WAVES IN THE OCEAN

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INTRODUCTION

In the ocean there is a great variety of waves of a mechanical nature – acoustic, surface, internal and inertial waves, Rossby waves, and so-called stability waves. The latter exist in the case when there is a stationary flow with a vertical velocity "shear" [1]. The periods of the various waves vary from 10^{-6} sec (high ultrasound) to months (Rossby waves).

Surfaces waves in the deep ocean and long waves in shallow water have the longest history of theoretical investigations. The classical works by Stokes, Nekrasov, Levi–Civita, Savarenskii, Korteweg–de Vries, et al., have allowed a mathematical device to be created along with a specific physics intuition, which have helped the development of other fields of science such a plasma theory.

The theory of internal waves and Rossby waves has been developing actively in recent times. This process goes on in parallel with the accumulation of experimental material.

Nonlinear interactions between various kinds of waves in the ocean are extremely essential. Thus, the interaction of surface and internal waves effectively influences the spectra of both types of waves [2, 3]. Surface waves, in interacting with one another, generate infrasonic waves in the ocean and in the atmosphere [4, 5]. An acoustic wave interacting with the surrounding noise acquires additional attenuation [6], etc.

The specifics of the theory of nonlinear waves in the ocean lies in the fact that it must essentially take into account the broad frequency and broad spatial spectral composition of waves which exist in nature. It is precisely this factor that determines the subject matter of the second and third lectures of the given course. In the first lecture, the simplest model and the language familiar to physicists are used to present a brief exposition of the linear theory of waves in the ocean (with the exception of stability waves). The contents of this lecture are required for understanding the two subsequent lectures and are likewise useful for establishing general concepts and terminology.

LECTURE 1. LINEAR THEORY OF WAVES IN THE OCEAN

In this lecture we shall consider waves of a hydrodynamic nature in the ocean – namely, acoustic, surface, internal and inertial waves, and also Rossby waves.

1. Original Equations

The original equations are: the Euler equation (the momentum-conservation equation)

$$\rho \frac{du}{dt} = -2\rho [\Omega u] - \nabla p - g\rho\nabla z, \tag{1}$$

where **u** is the velocity of the particles; **Ω** is the angular velocity of the earth; p is the pressure; ρ is the density; g is the acceleration of gravity; ∇z is the unit vector along the vertical coordinate axis; the matter-conservation equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0; \tag{2}$$

the equation of state (we neglect heat exchange and adiabatic processes)

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$$\rho = \rho(p), \quad s = \text{const}. \tag{3}$$

In general, it is necessary to solve Eqs. (1) and (2) on a sphere; however, if the wavelength is much shorter than the radius of the earth (which we shall assume), then they may be treated in the plane that is tangent to the spherical earth at the given point. The z axis of the rectangular coordinate system is directed vertically upward; the x axis is directed along the parallel from west to east; y is directed along the meridian from south to north. Let us linearize the equations for the relative quiescent state in which the density $\rho_0(z)$ and pressure $p_0(z)$ are functions solely of z. For this purpose, we replace p by $p_0(z) + p(x, y, z, t)$ and ρ by $\rho_0(z) + \rho(x, y, z, t)$ in (1)-(3) and shall assume that the quantities p, ρ and **u** are quantities of first-order smallness. Then we obtain the following results from (1) and (2):

$$\frac{\partial \boldsymbol{u}}{\partial t} = -2[\boldsymbol{\Omega}\,\boldsymbol{u}] - \frac{1}{\rho_0}\nabla p - g\frac{\rho}{\rho_0}\nabla z; \tag{4}$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \boldsymbol{u} + \boldsymbol{u}\nabla\rho_0 = 0, \tag{5}$$

Eq. (3) being written as

$$\frac{d}{dt}(\rho_0 + \rho) = \frac{1}{c^2}\frac{d}{dt}(p_0 + p) \tag{6}$$

under these conditions, where $c^2 = c^2(z) \equiv (\partial p/\partial \rho)_{s=\text{const}}$ is the adiabatic sound velocity. However, taking account of the well-known relationship

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{u}\nabla), \tag{7}$$

we obtain the following result by placing $\mathbf{u} = \{u, v, w\}$ and taking account of the fact that $dp_0/dz = g\rho_0$:

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c^2}\left(\frac{\partial p}{\partial t} - g\rho_0 w\right). \tag{8}$$

The boundary conditions on the bottom will be:

$$z = -H, \quad w = 0. \tag{9}$$

On the surface of the water, the pressure is constant – i.e., the right side of (8) is equal to zero, and, consequently,

$$z = 0, \qquad \frac{\partial p}{\partial t} - g\rho_0 w = 0. \tag{10}$$

In Eqs. (4), (5), and (8) we perform still another simplification: namely, we shall assume that $\rho_0(z)$ is constant everywhere where it is not differentiated (the Boussinesq approximation) and is equal, say, to $\rho_{00} \equiv \rho_0(0)$. Estimates show that for the problem considered by us below this assumption is substantiated.

The solution of Eqs. (4), (5), and (8) shall be sought on the assumption that the time dependence has the form $e^{-i\omega t}$ and that the variables x, y, and z are separated:

$$\begin{aligned} u &= \frac{1}{\rho_{00}}P(z)U(x, y)e^{-i\omega t}, & v &= \frac{1}{\rho_{00}}P(z)V(x, y)e^{-i\omega t}, \\ w &= i\omega W(z)\Pi(x, y)e^{-i\omega t}, & p &= P(z)\Pi(x, y)e^{-i\omega t}. \end{aligned} \tag{11}$$

Substituting (11) into (8), we fi

$$\rho = \left[\frac{P(z)}{c^2(z)} - \frac{\rho_{00}}{g}N^2(z)W(z)\right]\Pi(x, y)e^{-i\omega t}, \tag{12}$$

where

$$N^2 = -\left[\frac{g}{\rho_{00}}\frac{d\rho_0}{dz} + \frac{g^2}{c^2(z)}\right] \tag{13}$$

is the so-called Väisälä frequency or the frequency of free vertical oscillations of the particles of the liquid.

The substitution of (11) into Eqs. (4) and (5) yields the following results when (12) is taken into account:

$$U - iqV - s\rho_{00}\omega\frac{W}{P}\Pi = -\frac{i}{\omega}\frac{\partial \Pi}{\partial x},$$

$$V + iqU = -\frac{i}{\omega}\frac{\partial \Pi}{\partial y},$$

$$\frac{dP}{dz} + \frac{g}{c^2}P + \rho_{00}(\omega^2 - N^2)W - s\omega P\frac{U}{\Pi} = 0, \tag{14}$$

$$\frac{1}{c^2} + \frac{g\rho_{00}}{c^2}\frac{W}{P} - \rho_{00}\frac{1}{P}\frac{dW}{dz} = -\frac{i}{\omega}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\frac{1}{\Pi},$$

where the notation

$$q = \frac{2\Omega_z}{\omega} = \frac{2\Omega}{\omega}\sin\varphi, \qquad s = \frac{2\Omega_y}{\omega} = \frac{2\Omega}{\omega}\cos\varphi \tag{15}$$

has been introduced for the geographic latitude of the site. The boundary conditions (9) and (10) are written as follows with allowance for the notation (11):

$$W_{z=-H} = (P + g\rho_{00}W)_{z=0} = 0. \tag{16}$$

Equations (14) are separable in the variables x, y, and z on the basis of one of the two following assumptions, each of which is fairly substantiated for its case; we shall limit ourselves to them in our subsequent analysis:

1) q and s are assumed constant. This approximation is valid for acoustic, surface, internal, and inertial waves over whose length q and s vary little.

2) We neglect terms containing s (i.e., Ω_y). This allows us to analyze Rossby waves.

Turning to the first assumption, we place $(U, V, \Pi) = (U_0, V_0, \Pi_0)\exp[i(k_x x + k_y y)]$, where U_0, V_0, Π_0 are constant (without restricting generality, one may place $\Pi_0 = 1$). Then from (14) we find

$$U_0 = \frac{1}{\omega(1-q^2)}\left(k_x + iqk_y + s\omega^2\rho_{00}\frac{W}{P}\right); \tag{17}$$

$$V_0 = \frac{1}{\omega(1-q^2)}\left(k_y - iqk_x - isq\omega^2\rho_{00}\frac{W}{P}\right). \tag{18}$$

And, moreover, we obtain the equations for P and W (the prime denotes differentiation with respect to z):

$$P' + \left(\frac{g}{c^2} - s\frac{k_x + iqk_y}{1-q^2}\right)P + \rho_{00}\left(\frac{\omega^2 - 4\Omega^2}{1-q^2} - N^2\right)W = 0; \tag{19}$$

$$W' + \left(-\frac{g}{c^2} + s\frac{k_x - iqk_y}{1-q^2}\right)W + \frac{1}{\rho_{00}}\left[\frac{\xi^2}{\omega^2(1-q^2)} - \frac{1}{c^2}\right]P = 0. \tag{20}$$

Here

$$\xi^2 \equiv k_x^2 + k_y^2. \tag{21}$$

2. Acoustic Waves

By comparison with the frequency ω of the acoustic waves, Ω and N are negligibly small. The force of gravity for acoustic waves in the ocean is likewise negligible. Therefore, in Eqs. (19) and (20) one may place $s = q = N = g = 0$. Excluding W from (19) and (20), we obtain

$$P'' + (\omega^2/c^2(z) - \xi^2)P = 0$$

which is the basic equation in ocean acoustics.

The variation of the sound velocity c(z) in the thickness of the ocean is not large (no more than by 5%), but it is extremely essential. Specifically, the presence of a minimum of c(z) at a certain depth leads to the formation of an acoustic waveguide (an underwater acoustic channel) over which the sound at low frequencies (for which the absorption in the water is low) may propagate over exceedingly great distances. There are data from experiments on an investigation of acoustic fields at distances of 22,000 km from explosion sound sources and at a distance of 2800 km from tonal sources.

3. Surface, Internal, and Inertial Waves

Neglecting compressibility ($c = \infty$) and excluding P from (19) and (20), we obtain the equation for W:

$$W'' - \frac{1}{\omega^2}[4(\Omega\gamma)^2 + (\omega^2 - N^2)\xi^2]W = 0,$$
$$\gamma \equiv \gamma z\frac{\partial}{\partial z} + i\xi. \tag{22}$$

a) Surface Waves. In this case, neither stratification of the water nor the rotation of the earth play a role (s = q = N = 0). From (22) we obtain the equation

$$\frac{d^2 W}{dz^2} - \xi^2 W = 0. \tag{23}$$

Having expressed P in terms of W in the boundary conditions (16), we obtain

$$W_{z=-H} = \left(gW - \frac{\omega^2}{\xi^2}W'\right)_{z=0} = 0 \tag{24}$$

for this case. Writing the solution of (23) in the form

$$W = C_1 e^{kz} + C_2 e^{-kz}$$

and substituting it into the boundary conditions, we obtain the well-known dispersion equation for surface waves in an ocean of finite depth:

$$\omega^2 = \xi g \operatorname{th} \xi H \tag{25}$$

with the well-known limiting cases

$$\xi H \gg 1, \quad \omega^2 = \xi g; \quad \xi H \ll 1, \quad \omega/\xi = c = \sqrt{gH} = \text{const}.$$

b) Internal Waves. For the time being, we shall assume that $\omega \gg \Omega$ and shall neglect the earth's rotation. Equation (22) can be written in the form

$$\frac{d^2 W}{dz^2} - \xi^2\left(1 - \frac{N^2}{\omega^2}\right)W = 0. \tag{26}$$

Let us consider two cases here. In the first of them assume that the medium is unbounded and that N = const. Then Eq. (26) is satisfied by solutions of the form $W = W_0 \exp(\pm ik_z z)$ for which

$$k^2 = \xi^2 \frac{N^2}{\omega^2}, \qquad k^2 \equiv \xi^2 + k_z^2. \tag{27}$$

Using ϑ to denote the angle which is made by the vector **k** with the vertical, we write the dispersion equation (27) in the form

$$\sin\vartheta = \mu\omega/N \qquad (\mu = \pm 1). \tag{28}$$

Hence it follows that:

1) waves may exist only for $\omega < N$;

2) for a stipulated ϑ a frequency is uniquely determined by Eq. (28). The wavelength (and this means the phase velocity) may be arbitrary under these conditions.

Having begun by taking $\mu = 1$ in order to be specific (the wave propagates in the direction of positive z), we obtain the following result for the group velocity by differentiating the relationship $\omega = N(\xi/k)$:

$$v_{gr}\left(\frac{\partial\omega}{\partial\xi}, \frac{\partial\omega}{\partial k_z}\right) = \frac{\omega k_z}{k^2}\left(\frac{k_z}{\xi^2}\xi - \gamma z\right). \tag{29}$$

It is not difficult to verify the fact that $(v_{gr}k) = 0$ (i.e., v_{gr} is directed normal to k; see Fig. 1, where we have taken $k_y = 0$). For the pressure we obtain

$$p = -ik_z \rho_{00}\frac{\omega^2}{\xi^2}W_0 e^{i(kr-\omega t)}. \tag{30}$$

We have $\nabla p = -ikp$ which means the pressure gradient is directed along **k**.

Turning to Eqs. (11) and using (17) and (18), we obtain

$$u = \left(-\frac{k_z}{\xi^2}\xi - \gamma z\right)w, \qquad w = i\omega W_0 e^{i(kr-\omega t)}. \tag{31}$$

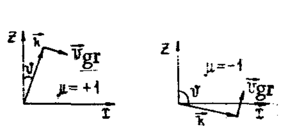

Fig. 1

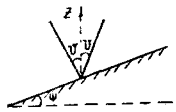

Fig. 2

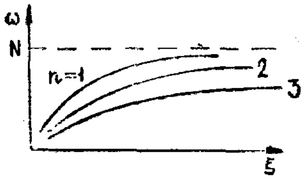

Fig. 3

for the velocity of the particles. It is not difficult to verify the fact that $(\mathbf{uk}) = 0$ (i.e., the particles move in the plane containing the vector $\mathbf{k}$ and the z axis along lines perpendicular to $\mathbf{k}$).

The energy flux averaged with respect to time is determined by the expression

$$I = \frac{1}{4}(pU^* + \text{c.c.}) = \frac{1}{2}\rho_{00} N^2 W_0^2 \mathbf{v}_{gr} \tag{32}$$

The energy flux is directed along the group-velocity vector. As is evident from Fig. 1, for a wave traveling upward the energy flux will be directed downward, and vice versa.

For reflection from the boundary, say, of the ocean bottom, the wave has interesting features if the bottom is inclined (Fig. 2). Since the frequency of the wave is conserved during reflection, the wave vector $\mathbf{k}_{ref}$ of the reflected wave must make the same angle ϑ with the vertical as the angle in the incident wave, and it is determined by Eq. (28). Thus, the angle of incidence here is equal to the angle of reflection, but relative to the vertical rather than to the normal to the surface.

Further, the reflected wave must always compensate the component of the particle velocity in the incident wave which is normal to the boundary. For this purpose it is necessary for the velocities of the traces of the incident and reflected waves along the boundary to be identical. This means that the projections of the wave vectors of the incident and reflected waves onto the boundary must be identical. For the case when both the normal to the boundary and the wave vector $\mathbf{k}_{inc}$ of the incident wave lie in the plane of the diagram, we have

$$k_{inc} \sin(\vartheta - \psi) = k_{ref} \sin(\vartheta + \psi), \tag{33}$$

in such a way that the wavelength (the wave number) changes during reflection; this does not contradict anything, since at the given frequency the wavelength may be arbitrary.

Let us now consider the case of a waveguide for internal waves. For the time being we shall not reject the assumption N = const, while the waveguide will be considered caused by the presence of the surface and bottom of the ocean (the horizontal bottom). The general solution of Eq. (26) will be

$$\begin{gathered} W = C_1 e^{ik_z z} + C_2 e^{-ik_z z}, \\ k_z = \xi \sqrt{\frac{N^2}{\omega^2} - 1}. \end{gathered} \tag{34}$$

Substituting it into the boundary conditions [which will again be written in the form (24)], we obtain the dispersion equation

$$g k_z \operatorname{tg} k_z H = N^2 - \omega^2, \tag{35}$$

from which we find the allowable values for k_z.

One of the solutions of Eq. (35) corresponds to small $k_z H$ and will be equal to (we assume $\tan k_z H \approx k_z H$, $\omega < N$)

$$k_{0z} = \sqrt{\frac{N^2 - \omega^2}{gH}}. \tag{36}$$

It will immediately be evident what kind of a wave this is if we find ξ_0 from (34):

$$\xi_0 = \frac{\omega}{\sqrt{gH}}.$$

This is a surface wave in shallow water having a propagation velocity $c = \sqrt{gH}$. Stratification of the medium in no way has any effect on it. In order to find the other roots of (35), we note that for N = const we have

$\rho(z) = \rho_{00}e^{-2\nu z}$, $N^2 = 2\nu g$, and since $\nu H \ll 1$, it follows that $g/H(N^2 - \omega^2) \sim 1/\nu H \gg 1$. Therefore, the roots will be very close to

$$k_{nz}H = n\pi \quad (n = \pm 1, \pm 2, \ldots). \tag{37}$$

For the horizontal wave number we obtain the following result using (34):

$$\xi_n = \frac{n\pi}{H}\left(\frac{N^2}{\omega^2} - 1\right)^{-1/2}.$$

The dispersion curves are shown schematically in Fig. 3.

In the general case $N = N(z)$ the problem can again be reduced to solution of Eq. (26) for the boundary conditions (24). In each case, one of the waves turns out to be of the surface type (maximum of $|W|$ at $z = 0$), and there is a set of waves having extremal points in the interval $0 < z < H$.

c) Inertial (gyroscopic) Waves. This type of wave may be obtained from Eq. (22) on the assumption that $N = 0$ (the liquid is homogeneous):

$$W'' + \left[\frac{4}{\omega^2}(\mathbf{\Omega k})^2 - \xi^2\right]W = 0. \tag{38}$$

For solutions in the form

$$W = W_0 \exp(\pm ik_z z) \tag{39}$$

we obtain

$$k_z^2 = \frac{4}{\omega^2}(\mathbf{\Omega k})^2 - \xi^2$$

from (38); i.e.,

$$\omega^2 = 4\left(\mathbf{\Omega}\frac{\mathbf{k}}{k}\right)^2, \tag{40}$$

or

$$\omega = 2\mu\Omega\cos\vartheta \qquad (\mu = \pm 1), \tag{41}$$

where ϑ is the angle between $\mathbf{k}$ and $\mathbf{\Omega}$. The value of μ is chosen from the condition $\mu\cos\vartheta > 0$. Thus, the angle ϑ is fixed for the given frequency. The wavelength may be arbitrary. We see that the properties of an inertial wave are very similar to the properties of the internal wave considered above. Specifically, for reflection from the boundary $\mathbf{k}_{ref}$ must make the same angle with $\mathbf{\Omega}$ as $\mathbf{k}_{inc}$ does. Just as in the case of an internal wave, we find

$$\mathbf{v}_{gr} = \frac{d\omega}{d\mathbf{k}} = \mu\frac{2\mathbf{\Omega}}{k} - \frac{\mathbf{k}}{k}\frac{\omega}{k}, \quad \mathbf{v}_{gr}\mathbf{k} = 0. \tag{42}$$

for the group velocity. The group velocity is normal to the direction of propagation of the wave $\mathbf{k}$.

Finding the two horizontal components of the particle velocity and p, we discover that the particles move in planes normal to $\mathbf{k}$ along circles having the radius $W_0/\sin\vartheta$. The energy flux is equal to

$$I = \frac{1}{2}\omega^2\rho_{00}\frac{W_0^2}{\sin^2\vartheta}\mathbf{v}_{gr}. \tag{43}$$

It may be shown that in an inertial wave motions take place in such a way that the component of the Coriolis force lying in the plane of rotation of the particles is balanced by the centrifugal force. The component in the direction $\mathbf{k}$ creates a pressure gradient in the wave.

In experiments it is inertial waves corresponding to $\xi = 0$ ($\omega = 2\Omega_z$) that are observed most frequently; these are frequently called inertial oscillations. It is possible that their prominence is caused by the fact that the waves of other forms are impeded by the presence of the stratification of the medium.

d) Gravitational-Gyroscopic Waves. Equation (22) allows simultaneous consideration of both the stratification of the ocean ($N \neq 0$) and the rotation effects ($\Omega \neq 0$). In the case $N = \text{const}$, we obtain

$$-k^2 + \frac{1}{\omega^2}[4(\mathbf{k\Omega})^2 + N^2\xi^2] = 0, \tag{44}$$

or

$$\omega^2 = N^2 \sin^2 \vartheta + 4\Omega^2 \cos^2 (k\Omega). \tag{45}$$

for a wave stipulated in the form (39); here ϑ is the angle between **k** and the z axis. In this case, we have waves which may be called gravitational-gyroscopic. In (44) the term $k_y\Omega_y$ may be neglected in the expression $\mathbf{k\Omega} = k_y\Omega_y + k_z\Omega_z$ if $k_y \ll k_z$ (i.e., if the scale of variation of the field in the z direction is considerably smaller than the wavelength in the y direction). Then, taking account of the fact that $k_z = k\cos\vartheta$, $\Omega_z = \Omega\sin\varphi$, we shall have

$$\omega^2 = N^2 \sin^2 \vartheta + 4\Omega^2 \cos^2 \vartheta \sin^2 \varphi \tag{46}$$

instead of (45). For the same reason, the quantity Ω_y is usually neglected in considering the ocean as a waveguide for gravitational-gyroscopic waves. In this case, we obtain the following equation for $W(z)$ for an arbitrary $N = N(z)$ from (22):

$$W'' - \xi^2 \frac{\omega^2 - N^2}{\omega^2 - 4\Omega_z^2} W = 0 \tag{47}$$

having the boundary conditions

$$W_{z=-H} = \left[gW - \frac{\omega^2}{\xi^2}(1-q^2)\,W'\right]_{z=0} = 0. \tag{48}$$

For N = 0, we have a solution in the form of the combination $\exp(\pm ik_z z)$, $k_z^2 = \xi^2/(1-q^2)$ for purely inertial waves. Substitution into the boundary conditions yields the characteristic equation for k_z: $(g/k_z)\tan k_zH = (\omega^2/\xi^2)(1-q^2)$. From this we obtain

$$\omega^2 = gH\frac{\xi^2}{1-q^2}, \quad \omega^2 - 4\Omega_z^2 = gH\xi^2 \quad \text{or} \qquad \omega^2 = \xi^2 gH + 4\Omega_z^2. \tag{49}$$

for a zero wave ($k_zH \ll 1$). Thus, long waves have a dispersion of the propagation velocity when the earth's rotation is taken into account.

For $\omega < 2\Omega_z$ we have $\xi^2 < 0$ which corresponds to nonpropagating waves which may be manifested locally and may propagate along the boundaries of depth differentials, along coastlines, etc.

4. Rossby Waves

The essential factor causing the presence of Rossby waves is the variation of a vertical component of the Coriolis force as a function of the latitude φ. For the simple description of these waves, one may again resort to the equations written in rectangular coordinates in the osculatory plane without assuming, as in the previous sections, that the Coriolis force $2\Omega\sin\varphi$ is constant but taking the next (linear) term of its expansion in powers of y (x = y = 0 is the point of osculation between the sphere and the plane):

$$2\Omega\sin\varphi = 2\Omega(\sin\varphi_0 + \cos\varphi_0\Delta\varphi) = 2\Omega\sin\varphi_0 + \beta y; \tag{50}$$

$$\beta = \frac{2\Omega\cos\varphi_0}{a}. \tag{51}$$

Here $\varphi_0 \equiv \varphi_{x=0,y=0}$; a is the radius of the earth. Henceforth the "0" subscript of φ will be dropped. Consideration of the term βy in (50) is frequently called consideration of the β-effect, and in this case the osculatory plane is called the β-plane, while the analysis of Rossby waves on the β-plane is called β-plane approximation. We shall likewise limit ourselves to this approximation.

The fact that in Eqs. (14) for $k_z \gg k_y$ the terms containing s (i.e., the horizontal component Ω_y of the angular-velocity vector Ω) may be neglected is a fact that greatly simplifies things. Then in the last equation in (14) the left side may depend only on z, while the right side may depend only on x. Setting each of them equal to the constant ε (the separation parameter), we obtain the following system of equations from (14) (it is likewise natural that $c = \infty$):

$$U - iqV = -\frac{i}{\omega}\frac{\partial \Pi}{\partial x}, \qquad V + iqU = -\frac{i}{\omega}\frac{\partial \Pi}{\partial y}, \tag{52}$$

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=i\,\varepsilon\omega\Pi;$$

$$P'+\rho_{00}(\omega^2-N^2)W=0, \tag{53}$$

$$W'+\frac{\varepsilon}{\rho_{00}}P=0.$$

From (52) we obtain the following equation for V by eliminating U and Π and assuming that $dq/dy = \beta/\omega$:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{i\beta}{\omega}\frac{\partial}{\partial x}+\varepsilon\omega^2(1-q^2)\right]V=0. \tag{54}$$

It may be shown that for operation in the β-plane we must assume $\omega \ll \Omega$, $q^2 \gg 1$. Therefore, unity may be neglected in comparison with q^2. Let us seek a solution for (54) in the form of plane waves

$$V=V_0\exp\left[i(k_x x+k_y y)\right]. \tag{55}$$

Then from (54) we obtain the dispersion equation

$$k_x^2+k_y^2+\frac{\beta}{\omega}k_x+4\,\varepsilon\Omega^2\sin^2\varphi=0, \tag{56}$$

or

$$\left(k_x+\frac{\beta}{2\omega}\right)^2+k_y^2=\left(\frac{\beta}{2\omega}\right)^2-4\,\varepsilon\Omega^2\sin^2\varphi. \tag{57}$$

Here ε is an eigenvalue of the system of equations (53) having the boundary conditions (16). Eliminating P from (53) and from the boundary conditions, we obtain the equation

$$W''+k_z^2W=0,\quad k_z^2=\varepsilon(N^2-\omega^2) \tag{58}$$

with the boundary conditions

$$W_{z=-H}=\left(gW-\frac{1}{\varepsilon}W'\right)_{z=0}=0. \tag{59}$$

The latter coincides with Eq. (26) and with the boundary conditions (24) if we place $\varepsilon = \xi^2/\omega^2$.

As a result, we have

$$\varepsilon_0=1/gH \tag{60}$$

for the zero mode and

$$\varepsilon_n=\frac{n^2\pi^2}{H^2(N^2-\omega^2)} \tag{61}$$

for the n-th mode. Let us now turn again for Eq. (57) from which many interesting corollaries derive.

In order for Eq. (55) to represent conventional propagating waves (k_x and k_y are real), the right side in (57) must be greater than zero: i.e.,

$$\frac{\beta}{2\omega}>2\sqrt{\varepsilon}\,\Omega\sin\varphi \tag{62}$$

or, taking account of the value of β from (51),

$$\operatorname{tg}\varphi<\frac{1}{2a\omega\sqrt{\varepsilon}}. \tag{63}$$

Thus, for a stipulated frequency ω waves may exist only in a band containing the equator for a site latitude satisfying condition (63). The higher ω, the narrower the band. For a stipulated latitude φ, the frequency ω must be lower than the critical frequency ω_{cr}, where

$$\omega_{cr}=\frac{1}{2a\sqrt{\varepsilon}\operatorname{tg}\varphi}. \tag{64}$$

For a zero ("barotropic") wave $\varepsilon = 1/gH$,

$$\omega_{cr}=\frac{\sqrt{gH}}{2a\operatorname{tg}\varphi}\quad(\omega<\omega_{cr}). \tag{65}$$

For waves of higher orders ("baroclinic" waves), ε is substantially greater, and consequently ω_{cr} is lower and causes the band in which waves exist to be substantially narrower.

Note further that Eq. (57) may be satisfied only for $k_x < 0$ (i.e., waves may propagate only from east to west). From (57) it is likewise clear that for a stipulated ω the tip of the vector **k** lies on a circle with its center at the point $k_y = 0$. $k_x = -\beta/2\omega$. The radius of the circle is equal to $(\beta/2\omega)^2 - 4\varepsilon\Omega^2\sin^2\varphi$. Assuming that ω and φ are far from the critical values, let us neglect the second term in this expression and consider a barotropic wave. It is not difficult to show that the velocity of the particles in the xy plane is normal to ξ. The dispersion equation (56) is written as

$$\omega = -\frac{k_x\beta}{k_x^2 + k_y^2}.$$

For the group velocity we have

$$v_{gr}\left(\frac{\partial\omega}{\partial k_x}, \frac{\partial\omega}{\partial k_y}\right) = \frac{\beta}{\xi^2}(\cos 2\gamma, -\sin 2\gamma), \tag{66}$$

where v_{gr} is directed from the tip of the ξ vector to the center of the circle. For $\gamma = 0$, v_{gr} is directed opposite to ξ.

LECTURE 2. MULTIWAVE INTERACTIONS IN THE OCEAN

Nonlinear interactions between waves play an important role in the formation of wave fields in the ocean. Thus, for example, one of the possible mechanisms for the generation of internal waves is their resonance interaction with surface waves. The existence of a resonance triad consisting of two surface waves and one internal wave may be proved by the following simple reasoning. In Lecture 1 it was demonstrated that the maximum possible frequency of the propagating internal waves is the maximum Väisälä frequency in the ocean: $N_m^2 = g\max\{(1/\rho_0)(d\rho_0/dz)\}$. This frequency is fairly low (10^{-2} Hz or lower), while the frequencies of surface waves are substantially higher (10^{-1} Hz or higher). It is obvious that the synchronism conditions may be fulfilled for the difference frequency $\omega_1(\mathbf{k}_1) - \omega_2(\mathbf{k}_2)$ between two surface waves that differ little in their lengths ($k_1 \approx k_2$) but propagate in arbitrary directions: $\mathbf{k}_3 = \mathbf{k}_1 - \mathbf{k}_2$, $\omega_3(\mathbf{k}_3) = \omega_1(\mathbf{k}_1) - \omega_2(\mathbf{k}_2)$. In [2] a synchronism curve was obtained (see Fig. 1) on which the tip of the vector $\mathbf{k}_2$ satisfying the synchronism conditions for the first mode of the internal wave must lie. The synchronism curves for the higher modes are situated inside the depicted curve and approach a circle having the radius k_1 with increasing mode number.

For the resonance triplet of waves $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ one may calculate the interaction coefficient and write a three-wave system of abridged equations. However, in oceanology, unlike optics, acoustics, and radio physics, monochromatic waves are encountered extremely rarely. The spectrum of the wave is usually continuous in **k**. Therefore, along with the original resonance triad, many other wave triplets will participate in the interaction (for example, any pair $\mathbf{k}_2$, $\mathbf{k}_3$ lying on the synchronism curve will interact in a resonance manner with the wave $\mathbf{k}_1$). Specular reflection of some resonance triads relative to one of the wave vectors will likewise lead to a new resonance triad of waves. Moreover, there exists a so-called "almost resonance" triads – i.e., triads of waves for which the synchronism conditions are fulfilled approximately: $\Delta = \omega_1 - \omega_2 - \omega_3 \approx 0$. The effect of these interactions is likewise substantial for small detunings. Actually, for the corresponding three-wave system of equations with the detuning Δ it is not difficult to show by analyzing the integrals of the system that the maximum possible energy loss of the energy-carrying decay wave is

$$E_m(\Delta) = H\left(1 - \frac{\Delta^2}{H\sigma_2\sigma_3}\right), \tag{1}$$

where H is the total system energy; σ_2 and σ_3 are the interaction coefficients in the equations for nondecay waves. The "almost resonance" criterion for the system of waves: $\Delta/\sqrt{\sigma_2\sigma_3 H} < 1/2$ derives from (1).

1. Spectral Form of the Equations

Thus, at least for oceanology problems, it is of interest to investigate multiwave interactions on the basis of the spectral method of solving nonlinear wave problems expounded in [3, 7, 8]. The principal idea of this method consists in the following. Let there be a certain nonlinear system of partial differential equations and in the linear approximation without allowance for dissipative processes let the solution, for example, of the initial problem be representable in the form of a superposition (spectrum) of harmonic waves propagating without any change in shape:

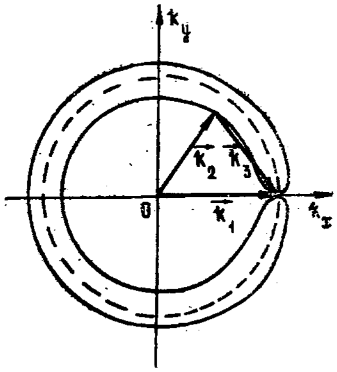

Fig. 1

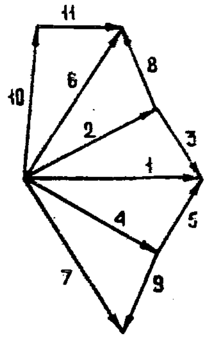

Fig. 2

$$w^{\nu}(\mathbf{r}, z, t) = \int \sum_{\nu=\pm 1, \pm 2, \ldots} a_{\mathbf{k}}^{\nu} \varphi_{\mathbf{k}}^{\nu}(z) \exp(i\mathbf{k}\mathbf{r} - i\omega_{\mathbf{k}}^{\nu} t)\, d\mathbf{k}. \tag{2}$$

Here we have written out the representation of a certain desired quantity (for example, the vertical velocity component of the particles of the liquid) in order to be specific; $\mathbf{r} \equiv \{x, y\}$ and z are the space coordinates and are assumed to be Cartesian, although depending on the symmetry of the problem they may also be expanded in spherical, cylindrical, and other harmonics; $\mathbf{k}$ is the horizontal wave vector; $a_{\mathbf{k}}^{\nu} d\mathbf{k}$ is the complex amplitude of the wave $(a_{\mathbf{k}}^{-\nu} = (a_{\mathbf{k}}^{\nu})^*)$; $\omega_{\mathbf{k}}^{\nu} (\omega_{\mathbf{k}}^{-\nu} = -\omega_{\mathbf{k}}^{\nu})$ and $\varphi_{\mathbf{k}}^{\nu} = \varphi_{\mathbf{k}}^{-\nu}$ are sets of natural frequencies and eigenfunctions of the corresponding boundary-wave problem for any fixed $\mathbf{k}$.

In the nonlinear case, assuming the set of eigenfunctions to be complete, the solution of the initial problem may likewise be sought in the form (2) but with amplitudes which depend on time t. The system of equations for the amplitudes which is equivalent to the original system of partial differential equations is written in the form

$$\begin{gathered} \dot{a}_{\mathbf{k}}^{\nu} = \varepsilon \int d\mathbf{k}_1 \sum_{\nu_1 \nu_2} D_{\mathbf{k}_1 \mathbf{k}_2 -\mathbf{k}}^{\nu_1 \nu_2 -\nu} a_{\mathbf{k}_1}^{\nu_1} a_{\mathbf{k}_2}^{\nu_2} \exp\left(-i \Delta_{\mathbf{k}_1 \mathbf{k}_2 -\mathbf{k}}^{\nu_1 \nu_2 -\nu} t\right)\Big|_{\mathbf{k}_2 = \mathbf{k} - \mathbf{k}_1}, \\ a_{\mathbf{k}}^{\nu}(0) = \alpha_{\mathbf{k}}^{\nu}, \end{gathered} \tag{3}$$

where ε is the nonlinearity parameter; $\Delta_{\mathbf{k}_1\mathbf{k}_2-\mathbf{k}}^{\nu_1\nu_2-\nu} = \omega_{\mathbf{k}_1}^{\nu_1} + \omega_{\mathbf{k}_2}^{\nu_2} - \omega_{\mathbf{k}}^{\nu}$ is the detuning; $D_{\mathbf{k}_1\mathbf{k}_2-\mathbf{k}}^{\nu_1\nu_2-\nu}$ are the interaction coefficients; the initial amplitudes $\alpha_{\mathbf{k}}^{\nu}$ are determined naturally from the initial conditions for the original system of equations. The principal difficulty in writing such a system resides in calculating the interaction coefficient $D_{\mathbf{k}_1\mathbf{k}_2-\mathbf{k}}^{\nu_1\nu_2-\nu}$. In [9, 10] these coefficients were identified with the coefficients of expansion of the Hamiltonian of the original system of equations in powers of $a_{\mathbf{k}}^{\nu}$. However, for specific calculations it is difficult to obtain the Hamiltonian, and additional difficulties develop during consideration of the attenuation of the waves. Therefore, it is preferable to derive the equations in spectral form directly from the original system of partial differential equations. In [3], this procedure was carried out for a system of hydrodynamics equations for an incompressible fluid in an oceanic waveguide with allowance for only three-wave interactions (i.e., in the quadratic approximation).

The system of equations (3) is substantially simplified in the case when for t = 0 the spectrum of $a_{\mathbf{k}}^{\nu}$ is discrete. Then and at subsequent times the spectrum remains discrete (only waves of the type $\mathbf{k}_l = \Sigma m_i \mathbf{k}_i$ develop, where m_i are arbitrary integers), and (3) goes over into a system of ordinary differential equations:

$$\dot{b}_j^{\nu} = \varepsilon \omega_j^{\nu} \sum_{l, \nu_1, \nu_2} D_{ls-j}^{\nu_1 \nu_2 -\nu} b_l^{\nu_1} b_s^{\nu_2} \exp\left(-i \Delta_{ls-j}^{\nu_1 \nu_2 -\nu} t\right)\Big|_{\mathbf{k}_s = \mathbf{k}_j - \mathbf{k}_l}. \tag{4}$$

In [3] the simplicity of applying asymptotic methods of solving partial differential equations (abridged equations) to the system (4) was noted. Let us demonstrate, for example, the method of obtaining the nonlinear correction to the frequency for a harmonic wave in a medium having strong dispersion. Assume that for t = 0 there is only one wave b_1^{ν}, and for simplicity assume there is no cubic nonlinearity. Then for large values of detuning $\Delta_{ls-j}^{\nu_1\nu_2-\nu}$ all combination harmonics of the wave b_1^{ν} will be of the order of ε, and the system (4) will be written as follows with allowance for just the principal terms:

$$\dot{b}_1^{\nu} = \varepsilon \omega_1^{\nu} \sum_{\nu_1} D_1^{\nu_1} b_2^{\nu_1} b_{-1}^{-\nu} \exp(-i \Delta_{\nu_1} t), \quad \Delta_{\nu_1} = \omega_2^{\nu_1} - 2\omega_1^{\nu},$$

$$\dot{b}_2^{\nu_1} = \varepsilon \omega_2^{\nu_1} D_2^{\nu_1} (b_1^{\nu})^2 \exp(i \Delta_{\nu_1} t), \quad D_1^{\nu_1} = D_{2-1-1}^{\nu_1-\nu-\nu}, \quad D_2^{\nu_1} = D_{11-2}^{\nu\nu-\nu_1}.$$

After integration of the second equation by parts, substitution into the first equation, and averaging, we obtain the following result accurate to ε^2:

$$\dot{b}_1^{\nu} = -i\varepsilon^2\omega_1^{\nu}\left|b_1^{\nu}\right|^2 b_1^{\nu}\sum_{\nu_1}\frac{\omega_2^{\nu_1}}{\Delta_{\nu_1}}D_1^{\nu_1}D_2^{\nu_1},$$

whose solution corresponds to a harmonic wave having the frequency

$$\tilde{\omega}_1^{\nu} = \omega_1^{\nu}\left(1 + \varepsilon^2\left|b_1^{\nu}\right|^2\sum_{\nu_1}\frac{\omega_2^{\nu_1}}{\Delta_{\nu_1}}D_1^{\nu_1}D_2^{\nu_1}\right).$$

The numerical solution of the system of equations (4) by the reduction method, in which the abridged system involving an ever greater number N of equations are solved successively until the solution is reproduced with a sufficient degree of accuracy, allows the process involving the interaction of waves over fairly large time intervals to be described. It is desirable to premise the section of an optimal set of N waves on a preliminary analysis such as that expounded, for example, in [3]. At present, there is a program which allows isolation of all possible interacting triplets $\mathbf{k}_m = \pm\mathbf{k}_l \pm \mathbf{k}_j$, calculation of the interaction coefficients between the waves of each triplet, and the solution of the stated Cauchy problem for any stipulated system of surface and internal waves $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n$ of an oceanic waveguide. We shall now go over to an exposition of the results of computation according to this program.

2. Instability of Three-Wave Interactions

As a result of the calculations, it was revealed that multiwave interaction may have a substantial influence on the redistribution of energy among waves of the resonance triad. We called this effect instability of three-wave resonance interactions.

The instability may be self-developing when it appears as a result of the excitation of combination waves. Its analysis was carried out on the following model. Assume that for t = 0 the amplitudes of only two surface waves 1, 2 which are in resonance with the different internal wave 3 are nonzero. During the interaction process, combination waves develop having $\mathbf{k} = \sum_{i=1}^{3} m_i\mathbf{k}_i$ among which there may be resonance triplets, "almost resonance" triplets and nonresonance triplets. If only nonresonance wave triplets develop, then their effect weakly influences the period of the process involving pumpover of energy between the waves of the resonance triad. The maximum amplitudes of the original wave triplet remain practically unchanged. Note that this process may be described in the first approximation by introducing a nonlinear correction to the frequencies of the original wave triplets. In [8] it is noted that in certain exceptional cases (for example, when the phase trajectory for the original wave triplet is close to a separatrix) the effect of nonresonance interactions is intensified and may result in a fundamental change in the character of the solution.

Additional resonance interactions lead to a substantial change both of the characteristic period of the three-wave process and of the limiting amplitudes of the waves. Under these conditions, the amplitudes of the displacements of the combination waves likewise become large.

Along with self-developing instability, a "priming" instability of a resonance system of waves relative to small perturbations in the initial spectrum is also possible. This instability is stronger and is more frequently encountered than the self-developing instability, since one can always find waves among the priming waves which are in resonance with the original ones.

As an example, let us consider the system of waves which is displayed in Fig. 2 and is obtained from the original resonance triad 1, 2, 3 by multiple specular reflections relative to the wave vectors 1 and 2. As has already been noted above, all of the triads obtained in this manner will be resonance triads. In Fig. 2 the waves 1, 2, 4, 6, 7, 10 are surface waves, while the remaining ones are internal waves. At the initial time, the unit amplitudes of the surface waves 1, 2 were stipulated, while the remaining surface waves had an amplitude of 0.001 (priming), and the amplitudes of all of the internal waves were equal to zero.

The time variation of the amplitudes of the waves is shown in Fig. 3 where the dashed curve depicts the same variation for interaction of only the waves of the principal triplet. From the dependences presented it follows that the three-wave interaction of surface and internal waves in the ocean is unstable, and the three-wave approximation remains valid only on a time interval of the order of the characteristic time T of the original process. This result may be clarified by considering the process of three-wave interaction of the principal triplet, since for this triplet it follows from the conservation integrals that at time t = T (see likewise Fig. 3) all of the energy of the wave system is mainly contained in the surface wave 2. At this time, the triplets

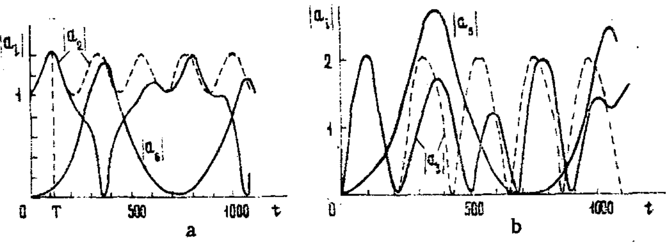

Fig. 3

(1, 2, 3) and (2, 6, 8) are practically of equal justification, and therefore henceforth the wave 6 is also intensively excited along with the wave 1. A more detailed exposition of this is given in [3]. Note that the complete transfer of the system energy to waves 2 is caused by the low ratio between the frequencies of the internal and surfaces waves ($\omega_3 / \omega_1 \ll 1$). In the case of interaction of waves having close frequencies (for example, plasma waves), this type of instability is conserved but will develop more slowly, since in this case the principal wave triplet remains isolated after one cycle T. But some portion of the energy will be transferred to other waves during each cycle.

Note likewise that multiwave interactions may lead to enrichment of the angular spectrum of the surface waves when they interact with internal waves, and also to more effective generation of internal waves (see Fig. 3).

3. Model One-Dimensional Equation for Nonlinear Wave Theory

Certain multiwave processes in the ocean may be investigated using the example of the simplest one-dimensional equation which we chose to be

$$\frac{\partial U}{\partial t} + LU = -\varepsilon U \frac{\partial U}{\partial x}, \quad U(x, 0) = F(x), \tag{5}$$

where L is a certain linear operator describing the possible dispersion and attenuation of the waves; ε is the nonlinearity parameter. The spectral form of Eq. (5) is obtained trivially (see [8]) and has the following form in the case of a discrete wave spectrum:

$$\dot{a}_n = -\gamma_n a_n - \frac{i\varepsilon}{2} k_n \sum_{m \neq 0} a_m a_l \exp(-i \Delta_{ml-n} t)|_{k_n = k_m + k_l}, \tag{6}$$

where γ_n is the attenuation coefficient. The simulation of any particular types of waves is accomplished by changing the dispersion law. Thus, for acoustic waves $\omega_n = c_0 k_n$, the Korteweg–de Vries (KDV) equations are $\omega_n = -\beta(k_n)^3$. Below we shall likewise use the dispersion laws shown in Fig. 4 (curve 1 corresponds approximately to the dispersion of an internal wave; curve 2 corresponds to the solitary resonance of the triplet of waves k_0, k_0, $2k_0$).

Using the example of the KDV equation, the efficiency of solving the spectral equations by the reduction method was checked in [8]. For this purpose, the distortion of a sinusoidal wave in time was traced, and the results of the calculation were compared with those obtained in [11] by direct numerical integration of the KDV equation. The solution of the spectral equations for 20 equations from (6) led to complete agreement with the calculations carried out in [11].

Note that the same type of result was obtained for surface and internal waves in shallow water on the basis of solving the system of equations (4) in the case of a unidirectional and one-mode spectrum of waves. The results of the calculation are fairly close to those obtained in [11]; however, for an increase in ocean depth ($kH > 1/20$) the difference became perceptible. Moreover, for a fairly pronounced nonlinearity ($ka > 1/50$) the picture was distorted due to the effect of nonresonance interactions with higher modes of internal waves.

Thus, the applicability limits of the KDV equation for surface and internal waves in shallow water (this equation has been used widely in a number of papers; see, for example, [12]) require further refinement.

4. Interaction of Wave Packets

Let us go over to an analysis of the interaction of initially quasimonochromatic wave packets. Assume that for $t = 0$ there is a system of waves having an energy concentrated near k_0. If the wave k_0 is at resonance with the wave $2k_0$, while the higher harmonics are nonresonance harmonics, then it is usually assumed that

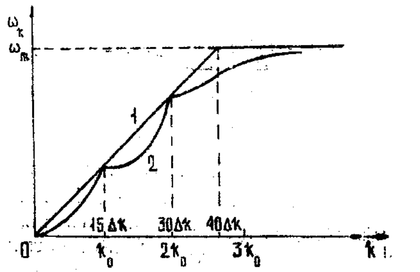

Fig. 4

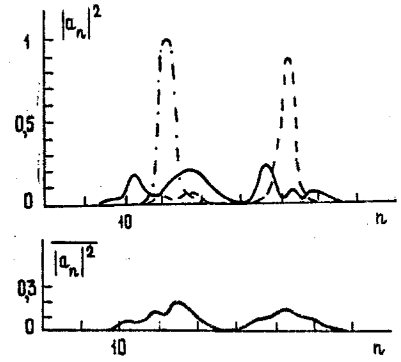

Fig. 5

the interaction of quasimonochromatic packets is analogous to resonance interaction of monochromatic waves k_0, k_0, $2k_0$. We considered this process using the following model. In the original equidistant spectrum of waves having a subdivision interval $\Delta k = k_0/15$, only the amplitudes of the waves having the numbers $12 < n < 18$ are nonzero at the initial time. Then the system of equations (6) was solved by the reduction method.

The results of the calculation showed the interaction process is complex and irregular in character: all the waves of the dispersion-free segment (resonance interactions) and from a certain interval having a low dispersion ("almost resonance" interactions) are excited to an equal extent. The time variation of the wave spectrum in the case of a dispersion law corresponding to curve 2 in Fig. 4 is illustrated by the spectrograms (Fig. 5), where the spectrum is depicted as continuous for the sake of clarity (the dashed-dot curve is for $t = 0$, the dashed curve is for $t = 25$, and the solid line is for $t = 100$). The time is measured in periods T_0 of the wave k_0. The bottom spectrograms shows the spectrum averaged over the entire calculated time interval (0 to 500). A comparison with the case of interaction of monochromatic waves demonstrated that the results coincide only on the initial time interval which is of the order of the characteristic interaction time of the resonance triad [(1/15) ε in our case]. Naturally, this interval widens when the width of the initial spectrum of the wave is reduced, the dispersion is intensified, and the nonlinearity is reduced. In the case of a dispersion law corresponding to curve 1 (Fig. 4), the average spectrum is practically uniform over the entire dispersion-free interval.

Note likewise that the spectral method allows the process by which equilibrium spectra of a large group of waves reach a steady state to be traced for excitation of large-scale motions by external sources in the presence of strong absorption in the shortwave range. The specific scheme for solving a problem of this type is given in [8] where the example of a one-dimensional equation (5) is used to investigate the establishment of the limiting wave spectrum.

Thus, in investigating nonlinear wave processes by asymptotic methods one should not forget the possible substantial effect of multiwave resonance and "almost resonance" interactions on the behavior of the process. This effect may lead to instability of the traditional three-wave interaction in a continuous wave spectrum, the transformation and expansion of spectrally narrow wave packets, etc.; this substantially limits the time interval suitable for the applicability of asymptotic methods. In this connection, the spectral method for the numerical solution of the equations of nonlinear wave theory would appear to be effective; this method allows the nonlinear interaction processes to be traced over fairly long time intervals.

LECTURE 3. INTERACTION OF WAVES WITH NOISE

1. Introduction

Waves of various types propagate in the ocean: surface and internal gravitational waves, surface capillary waves, acoustic waves, Rossby waves, and inertial waves. The nonlinear effects of the interaction of waves lead to an exchange of energy between the waves. As a result, energy fluxes from oscillations of one type to others develop in the ocean, and a certain energy distribution over the degrees of freedom is established whose knowledge is important for an understanding of the behavior of the ocean as a whole treated as some physical system.

An important aspect of this overall problem is the problem of the propagation of a weakly nonlinear wave in a region perturbed by noise that is treated as a random ensemble of waves. The present lecture is devoted precisely to an analysis of this problem. Based on the results obtained, one may in principle also consider a different problem: namely, the clarification of the character of the behavior of the noise spectrum in an inertial frequency interval.

Nonlinear wave-interaction effects which lead to an exchange in energy between waves may cause attenuation of a wave of finite amplitude propagating through a region with intense noise.

2. General Expressions for the Attenuation Coefficient

Let us consider the case in which a regular plane wave of finite amplitude $a_k(t)\exp(ikx - i\omega_k t)$ propagates in a region of the medium which is perturbed by intense noise. For simplicity, we shall assume that it is a discrete system of traveling plane waves of the form

$$c_q e^{iqx - i\omega_q t} \qquad (q_i = 0, \pm 1, \ldots, \pm \infty),$$

which propagates in all possible directions. The amplitudes are assumed to be real, so that

$$c_q^* = c_{-q}, \quad \omega_{-q} = -\omega_q, \tag{1}$$

and random, so that $\overline{c}_q = 0$, $\overline{c_q(t)c_q^*(t')} = R_q^2(t - t')$, where R_q^2 is a correlation function of such a form that for a sufficiently smooth function $f(t)$ the relationship

$$\int_0^\infty f(t) R_q^2(t)\, dt = N_q \int_0^{\tau_q} f(t)\, dt, \quad N_q = R_q^2(0), \tag{2}$$

is satisfied, where τ_q is the correlation time. It is assumed that there is one wave per solid angle $\Omega_0 = k_\theta^2/k_0^2$ (i.e., k_θ^2 is an area on the surface of a sphere having the radius k_0, whose size characterizes the scale of the angular correction). In order to be specific, we assume that the amplitude of the wave is a quantity of the same order of magnitude as are the spectral amplitudes of the noise.

In the second approximation, the nonlinear interactions of the waves may be described by an equation of the form [13]

$$\dot{y}_k = \sum_{\substack{k' \\ k'' = k - k'}} V_{kk'k''} y_{k'} y_{k''} \exp\left[i(\omega_{k'} + \omega_{k''} - \omega_k)t\right], \tag{3}$$

where $V_{kk'k''}$ is the interaction potential; $y_k(t)$ are slowly varying amplitudes which are assumed to be normalized in such a way that the relationship

$$y_k y_k^* = N_k = \mathcal{E}_k/\omega_k \tag{4}$$

is satisfied; here $\mathcal{E}_k$ is the energy density of the k-wave, and it is also convenient to introduce the Mach number:

$$\mathcal{E}_k k_\theta^2 k_0 = \mathcal{E}_k k_0^3 \Omega_0 = \rho c^2 M_k^2.$$

We shall assume that the noise field is stipulated while neglecting the variation of its average characteristics as a result of interaction with the isolated wave amplitude a_k (i.e., we shall treat it as a large reservoir whose energy is large in comparison with the energy of the considered wave). This approximation, which, as it were, corresponds to the parametric approximation, allows linearization of the nonlinear equation (4), since one of the multipliers of the quadratic terms turns out to be a well-known parameter – the stipulated amplitude of the noise-field wave. As a result, the problem stated can be reduced to the linear problem of propagation of a plane wave in a statistically inhomogeneous medium created by noise perturbations. In order to solve this problem, it is convenient to apply the Lifshits–Rozentsveig method [14] according to which the amplitude of the considered wave must be represented in the form of the sum of its average magnitude and a fluctuational correction:

$$a_k = \overline{a}_k + b_k,$$

where $\overline{b}_k = 0$.

Thus, the amplitude of a wave propagating in the direction q is, in general, made up of three parts:

$$y_q = \overline{a}_k \delta_{qk} + c_q + b_q, \tag{5}$$

where c_q is the stipulated amplitude of the noise wave. Substituting this expression into Eq. (3), we obtain the equation

$$\dot{y}_q = \sum_{\substack{q' \\ q''=q-q'}} V_{qq'q''} (\bar{a}_k \delta_{kq'} + c_{q'} + b_{q'}) (\bar{a}_k \delta_{kq''} + c_{q''} + b_{q''}) \exp [i(\omega_{q'} + \omega_{q''} - \omega_q) t]. \tag{6}$$

After averaging this expression with allowance for the fact that $\overline{c_{k'}} = 0$, $\overline{b_{q'}} = 0$, we obtain the following result for $\mathbf{q} = \mathbf{k}$, $\mathbf{q}' = \mathbf{k}'$, $\mathbf{q}'' = \mathbf{k}'' = \mathbf{k} - \mathbf{k}'$:

$$\dot{\bar{a}}_k = \sum_{\substack{k' \\ k''}} V_{kk'k''} \overline{b_{k'} c_{k''}} e^{i\Delta\omega_k t}, \tag{7}$$

where $\Delta\omega_k = \omega_{k'} + \omega_{k''} - \omega_k$. Note that the two terms which are on the average nonzero (namely, the terms proportional to $\overline{c_{q'} c_{q''}}$ and $\overline{b_{q'} b_{q''}}$) are not written out in the right side of this equation.

The first of them corresponds to interaction of the noise–noise type and therefore does not apply to the considered effect of interaction of a wave with noise; the second is a quantity which is of a higher order smallness in the wave amplitude.

Note that along with Eq. (7), which describes the variation of the regular part of the amplitude $\bar{a}_k$ of the considered wave due to its interaction with noise within the framework of the given approximation, one may write still another equation for the average amplitude; this equation is obtained after averaging of Eq. (6) in the same approximation as that used for Eq. (7) but for $\mathbf{q} = 2\mathbf{k}$, $\mathbf{q}' = \mathbf{q}'' = \mathbf{k}$ and describes the effect of the development and growth of the second harmonic of the considered $\mathbf{k}$-wave as a result of this self-action:

$$\dot{\bar{a}}_{2k} = V_{2k,k,k} a_k^2 \exp [i(2\omega_k - \omega_{2k}) t]. \tag{8}$$

Now subtracting Eqs. (7) and (8) from Eq. (6) written for $\mathbf{q}'' = \mathbf{k}$, $\mathbf{q}' = \mathbf{q} - \mathbf{k}$, we obtain the equation for the fluctuation part of the wave amplitude which develops as a result of its interaction with noise:

$$\dot{b}_q = V_{q,q-k,k} c_{q-k} \bar{a}_k e^{i\Delta\omega_q t}, \tag{9}$$

where $\Delta\omega_q = \omega_k + \omega_{q-k} - \omega_q$. Terms with factors of the type $a_k b_{q-k}$ and $c_k b_{q-k}$ are quantities of higher order of the wave amplitude and are therefore dropped. The term which is proportional to the product $c_k c_{q-k}$ and which describes the interaction of noise waves is mutually canceled with the term $\dot{c}_q$ in the left side of the equation within the framework of the considered approximation (the reaction of the wave on the noise field is neglected). Substituting the expression for $b_{k'}$ into Eq. (7), we rewrite it in the form

$$\dot{\bar{a}}_k = \sum V_{k,k',k''} e^{i\Delta\omega_k t} \left\langle c_{k''} \int_{-\infty}^{t} dt' V_{k',-k',k} c_{-k'} a_k(t') \right\rangle e^{i\Delta\omega_{k'} t}. \tag{10}$$

Here $\Delta\omega_k = \omega_{k'} + \omega_{k''} - \omega_k$, $\Delta\omega_{k''} = \omega_{-k'} + \omega_k - \omega_{k''}$.

By virtue of condition (1), $\omega_{-k} = -\omega_k$, and therefore we obtain

$$\Delta\omega_{k-k'} = -\Delta\omega_k. \tag{11}$$

Taking Eq. (2) which describes the correlation characteristics of the noise into account along with Eq. (11), we obtain the equation for $\bar{a}_k$ from (10) with allowance for the fact that $c_{-k'} = c^*_{k'}$:

$$\dot{\bar{a}}_k = \sum_{k'} V_{kk'k-k'} V_{k-k',-k',k} \int_{-\infty}^{t} dt' e^{i\Delta\omega_k (t-t')} R^2_{k'} \bar{a}_k(t). \tag{12}$$

Note that Eq. (12) is essentially the integral equation for the averaged wave amplitude; in principle, it allows the self-consistent solution to be derived for the amplitude of a wave propagating in the noise field but not at all the correction to its unperturbed value.

As is evident, $\bar{a}_k$ is determined by the correlation properties of the noise field and the form of the interaction potential characterizing the coupling between the wave and the noise. We seek the solution for $\bar{a}_k$ in the form $\bar{a}_k = a_k^0 e^{i\beta t}$, where a_k^0 is the unperturbed value of the wave amplitude.

Then from Eq. (9) we obtain the following result while likewise taking into account the form of the correlation function stipulated by Eq. (2):

$$i\beta_k = \sum_{k'} V^2 \int_{t-\tau_{k'}} dt' e^{i\Delta\omega_k (t-t')} N_{k'} e^{i\beta_k (t-t')}. \tag{13}$$

Here $V^2 = V_{kk'k-k'}, V_{k-k'k'k}$; for resonance interactions, $V^2 = |V_{kk'k''}|^2$. In view of the smallness of β_k which is proportional to the square of the interaction potential, one may assume that $e^{i\beta_k(t-t')} = 1$ which allows integration to be performed:

$$\int_0^{\tau_{k'}} e^{i\Delta\omega\xi} d\xi = \frac{\exp(i\Delta\omega_k\tau_{k'}) - 1}{i\Delta\omega_k}. \tag{14}$$

Isolating the real and imaginary parts of $\beta_{k'}$, we obtain the following general expressions for the dispersion correction to the frequency β_k' and for the absorption coefficient β_k'' which are caused by the interaction of the wave with the noise field:

$$\beta' = -\Sigma V^2 N_{k'} \frac{1 - \cos\Delta\omega_k\tau_{k'}}{\Delta\omega_k}. \tag{15}$$

The transition to a continuous spectrum presents no difficulty, since, for example, the expressions for the absorption coefficient may be written in the form

$$\beta' = \int dk^{3'} V^2 N_{k'} \frac{\sin\Delta\omega_k\tau_{k'}}{\Delta\omega_k}. \tag{16}$$

Equation (16) is the general expression for the absorption coefficient of a wave propagating in an intense noise field.

3. Attenuation of Low-Frequency and High-Frequency Waves

Let us emphasize the following two facts which are related to this expression. First, taking into account the fact that $N_k = \mathscr{E}_k/\omega_{k'}$, where $\mathscr{E}_{k'}$ is the spectral density of the noise energy, we arrive at the important conclusion to the effect that the absorption coefficient of the wave is determined by the noise intensity and is independent of the amplitude of the considered wave. The time $\tau_{k'}$ which enters into this equation characterizes the duration of an elementary interaction of three waves, unlike the time τ_x during which attenuation of the waves takes place as a result of multiple collisions. It is natural to adopt the growth time of the combination tone as a result of three-wave interaction as the time $\tau_{k'}$:

$$\tau_{k'}^{-1} = \frac{V_{kk'k''}}{\sqrt{\omega_k}} M. \tag{17}$$

In the case of acoustic waves, Eq. (16) leads to the following expression for the coefficients of absorption of a low-frequency wave having the frequency ω in nonequilibrium noise [15]:

$$\beta'' = \frac{2\pi\varepsilon^2\omega\mathscr{E}}{3\rho c^3} \approx \varepsilon^2\omega_k \frac{M^2}{\Omega_0}. \tag{18}$$

Here $\varepsilon = (\gamma + 1)/2$ is the nonlinearity index of the medium; $\mathscr{E}$ is the energy density of the noise. This formula coincides with the results obtained by Landau and Rumer [16] for the coefficient of sound absorption in metals; a similar expression, derived by a different method, was used by Krasil'nikov, Rudenko, and Chirkin [6] in discussing the nonlinear mechanism of the attenuation of low-frequency sound in the ocean.

In the case of a high-frequency (compared with the characteristic noise frequency ω_0) wave, the following result was obtained instead of Eq. (18):

$$\beta'' = \frac{2\pi\varepsilon^2\omega^2}{\rho c^2\omega_0}\mathscr{E}.$$

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THREE-MODE INTERACTION IN AN INCOMPRESSIBLE FLUID

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1. INTRODUCTION

Hydrodynamics is, incontrovertibly, a classical example of an essentially nonlinear field theory. Academician Gaponov-Grekhov [1], president of the Organizing Committee has already spoken of this in his introductory remarks at the second school. The exact solution of the one-dimensional problem of gasdynamics – the famous "simple wave" that was already discovered by Riemann more than one hundred years ago – is currently still the touchstone (the standard problem) for the development of many problems in the nonlinear theory of wave processes. Nonlinear acoustics, which makes full utilization of the arsenal of modern methods (specifically, the well developed theory of three-mode interactions), is developing successfully.

In problems in the hydrodynamics of an incompressible medium, the evolution of the system is likewise determined by three-mode interactions, since the equations of motion are quadratically nonlinear. Under these conditions, however, a series of specific difficulties associated with the isolation of "vortex modes" in specific problems develops, as well as difficulties caused by the very complicated nature of the "linkage" of the modes in practical systems having a large number of degrees of freedom (for example, in a turbulized fluid).

Nevertheless, the notion of three-mode interactions may also turn out to be useful in the dynamics of an incompressible fluid; this will be demonstrated using the example of the simplest hydrodynamic systems which allow realization under laboratory conditions.

2. APPROXIMATION OF THE HYDRODYNAMICS EQUATIONS ACCORDING TO THE GALERKIN METHOD. THE NOTION OF SYSTEMS OF THE HYDRODYNAMIC TYPE

Let us consider the motion of an incompressible fluid inside a closed vessel V which is bounded by a solid surface S. The state of the system at any time t is determined by the velocity field – a divergence-free vector field $\mathbf{v}(x, t)$ $(\operatorname{div} \mathbf{v} = 0)$ satisfying the boundary condition $v_n = 0$ on S. If at first we avoid the effect of viscosity, then the evolution of the system is described by the classical Euler equations:

$$\frac{\partial \boldsymbol{v}}{\partial t} = -(\boldsymbol{v}\nabla)\boldsymbol{v} - \nabla p; \tag{1}$$

here the density is taken to be unity; p is the scalar pressure field. The pressure in an incompressible fluid

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