian description of the motion, or wave-following curvilinear coordinates to obtain the correct solution for the steady streaming in the frictional boundary layers (Longuet-Higgins 1953). However, we can always reconstruct the full Lagrangian drift equation using known solutions for the steady streaming. As an example, we consider waves damped in time and space because of the viscous dissipation in the boundary layers at the surface and at the bottom. The irrotational part of the wave motion has a (complex) velocity potential $\varphi = A\cosh[\kappa(z + d)]\exp(i\kappa x + nt)$; see, for example, Weber (1998). Here $A = \text{const}$; $\kappa = k + i\alpha$; $n = -i\omega - \beta$; and α and β are the spatial and temporal damping rates, respectively (e.g., Jenkins 1986). Taking the real parts of the velocities $(\tilde{u}, \tilde{w}) = -\nabla\varphi$, we obtain from (25)

$$\frac{\partial\overline{u}}{\partial t} = \frac{\partial u_S}{\partial t} - \frac{\partial}{\partial x}\left[\frac{1}{H}\left(\frac{c_g}{c} - \frac{1}{2}\right)E\right]. \tag{38}$$

If we subtract (38) from the Lagrangian drift equation, the resulting equation is [see Jenkins 1986, his Eq. (4.11) with the Coriolis terms neglected]

$$\frac{\partial\overline{u}_E}{\partial t} - \nu\frac{\partial^2\overline{u}_E}{\partial Z^2} = 0. \tag{39}$$

The above result is particular to the case of constant eddy viscosity ν; for a vertically varying eddy viscosity we would have an additional term proportional to the vertical gradient of the eddy viscosity in (39); (see Jenkins 1987). A complete solution can be obtained by simultaneously solving Eqs. (38) and (39). The wave dissipation rates and the redistribution of momentum from the waves to the mean flow can either be determined from direct calculations (e.g., Jenkins 1986) or by using the mean wave momentum equation of Weber (2001) and Weber et al. (2006). Of course, the boundary layer solution must be known beforehand to obtain the full expression for $\overline{u}$, but also in order to formulate appropriate boundary conditions. Work along these lines is in progress.

The fact that the mean momentum equation becomes much simpler in the (X, Z) coordinates implies that a Lagrangian description of motion is physically more intuitive in wave drift problems. However, one of the main goals of this study is to derive equations valid for arbitrary depth. Since the depth is a function of x only, we need to use the mean momentum equation as formulated in Eq. (35), in conjunction with the integrated continuity Eqs. (36a) and (36b), in spite of the increase in complexity compared to a purely Lagrangian model.

a. *Integrated transports and radiation-stress theory*

Longuet-Higgins and Stewart considered vertically integrated mass, momentum, and energy fluxes in a series of papers (Longuet-Higgins and Stewart 1960, 1961, 1962, 1964; see also Whitham 1962). Phillips (1977) extended the validity of their model, obtaining equations that include fourth-order terms. It should be kept in mind that the mean flow was assumed independent of depth in all the papers referred to above, an assumption that considerably simplifies the problem.

The momentum equation in integrated form appears in (10), though we will show here that the correct equation can be derived from (25) by integrating in Z. As a first step we use (9) and (12) to rewrite the fluctuating pressure at the surface:

$$\tilde{p}(z = h) = -\sigma + \rho g \tilde{\eta} + O(a^2). \tag{40}$$

We define the atmospheric form stress responsible for wave growth as

$$\tau_D = -\left\langle \sigma \frac{\partial \tilde{\eta}}{\partial x} \right\rangle. \tag{41}$$

Using (8), (28), (40), and (41) we find that

$$\begin{aligned} \left\langle \tilde{p} \frac{\partial \tilde{\zeta}}{\partial x} \right\rangle_{Z=h} &= \tau_D + \frac{1}{2} \rho g \frac{\partial}{\partial x} \langle \tilde{\eta}^2 \rangle + O(a^3), \\ &= \tau_D + \frac{1}{2} \frac{\partial E}{\partial x} + O(a^3), \end{aligned} \tag{42}$$

where the second term on the right-hand side is the integrated contribution from the divergence effect (e.g., McIntyre 1988). We now integrate (25) from $Z = -d$ to h. Using (42) we obtain correct to $O(a^2)$:

$$\begin{aligned} \rho \frac{\partial \overline{U}}{\partial t} = \tau_D &+ \frac{1}{2} \frac{\partial E}{\partial x} - \frac{\partial}{\partial x} \langle \tilde{p}(z = h) \tilde{\eta} \rangle \\ &- \rho \frac{\partial}{\partial x} \int_{-d}^{h} \langle \tilde{u}^2 - \tilde{w}^2 \rangle \, dZ - \left\langle \tilde{p} \frac{\partial \tilde{\zeta}}{\partial x} \right\rangle_{Z=-d} \\ &+ \frac{\partial}{\partial x} \langle \tilde{p} \tilde{\zeta} \rangle_{Z=-d} - \rho g H \frac{\partial h}{\partial x}, \end{aligned} \tag{43}$$

where yet again we have interchanged the order of differentiation with respect to x, Z, and t. In the absence of external forces we find that

$$\rho \frac{\partial \overline{U}}{\partial t} = -\frac{\partial}{\partial x} \left[\left(\frac{2c_g}{c} - \frac{1}{2} \right) E \right] - \rho g H \frac{\partial h}{\partial x}, \tag{44}$$

which is just the result of Longuet-Higgins and Stewart (e.g., Longuet-Higgins and Stewart 1962). If we introduce the mean horizontal particle displacement $\overline{\xi}$, defined by

$$\overline{U} = \frac{\partial \overline{\xi}}{\partial t},$$

we can use (36a) to rewrite (44) as

$$\frac{\partial^2 \overline{\xi}}{\partial t^2} - c_0^2 \frac{\partial^2 \overline{\xi}}{\partial x^2} = -\frac{\partial}{\partial x} \left[\left(\frac{2c_g}{c} - \frac{1}{2} \right) \frac{E}{\rho} \right], \tag{45}$$

where $c_0 = \sqrt{gH}$. The homogeneous solution of (45) yields the shallow water waves (infragravity waves) that accompany the changes in mean surface elevation caused by a spatially nonuniform wave field (Benjamin 1970).

b. *Numerical modeling of wave drift*

Most numerical models use an Eulerian description of motion and, as mentioned in the introduction, the Stokes drift is a Lagrangian quantity. An expression for the Eulerian part of the drift velocity can be obtained by subtracting the horizontal wave velocity $\tilde{u}$ from the total velocity u in the governing equations. Using the definition of the Stokes drift velocity (26), the mean Eulerian drift velocity then becomes

$$\overline{u}_E = \frac{\partial}{\partial Z} \left\langle \int_{-d}^{\zeta} (u - \tilde{u}) \, dz \right\rangle = \overline{u} - u_S. \tag{46}$$

Substitution from (46) into the drift equation (25) yields an equation for the Eulerian part of the drift current $\overline{u}_E$ where the time derivative of the Stokes drift now appears as a forcing term (Jenkins 1989). If the drift equations are solved numerically, the particle drift is obtained by adding the Stokes drift u_S to the solution.

4. Summary and concluding remarks

In this paper we present a model for the wave-induced momentum fluxes and mass transport, and the vertical distribution of these quantities, in surface gravity waves for an ocean of arbitrary depth. The model is two-dimensional but an extension to three dimensions is straightforward. We use a Lagrangian description in the vertical direction in the sense that the mean vertical position of material surfaces are used as the vertical coordinate. As discussed previously, using a Lagrangian description of the motion in wave drift problems leads to simpler and physically more intuitive results. When the bottom topography varies, however, both the wave properties and the wave-induced forcing terms have a direct dependence on the Eulerian x coordinate. To model the effects of varying depth we therefore need to evaluate the horizontal Eulerian gradients in the governing equations.

In some respects our model is similar to that of Mellor (2003), who uses a nonorthogonal curvilinear coordinate system; that is, only the vertical coordinate is transformed to fit the waves. In orthogonal curvilinear coordinates many transformation terms in the governing equations cancel each other because of the inherent symmetry of orthogonal systems (e.g., Benjamin 1959). This is not the case if nonorthogonal curvilinear coordinates are used, and hence the resulting equations become algebraically complicated. Here we circumvent this problem by following material surfaces and using the kinematic condition (3). This approach gives us two advantages: First, when integrating the governing equations we can move temporal and spatial derivatives outside the integrals. As in Longuet-Higgins and Stewart (1964), each forcing term can easily be traced back to a specific physical mechanism, representing a divergence or convergence in the momentum flux due to either the pressure or bodily transport of momentum. Using the momentum Eq. (25), the depth dependence of each of these forcing terms can be investigated in detail. Second, the vertical coordinate Z can vary in space or time, allowing us to consistently model the mean drift and mass transport in spatially and temporally nonuniform surface waves.

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