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Wavetrains in inhomogeneous moving media

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When a slowly varying wavetrain of small amplitude propagates in a general medium, changes of frequency and wavenumber are determined along definite paths known as rays. It is shown that, for a wide class of conservative systems in fluid dynamics changes in amplitude along the rays may be computed from conservation of wave action, which is defined as the wave energy divided by the intrinsic frequency. The intrinsic frequency is the frequency which would be measured by an observer moving with the local mean velocity of the medium. This result is the analogue for continuous systems of the adiabatic invariant for a classical simple harmonic oscillator.

If the medium is time dependent or moving with a nonuniform mean velocity the intrinsic frequency is not normally constant, and wave energy is not conserved. Special cases include surface waves on a vertically uniform current in water of finite depth, internal gravity waves in a shear flow at large Richardson number, Alfvèn waves, sound waves, and inertial waves in a homogeneous rotating liquid in geostrophic mean motion.

1. Introduction

A *wavetrain* is a system of almost sinusoidal propagating waves with a recognizable dominant local frequency ω, vector wavenumber $\mathbf{k}$ and amplitude a. These may vary with position $\mathbf{x}$ and time t, but only slowly, in the sense that appreciable changes are apparent only over many periods and wavelengths. They are defined with a precision which increases the more slowly they vary. The dominant frequency and wavenumber may be derived from a *phase function* $\theta(\mathbf{x}, t)$ by

$$\omega = -\theta_t, \quad k_j = \theta_{x_j} \quad (j = 1, \ldots, m) \tag{1·1}$$

and the wave crests are surfaces of constant θ. At each point ω, $\mathbf{k}$ are connected by a *dispersion relation*

$$\omega = \Omega(\mathbf{k}, \lambda), \tag{1·2}$$

where the local properties of the medium are for convenience summarized in the parameter $\lambda(\mathbf{x}, t)$ (which may have several components), and are also assumed to be slowly varying. We will confine our attention to linearized waves of small amplitude for which ω, $\mathbf{k}$ are real (i.e. the medium is stable and non-dissipative).

The *group velocity* $\mathbf{c}$ is defined by

$$c_j = \frac{\partial \Omega}{\partial k_j} \quad (j = 1, \ldots, m) \tag{1·3}$$

and differentiation moving with the group velocity by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + c_j \frac{\partial}{\partial x_j}. \tag{1·4}$$

An observer always moving with the local value of $\mathbf{c}$ describes a path in space-time known as a *ray*.

In a uniform time-independent medium (λ = constant)

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}k_j}{\mathrm{d}t} = 0, \tag{1·5}$$

i.e. the frequency and wavenumber are constant along a ray. Then from equations (1·2), (1·3) $\mathbf{c}$ is also constant, and the ray is straight.

In a nonuniform medium

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial\Omega}{\partial\lambda}\frac{\partial\lambda}{\partial t}, \quad \frac{\mathrm{d}k_j}{\mathrm{d}t} = -\frac{\partial\Omega}{\partial\lambda}\frac{\partial\lambda}{\partial x_j}, \tag{1·6}$$

so that, if $\lambda(\mathbf{x}, t)$ is known, the frequency, wavenumber and group velocity vary in a predictable manner along a ray. The path of a ray through any given point is determined by integration of equations (1·6), together with

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = c_j = \frac{\partial\Omega}{\partial k_j} \tag{1·7}$$

and is in general curved. Equations (1·6), (1·7) are kinematic results, depending only on the existence of a phase function $\theta(\mathbf{x}, t)$ and a local dispersion relation (1·2). They are related to the theory of the Hamilton–Jacobi equation. Their derivation in a fluid dynamical context has been ably explained by Whitham (1960). A general survey of the concept of group velocity has been given by Lighthill (1965).

This paper is concerned with a general procedure for finding changes in the amplitude $a(\mathbf{x}, t)$ in a slowly varying moving or time dependent medium. Unlike the situation in a uniform medium, the maximum displacement of a material particle (for example) will in general vary in a different manner from the maximum velocity of the same particle, because the frequency will change. Either of these would be a suitable measure of wave amplitude, so general formulae directly in terms of a do not exist. However a convenient concept is that of *wave energy density* E. This is discussed in detail in § 3, but in any given problem it is normally straightforward to express it in terms of the local wavenumber, frequency and any convenient measure a of amplitude,

$$E = a^2 F(\omega, \mathbf{k}, \lambda). \tag{1·8}$$

The central result of this paper is that for a wide class of physical systems

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E}{\omega'}\right) + (\nabla \cdot \mathbf{c})\left(\frac{E}{\omega'}\right) = 0, \tag{1·9}$$

where ω' is the frequency relative to a frame of reference in which the mean state of the medium is locally in equilibrium at rest. This result was suggested but not proved by Garrett (1967). The frequency ω, which enters equations (1·2), (1·6) to determine the ray paths, is relative to a fixed observer. It is equal to the *intrinsic frequency* ω', which enters equation (1·9), plus an allowance for the Doppler shift. If the medium at the point under consideration is moving with velocity $\mathbf{U}$ relative to the observer,

$$\omega' = \omega - \mathbf{U} \cdot \mathbf{k}. \tag{1·10}$$

Equation (1·9) describes changes in wave energy density along a ray, in terms of ω' and the spatial divergence $\nabla \cdot \mathbf{c}$ of the rays. Changes in amplitude a then follow from equation (1·8). Ray divergence occurs even in a uniform time-independent

medium, for which $\omega' =$ constant, because the group velocities at neighbouring points in a wavetrain are not in general exactly the same, owing to slight differences in $\mathbf{k}$. Because E is an energy *density*, it is not constant down a ray, even if wave energy is conserved. However, in a time dependent and/or nonuniformly moving medium, ω' varies along a ray. If E/ω' is the *wave action density*, total wave action is conserved, whereas total wave energy is not.

The meaning of this is perhaps made clearer by reference to the concept of a *wave packet*. This is a wavetrain of which the amplitude is negligibly small, except within a certain moving region of space V. The dimensions of V are small compared to the scale of variation of the properties of the medium, but large compared to a wavelength. The dominant wavenumber is effectively uniform over the packet, so that it moves as a whole with a well-defined group velocity. Owing to the divergence of the rays, the volume occupied by V may change. However, viewed from the scale of the medium, a wave packet appears as a point associated with a definite position $\mathbf{x}(t)$ and a definite wavenumber $\mathbf{k}(t)$ which moves along a path predetermined by the variation of λ with $\mathbf{x}$ and t, and by the values of $\mathbf{x}$, $\mathbf{k}$ at one time t_0. In a uniform medium the path of every packet is a straight line, and the total wave energy associated with it is constant. In a slowly varying medium the path is in general curved, and the total wave energy is proportional to the instantaneous intrinsic frequency. This result is the analogue for a continuous system of the classical adiabatic invariant for a single discrete oscillator which is subject to a slow change in its defining characteristics (e.g. a pendulum consisting of a bob on a string of varying length). In the latter case the total energy divided by the frequency is constant (Einstein 1911).

The theory here is an approximate one analogous to the W.K.B. approximation and the relation of geometrical optics to electrodynamics. Such theories never describe partial reflexions at abrupt changes in the medium. Their mathematical basis is an asymptotic expansion in powers of a small parameter ϵ, which is the ratio of a typical wavelength to the length scale of the changes. Formally, the fractional changes in ω, $\mathbf{k}$, a over a period and over a wavelength are all proportional to ϵ. The above equations are for the lowest order terms. Although in practice some partial reflexion always occurs, it is exponentially small (e.g. $\exp\{-\epsilon^{-1}\}$) and does not emerge even from higher order corrections.

To establish our result we draw freely on ideas due to Whitham (1965). In a fundamental paper he suggested that, if the equations governing a dynamical system can be derived from a variational principle of a certain type, changes of amplitude in a slowly varying wavetrain (linear or nonlinear) are governed by a conservation equation. The conserved quantity, called by Whitham the adiabatic invariant, is obtained from a local average over a period of the integrand of the governing variational principal. We are concerned here mainly with its physical interpretation for waves which are small perturbations about a mean state, particularly when the latter is in motion. In a section on linear systems, Whitham came close to a statement of equation (1·9) with ω' replaced by ω. However, for linearized, as opposed to linear, systems great care is necessary over the definition of E, otherwise fallacious results may be obtained. This is discussed in § 3.

Two further difficulties must also be overcome before equation (1·9) may be established for general systems. Whitham gave no general method whereby a governing variational principle of the required type may be written down. In the example he discussed, long waves on water of finite depth, the principle was in terms of an Eulerian description of the instantaneous motion, and although it is correct, its genesis is not obvious. For our present purposes, Hamilton's principle may often be used, to obtain the required starting point. How this may be done for a wide class of fluid dynamical problems is discussed in § 4.

A second difficulty is that Whitham (1965) gave no justification for his averaged variational principle, other than by showing that it gave the correct averaged equations in one special case. This gap has been partly filled by Luke (1966). A complete treatment for linear systems will be given in the following paper (Bretherton 1968). In many problems the fields describing the wave motion are approximately sinusoidal only in some (longitudinal) directions. In others (the lateral directions) they may have a complex normal mode structure. It is necessary that the averaged quantities in the variational principle be integrated over the lateral coordinates, including possibly contributions from the lateral boundary conditions. For example, in waves on water of a finite depth, height is a lateral coordinate, the wave energy is the total average energy per unit horizontal area, including the integrated kinetic energy (which is distributed over depth) and the net potential energy (which is intimately connected with the free surface).

Before discussing these general arguments we shall first illustrate the concepts and methods involved by a simple, rather trivial, example, and then in the remainder of the paper show how these may be applied to a wider class of problems.

2. The stretched string

2·1. *Hamilton's principle*

We consider an infinitely long string of mass $\rho(x)$ per unit length, under tension $T(t)$. If $\eta(x,t)$ is the transverse displacement of each point of the string from its equilibrium position, the linearized equation of motion is

$$\rho\eta_{tt} - T\eta_{xx} = 0. \tag{2·1}$$

It is important that the mean state $\eta = 0$ is a possible solution of the dynamical equations of motion, even when $\rho(x)$, $T(t)$ are not constant. This is so if the modulus of elasticity of the string (which controls the speed of propagation of longitudinal vibrations) is taken to be infinite.

Equation (2·1) may be derived from the variational principle

$$\delta \int \{\tfrac{1}{2}\rho\eta_t^2 - \tfrac{1}{2}T\eta_x^2\}\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2·2}$$

subject to all suitably differentiable infinitesimal variations $\delta\eta$ which vanish for sufficiently large $|x|$, $|t|$. Equation (2·2) is clearly Hamilton's principle for a system defined by a continuum $\eta(x)$ of generalized coordinates. $\int \tfrac{1}{2}\rho\eta_t^2\,\mathrm{d}x$ and $\int \tfrac{1}{2}T\eta_x^2\,\mathrm{d}x$ are respectively the kinetic and potential energy for small perturbations.

2·2. *Local solutions*

If T, ρ are constant, equation (2·1) has solutions periodic in a phase function $\theta(x,t)$

$$\eta = a\sin\theta \tag{2·3}$$

if

$$\omega = \pm\sqrt{(T/\rho)}\,k, \tag{2·4}$$

where

$$\omega = -\theta_t, \quad k = \theta_x. \tag{2·5}$$

If $T(t)$, $\rho(x)$ are slowly varying equations (2·3) to (2·5) are still valid locally, the problem is to connect values of a, ω, k at widely different points (x,t).

2·3. *The averaged Lagrangian*

We define an averaged Lagrangian density $\mathscr{L}$ by substituting the elementary solution (2·3) into the Lagrangian density in equation (2·2), remembering that

$$\frac{\partial}{\partial t} = -\omega\frac{\mathrm{d}}{\mathrm{d}\theta} \quad \text{and} \quad \frac{\partial}{\partial x} = k\frac{\mathrm{d}}{\mathrm{d}\theta}$$

and integrating with respect to θ over a period

$$\begin{aligned}\mathscr{L} &= \frac{1}{2\pi}\int_0^{2\pi}(\tfrac{1}{2}\rho\omega^2\eta_\theta^2 - \tfrac{1}{2}Tk^2\eta_\theta^2)\,\mathrm{d}\theta \\ &= \tfrac{1}{4}(\rho\omega^2 - Tk^2)a^2.\end{aligned} \tag{2·6}$$

This is a function of parameters a, ω, k and also (via ρ, T) of x, t. Whitham (1965) suggested that for a slowly varying wavetrain, the dominant local amplitude, frequency, and wavenumber are governed by the variational principle

$$\delta\int\mathscr{L}(a, -\theta_t, \theta_x)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2·7}$$

subject to infinitesimal variations $\delta a(x,t)$, $\delta\theta(x,t)$ which vanish at infinity. Variation with respect to a yields

$$\partial\mathscr{L}/\partial a = \tfrac{1}{2}a(\rho\omega^2 - Tk^2) = 0. \tag{2·8}$$

This is equivalent to the dispersion relation (2·4). Variation with respect to θ yields

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\omega}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\mathscr{L}}{\partial k}\right) = 0. \tag{2·9}$$

This is a conservation equation for the quantity $\partial\mathscr{L}/\partial\omega$, subject to the flux $-\partial\mathscr{L}/\partial k$.

For linear waves $\mathscr{L}$ is proportional to a^2, so that the dispersion relation (2·8) is equivalent to

$$\mathscr{L} = 0. \tag{2·10}$$

The group velocity is then

$$c = -\mathscr{L}_k/\mathscr{L}_\omega, \tag{2·11}$$

so that equation (2·9) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{L}_\omega) + \frac{\partial c}{\partial x}\mathscr{L}_\omega = 0. \tag{2·12}$$

Whitham justified his simplified variational principle only by showing for one particular case that it was equivalent to simplified differential equations which could be obtained by averaging in a certain way (also heuristic) the exact differential equations obtained from the exact variational principle. His main interest was in nonlinear waves of finite amplitude. However, Luke (1966) has shown how wave-train solutions of the second order nonlinear Klein–Gordon equation

$$\eta_{tt} - \eta_{xx} + F(\eta) = 0$$

may be obtained as an asymptotic expansion in powers of a small parameter ϵ, the lowest order term being governed by the appropriate forms of equations (2·8), (2·9). He also partially justified this procedure for a general nonlinear second order partial differential equation derived from a variational principle, including equation (2·1) as a special case. It will be shown in the following paper (Bretherton 1968) how Luke's results may be generalized for a much wider class of systems of equations, and for linear systems a complete asymptotic expansion may be constructed.

2·4. *Wave energy*

We arrive at the concept of wave energy by considering the response of a string, which is initially in equilibrium, to arbitrary small external forces $f(x,t)$ per unit length applied transversely to it. The governing equation is

$$\rho\eta_{tt} - T\eta_{xx} = f. \tag{2·13}$$

To obtain the rate at which work is done by these forces, we multiply by the velocity η_t of the particles to which they are applied, and integrate. After a little manipulation

$$\dot{W} = \int_{x_1}^{x_2} f\eta_t \,\mathrm{d}x = \frac{\partial}{\partial t}\int_{x_1}^{x_2} \{\tfrac{1}{2}\rho\eta_t^2 + \tfrac{1}{2}T\eta_x^2\}\,\mathrm{d}x + [-T\eta_t\eta_x]_1^2 - \int_{x_1}^{x_2} \frac{1}{2}\frac{\partial T}{\partial t}\eta_x^2 \,\mathrm{d}x. \tag{2·14}$$

If T is constant, this is a conservation equation, changes in

$$\int_{x_1}^{x_2} \{\tfrac{1}{2}\rho\eta_t^2 + \tfrac{1}{2}T\eta_x^2\}\,\mathrm{d}x$$

being associated only with work done by external forces or with a flux $-T\eta_t\eta_x$ across the points $x = x_1, x_2$. Thus we unhesitatingly identify $\frac{1}{2}\rho\eta_t^2 + \frac{1}{2}T\eta_x^2$ with a perturbation energy per unit length, for any perturbation $\eta(x,t)$ however produced. If η is sinusoidal, according to equation (2·3), the average of the perturbation energy over a period is the wave energy density

$$E = \tfrac{1}{4}a^2(\rho\omega^2 + Tk^2) \tag{2·15}$$

and the wave energy flux is

$$\tfrac{1}{2}Ta^2\omega k. \tag{2·16}$$

It is easy to see that the latter is equal to the group velocity $c(= \pm\sqrt{(T/\rho)})$ times the wave energy density.

If $T(t)$ is not constant, however, the third term on the right-hand side of equation (2·14) is nonzero, and no amount of manipulation will turn the equation into a conservative form. $\frac{1}{2}(\partial T/\partial t)\eta_x^2$ thus describes the interaction between the waves and

the mean state. If $T(t)$ is slowly varying, it is still meaningful to use the local perturbation energy given by equation (2·15). It is still approximately conserved for times of a few wave periods, but over longer times the interaction terms cannot be ignored.

2·5. *The interpretation of* $\mathscr{L}_\omega$

Equation (2·12) does describe conservation of something, but it is not wave energy. To see this, we set up equation (2·14) again, but in terms of the Lagrangian density L.

The string may be regarded as a classical dynamical system; the position of every material particle in it being specified by the instantaneous values of a set of generalized coordinates $\{q_i\}$. In this case the value of η at each point is a coordinate q_i, and the set $\{q_i\} = \eta(x)$ forms a continuum. The set of possible configurations is limited by the requirement that $\eta(x)$ be differentiable. The present problem is typical of many in continuum mechanics in that the kinetic and potential energies of the complete string may be written as integrals of explicit expressions in η and its first derivatives only. Their difference is the Lagrangian

$$L_s(q_i, \dot{q}_i, t) = \int L(\eta, \eta_t, \eta_x; \lambda(x,t))\,\mathrm{d}x \tag{2·17}$$

$$= \int \{\tfrac{1}{2}\rho(x)\eta_t^2 - \tfrac{1}{2}T(t)\eta_x^2\}\,\mathrm{d}x. \tag{2·18}$$

If external forces $\{Q_i\} = f(x)$ (not included in the potential energy above) are applied to the system, their effect may be computed in the manner customary in the derivation of Lagrange's equations by considering an arbitrary infinitesimal virtual displacement $\{\delta q_i\} = \delta\eta(x)$ in which $\eta_t(x)$ is not varied and $\delta\eta$ vanishes for $|x|$ sufficiently large. The virtual work δW done by the external forces is then

$$\begin{aligned}\delta W &= \sum_i Q_i\,\delta q_i \\ &= \sum_i \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_s}{\partial \dot{q}_i}\right) - \frac{\partial L_s}{\partial q_i}\right\}\delta q_i \\ &= \int \left\{\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \eta_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \eta_x}\right) - \frac{\partial L}{\partial \eta}\right\}\delta\eta\,\mathrm{d}x.\end{aligned} \tag{2·19}$$

In this identification $\partial L_s/\partial \dot{q}_i$ is, of course, the partial derivative in which all the remaining $\dot{q}$'s and all the q's and t are held constant. For each i, $(\partial L_s/\partial \dot{q}_i)\,\delta\dot{q}_i$ is equivalent to the change in L_s due to a change $\delta\eta_t$ which is confined to a small unit neighbourhood of x; i.e. to $(\partial L/\partial \eta_t)\,\delta\eta_t$ (η, η_x, λ being held constant). Summation over i is replaced by integration with respect to x. The partial derivative $\partial/\partial t$ in $(\partial/\partial t)\,(\partial L/\partial \eta_t)$ implies that all the arguments η, η_t, η_x, λ of $\partial L/\partial \eta_t$ are regarded as functions of x and t, only x is held constant. $\sum_i \dfrac{\partial L_s}{\partial q_i}\delta q_i$, on the other hand, is the

change in L_s due to $\delta\eta(x)$ holding η_t and λ constant. This is

$$\delta L_s = \int \left\{ \frac{\partial L}{\partial \eta} \delta\eta + \frac{\partial L}{\partial \eta_x} \delta\eta_x \right\} \mathrm{d}x$$
$$= \int \left\{ \frac{\partial L}{\partial \eta} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \eta_x}\right) \right\} \delta\eta \, \mathrm{d}x,$$

where we have used the vanishing of $\delta\eta(x)$ at infinity.

It follows from equation (2·19) that the rate of working by any external forces on the string when it is moving transversely in any manner is

$$\dot{W} = \int \left\{ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \eta_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \eta_x}\right) - \frac{\partial L}{\partial \eta} \right\} \eta_t \, \mathrm{d}x. \tag{2·20}$$

By simple manipulations, this becomes

$$\dot{W} = \int \left\{ \frac{\partial}{\partial t}\left(\eta_t \frac{\partial L}{\partial \eta_t} - L\right) + \frac{\partial}{\partial x}\left(\eta_t \frac{\partial L}{\partial \eta_x}\right) + \frac{\partial L}{\partial \lambda} \lambda_t \right\} \mathrm{d}x. \tag{2·21}$$

This corresponds term by term to equation (2·14). In particular, in the absence of external forces and when $(\partial L/\partial\lambda)\,\lambda_t = 0$ it reduces to a conservation equation. Hence $\eta_t(\partial L/\partial\eta_t) - L$ is the perturbation energy per unit length of the string.

The purpose of this discussion is to emphasize the stages in the line of argument leading to the identification of $\eta_t(\partial L/\partial\eta_t) - L$. The basic postulate is equation (2·19), i.e. that, if

$$\frac{\delta L}{\delta \eta} = \frac{\partial L}{\partial \eta} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \eta_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \eta_x}\right)$$

is the functional derivative of L with respect to η, the virtual work done by the external forces on the system under any admissible variation $\delta\eta$ is given by

$$\delta W = -\int \frac{\delta L}{\delta \eta} \delta\eta \, \mathrm{d}x. \tag{2·22}$$

This is the property of Hamilton's principle which distinguishes it from other variational principles governing other nonphysical systems. It enables a specific interpretation to be given to functions derived from L. This interpretation does not depend on L being obtained as the difference between the kinetic energy density and the potential energy density. In systems including, for example, electromagnetic fields, the distinction between kinetic and potential energy is obscure. Equation (2·22) is basic to the derivation of an expression for the total energy density of the system. This cannot be defined (as in Whitham (1965)) simply as the quantity which is conserved when no external forces are acting and the Lagrangian does not depend explicitly on time, although such conservation is fundamental to the significance of the energy. With such a definition the Lagrangian could be multiplied everywhere by an arbitrary constant scalar without affecting the analysis at all, but the numerical values of the derived quantities would be altered. Furthermore, even when the right-hand side of equation (2·22) cannot be equated to the virtual work (as in systems linearized about a moving mean state, see § 3) a 'pseudo energy' may still formally be defined by $\eta_t L\eta_t - L$ and under certain circumstances it is conserved, but its interpretation must be examined very carefully.

When the perturbation has the sinusoidal form of equation (3·3), the Lagrangian density may be regarded as a function of the variable θ, with parameters ω, k, a, λ;

$$L = L(\eta, -\omega\eta_\theta, k\eta_\theta; \lambda), \quad \text{where} \quad \eta = a\sin\theta. \tag{2·23}$$

The frequency ω enters only via $-\omega\eta_\theta$ in lieu of the variable η_t so the average perturbation energy over a period is

$$E = \frac{1}{2\pi}\int_0^{2\pi}\left\{(-\omega\eta_\theta)\frac{\partial L}{\partial(-\omega\eta_\theta)} - L\right\}\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\omega\frac{\partial L}{\partial\omega} - L\right)\mathrm{d}\theta \tag{2·24}$$

$$= \omega\mathscr{L}_\omega - \mathscr{L}. \tag{2·25}$$

But since for waves of small amplitude $\mathscr{L} = 0$, (equation (2·10)), we have finally

$$\mathscr{L}_\omega = E/\omega. \tag{2·26}$$

Because the string is basically at rest, the observed frequency ω and the intrinsic frequency ω' are in this case equal.

3. Perturbation energy and wave energy

3·1. *General dynamical systems*

The dynamical system of § 2 was described by a single field $\eta(x,t)$, the transverse displacement of a material particle in the string. For more general systems more fields are required, and they will be functions of three space variables $\mathbf{x} = (x,y,z)$ and time t. Let them be the set

$$\{\Phi_\alpha + \phi_\alpha\} \quad (\alpha = 1, \ldots, N),$$

where $\Phi_\alpha(\mathbf{x},t)$ describes the basic state, and $\phi_\alpha(\mathbf{x},t)$ is a perturbation. The following discussion was conceived in the context of a class of problems in classical inviscid fluid dynamics, illustrated by the examples of § 6, for which the required fields are the components of the displacement $\Xi + \xi$ of fluid particles, together with the pressure $p + \pi$ which enters as a Lagrange multiplier associated with incompressibility. The treatment envisages a Lagrangian rather than an Eulerian specification of the system, i.e. the position of every material particle is known when the values of the fields are given. It may be extended to include the effects of a 'frozen in' magnetic field as in nondissipative magnetohydrodynamics. It will be assumed that Hamilton's principle can be written down exactly, in terms of the total fields and their first derivatives, and that from this an approximate version may be obtained. In the latter the basic state $\Phi_\alpha(x,t)$ is regarded as known and for convenience is formally subsumed into the composite parameter λ, but by considering variations $\delta\phi_\alpha(\mathbf{x},t)$ linearized equations for $\phi_\alpha(\mathbf{x},t)$ may be derived. Before variation the approximate Lagrangian density $L(\phi_\alpha, \phi_{\alpha\mathbf{x}}, \phi_{\alpha t}; \lambda)$ is homogeneous and quadratic in the ϕ_α's. Techniques for obtaining such an approximation to Hamilton's principle are described in § 4. The ideas of § 3 may be applicable in a wider context (e.g. relativistic situations) but some statements will require modification.

The objective of the remainder of § 3 is to define closely the concept of wave energy. For such a definition to be satisfactory, it must be widely applicable, and yield demonstrably unique results when applied to different formulations of the same problem. For example, an arbitrary divergence can be added to the Lagrangian density L in § 2. This must not affect the numerical value of E. Or in general fluid dynamical systems the linearized equations may be derived from an Eulerian, rather than a Lagrangian specification of the motion. The wave energy must be calculable and the same in either case. Finally, it must not involve consideration of second order corrections to linearized theory. These requirements are stringent, but in certain circumstances the authors believe they can be met, in a way which in fact corresponds to normal practice.

3·2. *In a medium in equilibrium at rest*

If the state of a system can meaningfully be described as a perturbation about some basic state, it is possible to define the perturbation energy as the difference between the total energy of the system and that of the basic state. This definition is only useful, however, if the basic state is prescribed with adequate precision. As the perturbation energy is a quadratic function of the amplitude a, an adequate description of the relation between the perturbed state and the basic state normally requires consideration of equations correct to second order in a.

There is one set of circumstances in which this is not necessary, and an adequate computation may be made from first order (linearized) theory alone. This is when the basic state is in approximate equilibrium at rest; i.e. the rates of change $\Phi_{\alpha t}$ (including particle velocities), and also the external generalized forces necessary to maintain them, are at most of order a^2. If now external forces $\mathbf{f}$ of order a are applied to particles in the system, the particles will move with velocity $\dot{\boldsymbol{\xi}}$ also of order a. The work done in setting up a given perturbation is correctly given (to order a^2) as the integral of $\mathbf{f}.\dot{\boldsymbol{\xi}}$ over space and time.

We shall restrict our attention to perturbations which can be excited from equilibrium by mechanical body or surface forces only (i.e. by generalized forces corresponding to displacements $\delta\boldsymbol{\xi}$). The generalized forces associated with other fields are assumed to vanish. For fluid dynamical problems this is not an unreasonable limitation (for an incompressible liquid the generalized force corresponding to the pressure perturbation $\delta\pi$ is the dilatation) and the set of permissible perturbations is very wide, if not completely general. For a frozen-in magnetic field, however, the flux linked with any material circuit is restricted to be equal to the equilibrium value.

Then, just as in § 2·5, the rate of working is

$$\begin{aligned} \dot{W} &= \int \mathbf{f}.\,\dot{\boldsymbol{\xi}}\,\mathrm{d}x \\ &= -\int \frac{\delta L}{\delta \phi_\alpha}\phi_{\alpha t}\,\mathrm{d}x, \end{aligned} \tag{3·1}$$

then

$$\dot{W} = \iint \left\{ \frac{\partial}{\partial t}\left(\phi_{\alpha t}\frac{\partial L}{\partial \phi_{\alpha t}} - L\right) + \frac{\partial}{\partial x_j}\left(\phi_{\alpha t}\frac{\partial L}{\partial \phi_{\alpha x_j}}\right) - L_\lambda \lambda_t \right\} \mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{3·2}$$

In a uniform medium in equilibrium at rest

$$L_\lambda \lambda_t = 0. \tag{3·3}$$

The quantity

$$\phi_{\alpha t}(\partial L/\partial \phi_{\alpha t}) - L, \tag{3·4}$$

where summation over α is understood, may be interpreted as a perturbation energy per unit volume. For a strictly sinusoidal wavetrain in a normal mode

$$\phi_\alpha = \mathscr{R}\{a\hat{\phi}_\alpha \mathrm{e}^{\mathrm{i}\theta}\}, \tag{3·5}$$

where $\hat{\phi}_\alpha(\mathbf{k};\lambda)$ are constants which depend on the mode under consideration. Then the Lagrangian density L may be expressed as a function of the single variable θ and of the parameters ω, $\mathbf{k}$, λ. Derivatives $\partial/\partial t$ are replaced by $-\omega\,\partial/\partial\theta$ and $\partial/\partial x_j$ by $k_j\,\partial/\partial\theta$. We *define* the wave energy density as the average of (3·4) over a period,

$$E = \frac{1}{2\pi}\int_0^{2\pi}\left\{-\omega\phi_{\alpha\theta}\frac{\partial L}{\partial(-\omega\phi_{\alpha\theta})} - L\right\}\mathrm{d}\theta \tag{3·6}$$

$$= \omega\mathscr{L}_\omega - \mathscr{L}. \tag{3·7}$$

This clearly has the form

$$E = a^2F(\omega, \mathbf{k};\lambda). \tag{3·8}$$

For a slowly varying wavetrain, the parameters a, $\mathbf{k}$, ω all vary slightly (by order ϵ) over a period, but if ϵ is sufficiently small the averaging procedure (3·6) may be applied with arbitrary accuracy (Bretherton 1968), and equation (3·8) applied locally at each point in the wavetrain.

Several points about this argument should be noted. The perturbation energy density (3·4) is not a well defined quantity, because the distribution between terms in equation (3·2) is not unique. If $\partial(\frac{1}{2}\eta^2)/\partial x$ is added to the Lagrangian density L the equations derived from Hamilton's principle are unaffected, but $\eta\eta_x$ must be subtracted from the perturbation energy density, and $\eta\eta_t$ added to the energy flux.

However, the wave energy obtained by averaging over a period is unique, provided that $L_\lambda\lambda_t = 0$. For then perturbation energy is conserved, and if

$$E = a^2F(\omega, \mathbf{k};\lambda); \quad E' = a^2F'(\omega, \mathbf{k};\lambda) \tag{3·9}$$

are two different expressions for the perturbation energy density of the same system, but obtained by different methods (e.g. with a different Lagrangian L or with an Eulerian representation), we may consider the work W which must be done by external forces to set up from rest a wave packet of slowly varying amplitude $a(\mathbf{x},t)$, but with effectively constant ω, $\mathbf{k}$, λ over the volume for which $a \neq 0$. This work is the same for either method of computation, for it depends only on products of first order quantities, and cannot involve second order corrections. Then

$$W = F(\omega, \mathbf{k}, \lambda)\int a^2\,\mathrm{d}\mathbf{x}$$

$$= F'(\omega, \mathbf{k}, \lambda)\int a^2\,\mathrm{d}\mathbf{x},$$

where the integration is over the complete volume V occupied by the packet. Hence

$$F = F'. \tag{3·10}$$

Conservation of perturbation energy is essential to this argument. If interaction terms are admitted in equation (3·2), then, for example, $\partial(\frac{1}{2}\eta^2 t)/\partial t$ may be added to L, implying subtraction of $\frac{1}{2}\eta^2$ from the perturbation energy, and the addition of $\eta\eta_t$ to the terms $L_\lambda \lambda_t$. The average value of $\frac{1}{2}\eta^2$ does not vanish, so the value of E would be affected. However, if the basic state is in equilibrium at rest, it is possible to group terms in equation (3·2) in such a way that the interaction terms are identically zero. If the computation for W is reworked in terms of different fields, presumably a similar conservation equation may also be found, and an expression given for the perturbation energy. Having found it, the wave energy for an approximately sinusoidal wavetrain may be computed in the form (3·9) by averaging, and equation (3·10) shows that the result will be independent of the actual linearized fields used. This remark is very important in applications of equation (1·9), because the wave energy is frequently most easily computed from an Eulerian specification of the motion (cf. § 4·2), whereas equation (3·7), which is vital for the proof of equation (1·9), can only be derived by the methods used here in terms of a Lagrangian specification.

The passage from equation (3·6) to (3·7), which is a generalization of equation (2·24), appears to rely on the structure constants $\hat{\phi}_\alpha$ in equation (3·5) being independent of ω, so that the only dependence of L on ω in equation (3·6) is multiplied by $\phi_{\alpha\theta}$ in precisely those places where η_t occurs in (3·4). In each normal mode, the dispersion relation $\omega = \Omega(\mathbf{k}; \lambda)$ may always be used formally to eliminate any such dependence $\hat{\phi}_\alpha(\omega)$ and, for given $\mathbf{k}$, equation (3·5) can be treated as a *definition* of $\phi_\alpha(\theta; \mathbf{k}, \omega, a; \lambda)$ for values of $\omega \neq \Omega$. This device enables $\mathscr{L}$ to be defined even in the physically uninteresting situations when the wavenumber and frequencies are assigned values incompatible with the existence of a solution to the governing equations. Such definition is necessary before meaning can be attached to the partial derivatives $\mathscr{L}_\omega$, $\mathscr{L}_{k_i}$ although of course these are only used when $\omega = \Omega$. This device is simple but arbitrary. Whitham (1965) allowed the wave period to be flexible in order to achieve the same object. The difference in approach is immaterial, because in fact the values of $\mathscr{L}_\omega$ and $\mathscr{L}_{k_i}$ when $\omega = \Omega(\mathbf{k})$ are quite insensitive to the definition of $\phi_\alpha(\theta, \mathbf{k}, \omega)$ off the surface $\omega = \Omega$, provided only that it is differentiable and correct on the surface (Bretherton 1968; Lighthill 1965). The averaged variational principle may be justified in either case.

If, as in §§ 4·1, 4·2, the fields $\{\phi_\alpha(x, y, z, t)\}$ are sinusoidal only in (x, y, t) but depend also on a lateral coordinate z, the averaged Lagrangian $\mathscr{L}$ is obtained by integrating over z as well as over a wavelength, using the appropriate lateral structure for a normal mode, and including contributions from the lateral boundary conditions. The wave energy density E is then the average perturbation energy per unit horizontal area, integrated with respect to depth, but the formula (3·7) still applies.

The wave energy flux may similarly be defined as the average of $\phi_{\alpha t}\partial L/\partial\phi_{\alpha x_j}$ and is given by the expressions

$$\begin{aligned} G_j &= a^2 H_j(\omega, \mathbf{k}, \lambda) \\ &= -\omega \mathscr{L}_{k_j}. \end{aligned} \tag{3·11}$$

The uniqueness of $G_j(\omega, \mathbf{k}; \lambda)$ follows from consideration of a wavetrain which

occupies a volume V which is small compared to the scale of variation of ω, $\mathbf{k}$, λ but large compared to a wavelength. For this wavetrain, unlike a wave packet, the amplitude a does not vanish on the boundary S of V. The rate of working by external forces, and the rate of change of wave energy within V are uniquely defined, so from equations (3·2), (3·3),

$$\int_S a^2(\mathbf{x}, t)\,(H_j - H'_j)\,\mathrm{d}S_j = 0 \tag{3·12}$$

$H_j - H'_j$ may be taken outside the integral sign, and $\partial a^2/\partial x_j$ is arbitrary, so

$$H_j(\omega, \mathbf{k}; \lambda) - H'_j(\omega, \mathbf{k}; \lambda) = 0. \tag{3·13}$$

For this uniqueness it is essential that the energy flux be in the form (3·11). For Rossby waves, Longuet-Higgins (1964) found that the average $\overline{\pi\dot{\xi}}$ of the perturbation energy flux for a strictly sinusoidal wavetrain is not equal to the group velocity times the wave energy density. Indeed it is in a different direction. However, for a slowly varying wavetrain an additional term of comparable magnitude appears in the expression for $\overline{\pi\dot{\xi}}$ proportional to the gradient of a^2. The difference between this revised expression and $\mathbf{c}E$ is a non-divergent vector, so there are two genuinely alternative forms for the energy flux. However, only the latter does not involve derivatives of a.

Finally, it should be remarked that computations of 'wave energy' as the average value of the term proportional to a^2 in an expansion of the total energy of the system in powers of a, naïvely ignoring second order corrections to linearized theory, can yield different answers if applied in slightly different ways. For example, given a certain field with basic value Φ,

$$\tfrac{1}{2}\overline{(\Phi+\phi)^2} = \tfrac{1}{2}\Phi^2 + \Phi\overline{\phi} + \tfrac{1}{2}\overline{\phi^2}.$$

If the perturbation ϕ is taken to be sinusoidal, the contribution to the 'wave energy' would be

$$E = \tfrac{1}{2}\overline{\phi^2}. \tag{3·14}$$

But suppose $\Phi + \phi = (\Psi + \psi)^2$

so that $\phi = 2\Psi\psi$

to first order in a, then

$$\tfrac{1}{2}\overline{(\Psi+\psi)^4} = \tfrac{1}{2}\Psi^2 + 2\Psi^3\overline{\psi} + 3\Psi^2\overline{\psi^2} + O(a^3).$$

If now ψ is assumed to be sinusoidal, the 'wave energy' is

$$E' = 3\Psi^2\overline{\psi^2} = \tfrac{3}{2}E. \tag{3·15}$$

In practice, there would often be no reason other than prejudice for preferring to develop the linearized theory in terms of ϕ rather than ψ. The quantities $\overline{\phi^2}$ and $\overline{(2\Psi\psi)^2}$ which are obtained from products of first order quantities are, of course, strictly comparable, at least to $O(a^2)$. The paradox depends on the $O(a^2)$ difference between the assumed mean states in the two cases.

The definition (3·6) of wave energy is restricted to a basic state in approximate equilibrium at rest. We are interested in basic states which are moving.

3·3. *Difficulties in a general moving basic state*

In § 4, we will show how to obtain an approximate form of Hamilton's principle which describes linearized perturbations about any changing basic state (uniform or nonuniform) which is an exact solution of the equations of motion (i.e. $\phi_\alpha = 0$ is a dynamically consistent state). However it is not necessary that this standard solution be exact; an error of order a^2 in it would not affect the total Lagrangian in equation (4·16) to order a^2, because such an error may be regarded as a variation about an exact solution. The mean state of a system with waves on it is not normally known more accurately than this. Thus correctly linearized equations of motion may be derived from equation (4·16), even though the basic mean state is not precisely specified. Difficulties arise, however, with the energy.

These difficulties may be illustrated by the example of § 2. If the mean state of the stretched string could only be maintained by application of an external transverse force F per unit length, or if in the mean state the particles have a transverse velocity V, the total rate of working by the external forces is not given by equation (2·14) but is

$$\int_{x_1}^{x_2} (F+f)\,(V+\eta_t)\,\mathrm{d}x,$$

compared with

$$\int_{x_1}^{x_2} FV\,\mathrm{d}x$$

for the basic state. The additional energy supplied can be evaluated correctly to order a^2 from a theory which is linear in f and η_t only if F and V are themselves of order a^2 or smaller. Otherwise a second order theory has to be considered.

Thus although the term of order a^2 in an expansion of Hamilton's principle for the total fields yields a variational principle which gives the correct linearized equations for the perturbations, if the basic state is one of motion the right-hand side of equation (3·1) can no longer be equated to the increment of the rate of working by the external forces. Although the external force $\mathbf{F}$ necessary to maintain the basic state $\Phi_\alpha(\mathbf{x}, t)$ is negligible (because to order a^2 the basic state is a solution of the equations of motion), the particle velocities $\mathbf{U}(\mathbf{x}, t)$ are not.

A further complication arises because the perturbation velocity $\dot{\boldsymbol{\xi}}$ is not the partial derivative $\partial\boldsymbol{\xi}(\mathbf{x}, t)/\partial t$, but is rather the derivative $([\partial/\partial t] + \mathbf{U}.[\partial/\partial\mathbf{x}])\,\boldsymbol{\xi}$ following a fluid particle with the velocity of the basic state. It is still formally possible to define quantities like expression (3·4) and the right-hand side of equation (3·7), but the ideal generalization of equation (2·20) has been lost, and their significance is obscure.

3·4. *Definitions in a slowly varying mean state*

However, if the mean state is to a sufficient approximation in uniform constant motion $\mathbf{U}$ relative to a fixed observer and otherwise independent of time, it is still possible to make meaningful calculations of perturbation energy and wave energy, but only relative to a frame of reference which is itself moving with velocity $\mathbf{U}$. In this frame, the rate of working by any external forces is $\mathbf{f}.\dot{\boldsymbol{\xi}}$ per unit volume, and we may proceed exactly as in § 3·2. However, any attempt to relate the wave energy so

computed to a difference between the perturbed and mean states in a stationary frame founders because the momentum of the mean state is not known from first order theory with adequate precision.

If the mean velocity is not uniform, no frame of reference can be found in which the mean state of all parts of the system is one of rest. However, if the mean velocity $\mathbf{U}(\mathbf{x}, t)$ varies only slightly over a large number of wavelengths and periods of a wave packet centred at the point $\mathbf{x}_0, t_0$, this may be achieved *locally* in space and time over a region sufficiently large to envisage the wave packet being set up from relative rest within the region by suitable external forces $\mathbf{f}$ of magnitude of order a. The relevant properties of the basic state must also be approximately constant over the time it takes to do this, which will be a large number of wave periods. Over the region $L_\lambda \dot{\lambda}$ is small, and perturbation energy in a frame moving with velocity $\mathbf{U}(\mathbf{x}_0, t_0)$ is approximately conserved. Thus we are led to *the wave energy E being the average over a period or wavelength of the perturbation energy in a locally co-moving frame of reference.* Uniqueness follows as in § 3·2. No valid interpretation can be made of global integrals of wave energy, but locally it is well defined, asymptotically as the parameter ϵ which measures both the ratio of a wavelength to the scale of variation of the basic state, and of a wave period to the basic time scale tends to zero.

To view this analytically we note that

$$\begin{aligned}\iint \mathbf{f}.\dot{\boldsymbol{\xi}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \iint \frac{\delta L}{\delta \phi_\alpha}\{\phi_{\alpha t} + U_j \phi_{\alpha x_j}\}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &= \iint\left[\left\{\frac{\partial}{\partial t}\left(\phi_{\alpha t}\frac{\partial L}{\partial \phi_{\alpha t}} - L\right) + \frac{\partial}{\partial x_j}\left(\phi_{\alpha t}\frac{\partial L}{\partial \phi_{\alpha x_j}}\right) + \frac{\partial L}{\partial \lambda}\lambda_t\right\}\right. \\ &\quad \left. + U_j\left\{\frac{\partial}{\partial t}\left(\phi_{\alpha x_j}\frac{\partial L}{\partial \phi_{\alpha t}}\right) + \frac{\partial}{\partial x_i}\left(\phi_{\alpha x_j}\frac{\partial L}{\partial \phi_{\alpha x_i}}\right) + \frac{\partial L}{\partial \lambda}\lambda_{x_j}\right\}\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t. \quad (3{\cdot}16)\end{aligned}$$

In a medium of which the properties are independent of space and time, i.e. $\lambda_t = \lambda_{x_j} = 0$, and in which there are no lateral coordinates, the perturbation equations have sinusoidal solutions like equation (3·5). The averages over a period of

$$\phi_{\alpha t}\frac{\partial L}{\partial \phi_{\alpha t}} - L, \quad \phi_{\alpha t}\frac{\partial L}{\partial \phi_{\alpha x_j}}, \quad \phi_{\alpha x_j}\frac{\partial L}{\partial \phi_{\alpha t}}, \quad \phi_{\alpha x_j}\frac{\partial L}{\partial \phi_{\alpha x_i}},$$

are respectively $\quad \omega \mathscr{L}_\omega - \mathscr{L}, \quad -\omega \mathscr{L}_{k_j}, \quad -k_j \mathscr{L}_\omega, \quad k_j \mathscr{L}_{k_i}.$

Integrating equation (3·16) by parts to include time and space derivatives of U_j and replacing the integrand everywhere by its averaged value, we have for a slowly varying wavetrain

$$\begin{aligned}\dot{W} &= \iint \mathbf{f}.\dot{\boldsymbol{\xi}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &= \iint\left[\frac{\partial}{\partial t}\{(\omega - k_j U_j)\mathscr{L}_\omega - \mathscr{L}\} - \frac{\partial}{\partial x_i}\{(\omega - k_j U_j)\mathscr{L}_{k_i}\}\right. \\ &\quad \left. + \left\{\mathscr{L}_\lambda(\lambda_t + U_j \lambda_{x_j}) + \frac{\partial U_j}{\partial t}k_j \mathscr{L}_\omega - \frac{\partial U_j}{\partial x_i}k_i \mathscr{L}_{k_j}\right\}\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad (3{\cdot}17)\end{aligned}$$

to an approximation which is arbitrarily good as $\epsilon \to 0$. The approximation involved in replacing the integral with respect to space and time of an approximately

sinusoidal function by the integral of its average over a period is discussed in Bretherton (1968).

As $\epsilon \to 0$ the last group of three terms in equation (3·17) becomes small compared to the first two. Thus, as in § 3·2, we identify the wave energy density

$$E = \omega' \mathscr{L}_\omega - \mathscr{L}, \tag{3·18}$$

and the wave energy flux in direction x_j as

$$G_j = -\omega' \mathscr{L}_{k_j}, \tag{3·19}$$

where ω' is the intrinsic frequency

$$\omega' = \omega - U_j k_j. \tag{3·20}$$

Alternatively, these results can be obtained by considering a local application of the definition of § 3·2 and equations (3·7), (3·11) in a frame of reference moving with velocity $\mathbf{U}(x_0, t_0)$ and showing that $\mathscr{L}$, $\mathscr{L}_\omega$, $\mathscr{L}_{k_i}$ are unaltered under Galilean transformations.

According to the averaged variational principle, for linear waves the dispersion relation is

$$\mathscr{L} = 0$$

so that we have finally that the adiabatic invariant may be equated with the wave action density

$$\mathscr{L}_\omega = E/\omega'. \tag{3·21}$$

Using the dispersion relation, $c_j = -\mathscr{L}_{k_j}/\mathscr{L}_\omega$

so that the wave energy flux $-\omega' \mathscr{L}_{k_j}$ is indeed equal to the group velocity times the wave energy density. From Whitham's conservation equation

$$\frac{\partial}{\partial t} \mathscr{L}_\omega - \frac{\partial}{\partial x_j} \mathscr{L}_{k_j} = 0$$

we derive equation (1·9).

3·5. *Summary of* § 3

This definition of wave energy by reference to the apparent work done by virtual localized external forces moving with the perturbation velocity of fluid particles is a natural one in many fluid dynamical problems, and the authors believe that it corresponds to normal practice. It is important, however, to recognize the conditions which are necessary for it to be unambiguous. The first is that the external forces be mechanical ones associated with specific fluid particles of which the perturbation velocity is well defined, and no constraints on the system are violated.

Secondly, there must exist an equation expressing conservation of perturbation energy, at least over a local region (cf. discussion in § 3·2). However, equation (3·17) shows that a conservation equation without interaction terms does exist in a suitably moving frame to an arbitrarily good approximation over a local region, provided that the governing equations for the perturbation may be derived from an approximate form of Hamilton's principle, and provided that the variations in the mean state are sufficiently small over the region. Granted its existence, the wave energy may then be calculated from any convenient form of the linearized

equations, in particular from an Eulerian formulation (cf. § 4·2). The Lagrangian specification assumed here is, however, necessary to establish equation (3·18), because the perturbation velocity must be identified with $\phi_{\alpha t}+U_j\phi_{\alpha x_j}$ for the relevant values of α.

Two more conditions concern the assumption that the basic state is slowly varying. If the wavetrain is slowly varying only in horizontal directions (x, y) and in time but depends on a lateral coordinate z, it is essential that $U(x, y, t)$ be uniform in z, with error no larger than $O(\epsilon)$. Otherwise, the passage from equation (3·16) to (3·17) fails, because $\mathbf{U}$ cannot be taken outside the integral sign when $\phi_{\alpha t}(\partial L/\partial\phi_{\alpha t})-L$ etc. are integrated with respect to z as part of the averaging procedure. Thus we cannot discuss surface waves on a current which varies with depth within the range where there are significant velocity perturbations. In such a case there would be no unambiguous value of the intrinsic frequency ω'.

Finally, the statement that the basic state is slowly varying does not mean that variations of every Φ_α are small; only those fields and those derivatives are restricted which enter into the specification of the parameter λ which sums up the basic state from the point of view of the perturbation. In particular, normally only the particle velocities $\mathbf{U}$ are relevant, not their absolute displacements. However, when ϵ is small, these velocities must become the same to within an error which tends to zero with ϵ for all particles in a region which contains a number which tends to infinity of wavelengths and wave periods. This condition may be very restrictive, as for example for Rossby waves, of which the period is a decreasing function of wavelength. The only type of basic flow on a β-plane for which the length scale is very much larger than a wavelength and the time scale very much longer than a period appears to be one which is everywhere round circles of latitude. For internal gravity waves in a uniformly stratified medium the period is independent of wavelength, but depends only on the direction of the wavenumber. The condition then appears to imply that the ratio of the horizontal to the vertical scale of variation of the basic flow is much larger than the ratio of a horizontal to a vertical wavelength. Otherwise the basic flow itself must be changing over a time comparable with a wave period. For sound waves, on the other hand, the relevant condition seems to be simply that the wavelength is small compared to the spatial scale of the medium, the temporal condition is then implied. These restrictions are necessary if λ, $\partial U_j/\partial t$ and $\partial U_j/\partial x_i$ in equation (3·17) are to be negligibly small (say $O(\epsilon)$). It is possible to envisage a wave packet containing $\epsilon^{\frac{1}{2}}$ wavelengths being set up by nearly sinusoidal external forces in a time of $\epsilon^{\frac{1}{2}}$ periods. Over the volume occupied by the packet the error involved in neglecting the interaction terms in equation (3·17) is then $O(\epsilon^{\frac{1}{2}})$ smaller than the rate of change of wave energy and the energy flux separately.

4. Hamilton's principle

4·1. *Surface waves on water*

Before the arguments of the previous section can be seen to apply to any particular problem, an appropriate approximation to Hamilton's principle must be found, from which the complete linearized equations for the system may be derived without

any approximation about the basic state being slowly varying. As in equation (2·2) the integrand must be a quadratic function of the fields $\{\phi_\alpha\}$ and their first derivatives, and must be integrated with respect to space coordinates and time. Some of the fields $\{\phi_\alpha\}$ must be the displacements $\boldsymbol{\xi}$ of material particles. For the stretched string, such a principle was easily written down. A similar version is also known for small displacements in an elastic solid (Morse & Feshbach 1953, p. 322). For electromagnetic radiation in free space (Morse & Feshbach 1953, p. 327) the corresponding principle does not involve particle displacements. However, when the equations for the waves involve linearization about a non-uniform state of mean motion more sophisticated treatment is required. In this subsection we will illustrate with the problem of surface waves of small amplitude on water of depth $h(x, y, t)$ which is moving with mean velocity $\mathbf{U}(x, y, z) = (U, V, W)$. Here (x, y, z) are rectangular coordinates with z vertically upwards. This is a good prototype for problems in classical fluid dynamics, because it involves lateral boundary conditions, at the free surface and at the bottom, and also the condition that the water be incompressible. This last is a non-holonomic constraint on possible displacements.

The basic idea of this section is due to Eckart (1963). He showed how the equations of motion for small oscillations about a moving mean state for an unbounded, inviscid, adiabatic, compressible fluid in a gravitational field may be obtained from Hamilton's principle. However, here we have an incompressible liquid with a rigid boundary below and a free surface above, and the limit of incompressible motion is not trivial. Also Eckart's treatment is for our purpose unnecessarily complicated by his simultaneously transforming into curvilinear coordinates. We start by writing down Hamilton's principle for fully nonlinear incompressible motions in a Lagrangian specification, the boundary conditions being included. To see how this may be done for a compressible fluid see Herivel (1955). The pressure then no longer enters as an independent field. For the inclusion of electromagnetic fields see Lundgren (1963).

In the usual Lagrangian specification of the motion of a fluid, the instantaneous position vector of each material particle $\mathbf{X} = (X, Y, Z)$ is expressed as a function of its initial position $\mathbf{X}_0$ and time t

$$\mathbf{X} = \mathbf{X}(\mathbf{X}_0, t). \tag{4·1}$$

The kinetic energy is $\frac{1}{2}|\dot{\mathbf{X}}|^2$ per unit mass, the gravitational potential energy is gZ. If the fluid is incompressible, there is an additional constraint that

$$\frac{\partial(X, Y, Z)}{\partial(X_0, Y_0, Z_0)} = 1. \tag{4·2}$$

If it is bounded below by a rigid boundary $b(x, y)$, then

$$Z(\mathbf{X}_0, t) = b(X, Y) \quad \text{if} \quad Z_0 = b(X_0, Y_0). \tag{4·3}$$

Otherwise there are no constraints on the motion, in particular the upper surface

$$Z_0 = s(X_0, Y_0) \tag{4·4}$$

is free.

The exact form of Hamilton's principle is then

$$\delta\int \mathrm{d}X_0\,\mathrm{d}Y_0\,\mathrm{d}t\Bigg[\int_b^s\left\{\tfrac{1}{2}\rho\,|\dot{\mathbf{X}}|^2-\rho gZ+\lambda_1\left(\frac{\partial(X,Y,Z)}{\partial(X_0,Y_0,Z_0)}-1\right)\right\}\mathrm{d}Z_0 \\ +\lambda_2\{Z-b(X,Y)\}_{Z_0=b(X_0,Y_0)}\Bigg]=0 \quad (4\cdot5)$$

for all variations $\delta\mathbf{X}(X_0,t)$, $\delta\lambda_1$, $\delta\lambda_2$ which vanish for sufficiently large $|X_0|$, $|Y_0|$, $|t|$. In this equation the integration over the initial coordinates X_0, Y_0, Z_0 corresponds to a summation over material particles to form the Lagrangian. The further integration over t forms the action. λ_1, λ_2 are Lagrange multipliers, their variation implying the incompressibility condition (4·2) and the bottom condition (4·3) respectively. However, as usual with constraints in Lagrange's equations, λ_1 and λ_2 may be interpreted as the associated generalized forces. Thus λ_1 is the generalized force corresponding to a change in volume of a fluid element, i.e. the pressure P. λ_2 is associated with the vertical component of the stress on the bottom; in fact

$$\lambda_2=\left\{P\frac{\partial(X,Y)}{\partial(X_0,Y_0)}\right\}_{Z_0=b(X_0,Y_0)}, \quad (4\cdot6)$$

a condition which also follows from the variational principle by varying Z. The Jacobian in equation (4·6) is evaluated on $Z_0=b(X_0,Y_0)$ and is thus given by

$$\frac{\partial(X,Y)}{\partial(X_0,Y_0)}\bigg|_{Z_0=b}=\begin{vmatrix}\dfrac{\partial X}{\partial X_0}&\dfrac{\partial X}{\partial Y_0}\\[2mm]\dfrac{\partial Y}{\partial X_0}&\dfrac{\partial Y}{\partial Y_0}\end{vmatrix}+\frac{\partial b}{\partial X_0}\begin{vmatrix}\dfrac{\partial X}{\partial Z_0}&\dfrac{\partial X}{\partial Y_0}\\[2mm]\dfrac{\partial Y}{\partial Z_0}&\dfrac{\partial Y}{\partial Y_0}\end{vmatrix}+\frac{\partial b}{\partial Y_0}\begin{vmatrix}\dfrac{\partial X}{\partial X_0}&\dfrac{\partial X}{\partial Z_0}\\[2mm]\dfrac{\partial Y}{\partial X_0}&\dfrac{\partial Y}{\partial Z_0}\end{vmatrix}. \quad (4\cdot7)$$

Besides equation (4·6), variation of Z gives the Z equation of motion and the free surface condition

$$P(X_0,Y_0,s(X_0,Y_0))=0. \quad (4\cdot8)$$

The other equations of motion follow simply from variation of X, Y. However, since each particle has to be identified by its initial position, the equations derived from the variational principle are not in a very convenient form.

We now envisage one standard solution of these equations

$$\mathbf{X}=\mathbf{x}(\mathbf{X}_0,t), \quad P=p(\mathbf{X}_0,t) \quad (4\cdot9)$$

and express the *independent* variables in terms of it

$$\mathbf{X}_0=\mathbf{X}_0(\mathbf{x},t). \quad (4\cdot10)$$

This is merely a convenient way of relabelling each material particle in terms, not of its original position, but of the position $\mathbf{x}$ it would have had, had it moved for time $t-t_0$ according to the particle paths of the standard solution. We denote the standard particle velocity

$$(\partial\mathbf{x}/\partial t)_{\mathbf{X}_0}=\mathbf{U} \quad (4\cdot11)$$

and regard all fields as functions of $\mathbf{x}, t$.

From equation (4·2)

$$\frac{\partial(x,y,z)}{\partial(X_0,Y_0,Z_0)}=1, \quad (4\cdot12)$$

and equation (4·5) becomes

$$\delta\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\left[\int_b^s \left\{\tfrac{1}{2}\rho\,|\dot{\mathbf{X}}|^2-\rho g Z+P\left(\frac{\partial(X,Y,Z)}{\partial(x,y,z)}-1\right)\right\}\mathrm{d}z \right.$$
$$\left. +\left\{P\frac{\partial(X,Y)}{\partial(x,y)}(Z-b(X,Y))\right\}_{z=b(x,y)}\right]=0, \quad (4\cdot13)$$

where
$$\dot{\mathbf{X}}=\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}+W\frac{\partial}{\partial z}\right)\mathbf{X}. \quad (4\cdot14)$$

Equation (4·13) holds for any solution $\mathbf{X}(\mathbf{x},t)$, $P(\mathbf{x},t)$ of the equations of motion. In particular we may put
$$\mathbf{X}=\mathbf{x}+\boldsymbol{\xi}, \quad P=p+\pi, \quad (4\cdot15)$$

where $\boldsymbol{\xi}$, π are small, and consider infinitesimal variations $\delta\boldsymbol{\xi}$, $\delta\pi$. If equation (4·13) is expanded in powers of $\boldsymbol{\xi}, \pi$ and then varied, the lowest order term in the expansion does not involve $\boldsymbol{\xi}$, π at all and hence has no variation. The linear term involves $\mathbf{x}$, p and $\delta\boldsymbol{\xi}$, $\delta\pi$ only, and it vanishes precisely because $\mathbf{x}$, p is a solution of the equations of motion. The quadratic term yields a variational principle governing $\boldsymbol{\xi}$, π. This gives homogeneous Euler equations for perturbations of small magnitude about the standard flow. Picking out these quadratic terms, (4·9) becomes

$$\delta\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\left[\int_b^s \{\tfrac{1}{2}\rho\,|\dot{\boldsymbol{\xi}}|^2+\pi(\xi_x+\eta_y+\zeta_z)+pq(\boldsymbol{\xi})\}\,\mathrm{d}z \right.$$
$$\left. +\{\pi+p(\xi_x+\eta_y+b_x\,\xi_z+b_y\eta_z)\}\{(\zeta-b_x\,\xi-b_y\,\eta)-pr(\boldsymbol{\xi})\}_{z=b}\right]=0, \quad (4\cdot16)$$

where
$$q(\boldsymbol{\xi})=\eta_y\zeta_z-\zeta_y\eta_z+\zeta_z\xi_x-\xi_z\zeta_x+\xi_x\eta_y-\eta_x\xi_y, \quad (4\cdot17)$$

and
$$r(\boldsymbol{\xi})=\tfrac{1}{2}(b_{xx}\xi^2+2b_{xy}\xi\eta+b_{yy}\eta^2). \quad (4\cdot18)$$

The integration is strictly over the region occupied by the standard flow, and the representation is quasi-Lagrangian in that the displacements $\boldsymbol{\xi}$ are of fluid particles from the instantaneous value of a moving reference position. $\mathbf{U}$, p, s, b are regarded as known.

Judicious integration by parts and careful consideration of the boundary terms enables one to cast equation (4·16) into the form

$$\delta\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\left[\int_b^s \{\tfrac{1}{2}\rho\,|\dot{\boldsymbol{\xi}}|^2+\pi'(\xi_x+\eta_y+\zeta_z)-\tfrac{1}{2}(p_{xx}\xi^2+p_{yy}\eta^2+p_{zz}\zeta^2 \right.$$
$$+2p_{yz}\eta\zeta+2p_{zx}\zeta\xi+2p_{xy}\xi\eta)\}\,\mathrm{d}z+\{\tfrac{1}{2}p_z(\zeta-\xi s_x-\eta s_y)^2\}_{z=s}$$
$$\left.+\{\tfrac{1}{2}p_z(\zeta-\xi b_z-\eta b_y)^2+\pi'(\zeta-\xi b_x-\eta b_y)\}_{z=b}\right]=0, \quad (4\cdot19)$$

where
$$\pi'=\pi-\xi p_x-\eta p_y-\zeta p_z. \quad (4\cdot20)$$

The basic flow is thus relevant to the perturbation only through the convection terms $\mathbf{U}.\partial/\partial\mathbf{x}$ in the time derivative, and through the pressure gradients. The position $z=s(x,y,t)$ of the free surface is also determined by the fact that the basic flow is a solution of the equations of motion. The volume integral in equation (4·19) is made up of the perturbation kinetic energy $\frac{1}{2}\rho\,|\dot{\boldsymbol{\xi}}|^2$ minus two forms of potential energy. The first, $\pi'(\xi_x+\eta_y+\zeta_z)$ is purely virtual; variation of the perturbation

pressure π' at any point implies incompressibility, so for real perturbations there is no contribution from this term. The second form is the potential energy of a displaced particle in the basic pressure field, gradients of pressure at the equilibrium position being balanced against basic mass accelerations. For the present problem this term is negligibly small if the basic flow is slowly varying. The surface integral over the free surface describes the only non-zero perturbation potential energy in a slowly varying flow. Then, the basic vertical accelerations are negligible and p_z may be replaced by $-\rho g$. Variation of π' also implies the bottom boundary condition, and contributions to the energy from the bottom integral are purely virtual.

Equation (4·16), subject to infinitely differentiable variations $\delta\boldsymbol{\xi}$, $\delta\pi$ which vanish as $|x|, |y|, |t| \to \infty$ but which are otherwise unrestricted, is the required approximate form of Hamilton's principle, to which the averaging procedure and the arguments of § 3 may be applied. Alternatively equation (4·19) could be used; the two are entirely equivalent. The explicit appearance in equation (4·19) of a perturbation potential energy is not an essential part of the argument, merely a comforting check on the calculations.

4·2. *Wave energy for surface waves*

To use equation (1·9), rather than to prove it, this procedure is unnecessary.

If $\mathbf{U}$, h are uniform ($W = 0$), the governing equations are separable and we may find sinusoidal solutions in which the vertical displacement of a fluid particle is given by

$$\zeta = a \frac{\sinh |\mathbf{k}|\,(z-b)}{\sinh |\mathbf{k}|\,h} \cos{(kx + ly - \omega t + \beta)} \tag{4·21}$$

provided the dispersion relation

$$\omega = \mathbf{U}.\mathbf{k} \pm \{g\,|\mathbf{k}| \tanh |\mathbf{k}|\,h\}^{\frac{1}{2}} \tag{4·22}$$

is satisfied. Here the bottom is at $z = b$, so the mean position of the free upper surface is

$$s = b + h.$$

The wavenumber $\mathbf{k}$ has two components (k, l) and there are for each $\mathbf{k}$ two normal modes corresponding to the choice of sign in equation (4·22). The state of motion of the system is completely determined when we specify the mode, the wavenumber $\mathbf{k}$, the amplitude a, and the arbitrary constant β in the phase. If the total velocity at the point (x, y, z, t) is $(U+u, V+v, w)$ and ρ is the density of the water (assumed constant), the wave energy density is the average over a wavelength of

$$\int_b^s \tfrac{1}{2}\rho(u^2 + v^2 + w^2)\,\mathrm{d}z + \tfrac{1}{2}\rho g \zeta^2|_{z=s}, \tag{4·23}$$

which is easily shown to be

$$E = \tfrac{1}{2}\rho g a^2. \tag{4·24}$$

The intrinsic frequency is

$$\omega' = \omega - \mathbf{U}\mathbf{k} = \pm \{g\,|\mathbf{k}| \tanh |\mathbf{k}|\,h\}^{\frac{1}{2}}, \tag{4·25}$$

and the wave action density is $\frac{1}{2}\rho g a^2/\omega'$.

If $\mathbf{U}$, h are not precisely uniform, but vary substantially in a known manner in a horizontal direction over a scale of many wavelengths (but with $W \ll U, V$, and with U, V varying only slightly with z), then we may still determine the motion locally in the same way. However, the values of $\mathbf{k}$, a, β at different places will be

different but interconnected. The frequency and wavenumber are connected along rays according to equations (1·6) and (1·7) and the dispersion relation (4·22), and the amplitude is determined by conservation of wave action, according to equations (1·9), (4·24) and (4·25). The phase β requires analysis to a higher order of approximation than considered in this paper.

4·3. *Hamilton's principle for a frozen-in magnetic field*

A statement of Hamilton's principle for general magnetohydrodynamic flows has been given by Lundgren (1963). He extends the classical non-magnetic treatment due to Herivel (1955) to include Galilean invariant electric and magnetic fields in a system consisting of perfectly conducting fluid, vacuum and perfectly conducting solid parts. This is accomplished by subtracting the magnetic energy from the integrand in the statement of Hamilton's principle for non-magnetic systems, and Lundgren shows that this leads to the correct equations of motion for the fluid, and boundary conditions at the fluid-vacuum interface.

The approach of § 4·1 may then be used to derive a variational principle describing small perturbations about any state which is an exact solution of the governing equations. It should be noted that perturbations in the electromagnetic field in any nonconducting regions can only be induced by particle displacements in conductors. The methods of § 3 can only be used to define the wave energy density by reference to the velocities of these conductors. This situation will only arise when adjacent conducting and nonconducting regions are acting as a waveguide implying integration over lateral coordinates.

5. Further remarks

The class of problems for which equation (1·9) may be considered established is circumscribed in three ways. First it is necessary that the appropriate approximate form of Hamilton's principle may be obtained. The methods of § 4 appear to be generally applicable to nondissipative problems in nonrelativistic fluid dynamics and magnetohydrodynamics with conservative boundary conditions, and yield a formulation explicitly in terms of particle displacements together with other fields such as the pressure and the magnetic field as appropriate. For compressible fluids the internal energy per unit mass should be expressed as an explicit function of the dilatation, and the pressure no longer appears as an independent field. External gravity fields and nonuniformities in initial density are all easily included.

Secondly, it is necessary that the averaged variational principle may be applied to the approximate form of Hamilton's principle to describe slowly varying wavetrains. The mathematical requirements for this are discussed fully in Bretherton (1968). Here it suffices to remark that the locally valid normal modes, in which an almost sinusoidal oscillation must be found, are required to be well-behaved functions of position and wavenumber, discrete and nondegenerate. This ensures that a wavetrain can indeed propagate between two widely separated points with a continuously varying but well defined internal structure imposed by the appropriate mode at each point along its path. It excludes the possibility of energy being transferred en route to another mode. The normal modes at each point are not required to form a complete set, although a weaker condition of this general type

is needed. With a few minor qualifications, the averaging procedure can then be made precise as the zero order approximation in an asymptotic sense as $\epsilon \to 0$, and for linearized equations an infinite sequence of higher order approximations in powers of ϵ can be constructed.

Thirdly, even if the adiabatic invariant $\mathscr{L}_\omega$ can be obtained, it cannot always be identified with the wave action density E/ω'. The discussion of § 3 indicates the conditions under which this appears to be possible. Noteworthy are the restrictions implied by the basic state being slowly varying. The train of argument is full of pitfalls for the unwary. The present authors fell at some stage into all of them, some of them several times.

6. Application to particular problems

6·1. *Surface gravity waves*

An important application of the result of this paper is to the propagation of surface gravity waves on a slowly varying nonuniform current, which is approximately independent of the vertical coordinate. This is the prototype problem of § 4.

Longuet-Higgins & Stewart (1961, 1964) have investigated special cases of this problem, and show that the results of their detailed perturbation analyses in these cases may be interpreted in terms of the interaction between the rate of strain of the basic current and the radiation stress of the waves. They thus infer from these special cases a general equation governing the energy propagation of surface waves on a nonuniform current

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \nabla . \mathbf{c}\, E + \tfrac{1}{2} S_{ij} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) = 0, \tag{6·1}$$

where $E = \frac{1}{2}\rho g a^2$, a is the amplitude of the surface elevation, and S_{ij} is the radiation stress tensor. Whitham (1962) derived equation (6·1) for special cases by a different method.

It is shown in the appendix that (6·1) may be cast into the form of equation (1·9) using equations (1·6) and the continuity equation of the basic flow. The same equivalence may be demonstrated if the effects of capillarity are included, though the wave energy density is then given by $E = \frac{1}{2}\rho g a^2 (1 + T\mathbf{k}^2/\rho g)$, where T is the surface tension.

We have also shown the equivalence of equation (6·1) and (1·9) for many other types of wave motion in fluid dynamics; some of these are discussed further in the ensuing paragraphs. This equivalence suggests certain general properties of radiation stress which are still under consideration.

6·2. *Sound waves*

Blokhintsev (1946) considers the problem of sound propagation in a slowly varying steady nonuniform moving medium. Using a W.K.B. approximation he derives an energy equation (his (2·30)) which may be written with our notation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega E}{\omega'}\right) + (\nabla . \mathbf{c})\left(\frac{\omega E}{\omega'}\right) = 0. \tag{6·2}$$

This agrees with our result (1·9) when the medium is not changing with time, as we then have $\mathrm{d}\omega/\mathrm{d}t = 0$. Extension of Blokhintsev's analysis gives (1·9) as the

correct energy equation even when the basic state is not steady. The same result may also be reached from consideration of the radiation stress for sound waves.

6·3. *Internal gravity waves*

Both types of wave motion discussed so far in this section have their group velocity and phase velocity parallel. Internal gravity waves provide an example for which this is not so. Bretherton (1966) has considered the problem of internal wave propagation in a shear flow $(U(z), V(z), 0)$ with Brunt–Väisälä frequency $N(z)$ and large Richardson number $N^2/(U_z^2+V_z^2)$. He derives equation (1·9) both by using a W.K.B. approximation, and also by considering the work done by the relevant Reynolds stresses. Hines & Reddy (1966) also arrive at (1·9) for this problem, though by a different method.

6·4. *Alfvèn waves*

From the extension of Hamilton's principle to include magnetic fields, mentioned in §4·3, we see that equation (1·9) applies also to the propagation of Alfvèn and magneto-acoustic waves in a compressible or incompressible fluid with a frozen-in magnetic field, provided that the slowly varying approximation is satisfied. For Alfvèn waves this result has also been verified by consideration of the effect of radiation stress.

6·5. *Inertial waves*

We have also considered the action of radiation stress for wave motions which do not have equipartition between kinetic and potential energy. Inertial waves afford the prime example of this, the wave energy in this case being entirely kinetic. For inertial waves in an incompressible homogeneous liquid rotating about Oz with a constant angular velocity, and with a superimposed geostrophic shear flow $(U(x,y), V(x,y), 0)$, equation (6·1) may again be shown to be equivalent to equation (1·9). For waves of this type, the frequency is independent of the magnitude of the wavenumber (though not its direction). The conditions for slow variation, both spatial and temporal must be satisfied separately. It is not sufficient merely to take the limit of small wavelength.

6·6. *Standing surface waves*

Taylor (1962) has investigated the effect of slowly changing currents of uniform divergence on standing surface waves. In each of the three cases he considers, his results are consistent with the statement that the energy of an area of standing waves expanding or contracting with the current varies in proportion to the frequency of the waves. If we consider a standing wave train to be made up of two progressive wave trains travelling in opposite directions with the same amplitude and frequency, then this result is only to be expected in view of the reversibility of equations (1·6) and (1·9).

6·7. *Wave action density*

It is noteworthy that the quantity E/ω', designated here the 'wave action density', is also of fundamental importance in nonlinear wave interactions. If the resonance

conditions are satisfied, a unit of wave action density from one wave train may react with a unit of wave action density from another wave train to produce one unit of wave action density in each of one or more further wave trains, very much in the manner of chemical reaction between different molecules. This is discussed further by Hasselmann (1963) who calls E/ω' the 'wave number density', and by Bretherton (1964).

6·8. *Further generalizations*

The present analysis is essentially confined to nonrelativistic classical dynamical systems. However, a result similar to equation (1·9) is known for gravitational waves in general relativity (Isaacson 1967) and electromagnetic waves in a relativistic gravitational field (Kristian & Sachs 1966).

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Appendix

We wish to show that the two energy propagation equations (1·9) and (6·1) are equivalent for surface gravity waves on water of depth h. We thus require

$$\tfrac{1}{2}S_{ij}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right) = -\frac{E}{\omega'}\frac{\mathrm{d}\omega'}{\mathrm{d}t}.$$

Now
$$\frac{\mathrm{d}\omega'}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{U}.\mathbf{k}),$$

and equations (1·6) hold equally well for a moving medium as for a stationary medium, as they are merely a consequence of the definitions $\omega = -\theta_t$, $k_i = \theta_{x_i}$ in (1·1). Here we have $\omega = \mathbf{U}.\mathbf{k} + \Omega'(\mathbf{k}, h)$.

Thus
$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \mathbf{k}.\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \Omega'}{\partial h}\frac{\partial h}{\partial t},$$

$$\frac{\mathrm{d}k_j}{\mathrm{d}t} = -k_i\frac{\partial U_i}{\partial x_j} - \frac{\partial \Omega'}{\partial h}\frac{\partial h}{\partial x_j},$$

$$\frac{\mathrm{d}U_j}{\mathrm{d}t} = \frac{\partial U_j}{\partial t} + U_i\frac{\partial U_j}{\partial x_i} + c_i'\frac{\partial U_j}{\partial x_i},$$

where
$$c_i' = \frac{\partial \Omega'}{\partial k_i}.$$

Hence
$$\frac{\mathrm{d}\omega'}{\mathrm{d}t} = \frac{\partial \Omega'}{\partial h}\left(\frac{\partial h}{\partial t} + \mathbf{U}.\nabla h\right) - k_j c_i'\frac{\partial U_j}{\partial x_i}$$

$$= -h\frac{\partial \Omega'}{\partial h}\nabla.\mathbf{U} - k_j c_i'\frac{\partial U_j}{\partial x_i},$$

using the continuity equation for the basic flow

$$\frac{\partial h}{\partial t} + \mathbf{U}.\nabla h + h\nabla.\mathbf{U} = 0.$$

If $\mathbf{U}$ is arbitrary, we thus require, using the dispersion relation (4·25),

$$S_{ij} = E\left(\frac{c_l' k_l}{\omega'} - \frac{1}{2}\right)\delta_{ij} + E\frac{c_i' k_j}{\omega'},$$

which is the general form of the radiation stress tensor derived by Longuet-Higgins & Stewart.

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