LXXXII. *The Effect of Eddy Viscosity on Ocean Waves.*

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Abstract.

The effect of turbulence on ocean waves appears to call for an eddy viscosity which, while being large compared with the molecular viscosity, is small compared with the eddy viscosity found to apply to currents. A recent suggestion that the eddy viscosity should be taken as proportional to $\lambda^{4/3}$ where λ is the wavelength, seems to the author to be unsatisfactory. It is proposed, instead, that the coefficient of eddy viscosity N, applicable to waves, should be of the form $N=Kca$, where c is the speed of propagation, a is the amplitude and K is a constant. This form is indicated on dimensional grounds and is shown to be in conformity with v. Karman's similarity hypothesis for shearing flow. With K of the order of 5×10^{-5}, the eddy viscosity would be large enough to account entirely for the observed rate of decay of ocean swell, while its effect on the attenuation of waves with depth would still be negligible. In the initial formation of waves, only the molecular viscosity need be considered. A possible effect of eddy viscosity on waves in the generating area would be to limit the steepness of the longer waves to a value less than the breaking steepness, even when the duration and fetch were unlimited.

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§ 1. Introduction.

While molecular viscosity appears to be an important factor in the initial formation of surface waves on water, its dissipative effect on the long waves of ocean swell, for example, is negligibly small. If an eddy viscosity of the order of magnitude found to apply to ocean currents, on the other hand, were operative in the case of waves, their loss of energy would be much more rapid than is observed. Since it is well recognized in meteorology and oceanography that the eddy viscosity applicable to a particular motion depends on its scale, the suggestion has been made that waves may be affected by turbulence but that a smaller, and probably variable, eddy viscosity is applicable.

§ 2. Turbulence and Eddy Viscosity.

The first quantitative expression of this idea would appear to be that made recently by Groen and Dorrestein (1950), who assumed an eddy viscosity proportional to $\lambda^{4/3}$, where λ is the wavelength. This form was based on a result found by v. Weizsäcker (1948) in the theory of locally

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isotropic turbulence, and on an empirical law found by Richardson (1926) from diffusion experiments. V. Weizsäcker was considering the transfer of energy from turbulence on a scale characterized by a linear dimension l to turbulence of scales smaller than l. By similarity considerations, he found the effective eddy viscosity to be proportional to $l^{4/3}$ for values of l within a certain range. Turbulence on a scale so small that dissipation by molecular viscosity is the dominating factor, or so large that transfer of energy from the mean motion to the turbulence predominates is excluded from the range to which the similarity law applies. In the case of the dissipation of waves, we are dealing with the loss of energy from the mean motion (*i. e.* the wave motion) and it is not clear, therefore, that the eddy viscosity law deduced by v. Weizsäcker's theory will apply to this problem.

Richardson's result, which was obtained originally for the atmosphere and has been found by Richardson and Stommel (1948) to apply also to the ocean in certain conditions, was that the coefficient of diffusion applicable to the dispersal of a group of particles is proportional to $d^{4/3}$, where d is the separation of a pair of particles. The process considered in this case can take place in turbulence which does not necessarily involve shear in the fluid and it is not apparent that the dissipation of energy in a problem involving shearing stresses can be deduced from it.

A further difficulty is that, even if we assume an eddy viscosity proportional to $l^{4/3}$, the question arises : what is the characteristic length l for wave motion : Groen and Dorrestein have taken it to be λ, but one might expect the amplitude a also to be a characteristic length for a given set of waves.

§ 3. Wave Motion as affected by Molecular and Eddy Viscosity.

Let us consider two-dimensional surface waves in deep water. Let rectangular axes be taken, with Ox and Oy in the undisturbed sea surface and Oz vertically upwards. Let the waves be travelling in the direction of the negative x-axis, and U and W be the horizontal and vertical components of the particle velocity at any point. Then for irrotational waves of small amplitude

$$\left.\begin{aligned} U&=-\omega a e^{kz}\cos(\omega t+kx),\\ W&=-\omega a e^{kz}\sin(\omega t+kx),\end{aligned}\right\} \quad \ldots \quad (1)$$

where $\omega=2\pi/T$, T being the period, $k=2\pi/\lambda$, λ being the wavelength and a is the amplitude of the surface elevation. t is the time.

The rate of dissipation of energy by molecular viscosity may be shown to be (Lamb 1932, p. 623)

$$D=2\mu k^3c^2a^2 \quad \ldots \quad (2)$$

where μ is the coefficient of viscosity and $c=\lambda/T$ is the velocity of propagation. If no external forces are acting, the amplitude of the waves will decay at a rate given by the factor $\exp(-2\nu kt)$, where $\nu=\mu/\rho$ is the kinematic viscosity, ρ being the density.

To a first approximation, the components of the particle velocity are still given by (1), provided that

$$\theta \equiv \frac{\nu k}{c} \ll 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By inserting numerical values in (3), we find that θ is, in fact, negligibly small, except for very short waves.

To consider the effect of eddy viscosity, we note that a wave motion is specified by the three quantities λ, T and a. We will assume, therefore, that if there exists an eddy viscosity N, applicable to wave motion, it is a function of λ, T and a. If we assume a power law, $N = K\lambda^{\alpha}a^{\beta}T^{\gamma}$ where K is a non-dimensional constant, then, since the dimensions of N are L^2T^{-1}, we find

$$\alpha + \beta = 2, \qquad \gamma = -1.$$

Taking $\alpha = \beta = 1$ as the simplest solution including λ and a,

$$N = K\frac{\lambda a}{T} = Kca. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is suggested that equation (4) gives the appropriate form of eddy viscosity to apply to wave motion. We need not expect K to be an absolute constant but rather a non-dimensional parameter which varies only slowly compared with the other quantities involved. The above treatment is still useful if we can take K as constant over a limited range of the quantities λ, a and T.

The rate of dissipation of energy by eddy viscosity, replacing μ by ρN in equation (2), becomes,

$$W = 2\rho K k^3 c^3 a^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The particle velocity components will still be given to a first approximation by (1), provided that the criterion (3) is satisfied with N replacing ν, *i. e.*

$$\theta_N \equiv \frac{Nk}{c} = Kka \ll 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It will be shown later that, with a value of K large enough for eddy viscosity to account entirely for the observed rate of decay of ocean swell, the condition (6) is still satisfied.

§ 4. V. Karman's Similarity Hypothesis applied to Wave Motion.

An alternative approach to the problem of the dissipation of wave motion by turbulence may be made by v. Karman's similarity hypothesis for shearing flow, which has found successful application to turbulent flow through pipes and channels. Let u, v and w be the components of turbulent velocity and let square brackets [ ] denote values of functions of these components. Then if the mean velocity U is a function of z only, while $V = W = 0$, the rate of dissipation of energy of the mean motion by turbulence is given by

$$\psi = \rho[uw]\frac{dU}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

v. Karman assumed dynamical similarity in the turbulence at various points in the field of motion and deduced that

$$[uw]=l^2\left|\frac{d\mathrm{U}}{dz}\right|\frac{d\mathrm{U}}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where

$$l=\mathrm{K}_1\left|\frac{d\mathrm{U}}{dz}\right|\Big/\left|\frac{d^2\mathrm{U}}{dz^2}\right| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where K_1 is a non-dimensional constant.

In the case of two-dimensional motion, in the xz-plane, (7) is replaced by (Lamb, *loc. cit.*, p. 676)

$$\psi=\rho\left\{[u^2]\frac{\partial\mathrm{U}}{\partial x}+[w^2]\frac{\partial\mathrm{W}}{\partial z}+[uw]\left(\frac{\partial\mathrm{U}}{\partial z}+\frac{\partial\mathrm{W}}{\partial x}\right)\right\}. \quad . \quad . \quad (10)$$

In applying this equation to wave motion, we assume that mean values of $\partial\mathrm{U}/\partial x$ etc., as well as $[u^2]$ etc. are taken over an element of volume of dimensions small compared with the wavelength and over a time small compared with the period of the wave motion. If U and W are given by equations (1),

$$\psi=\rho k\omega a e^{kz}\{([u^2]-[w^2])\sin(\omega t+kx)-2[uw]\cos(\omega t+kx)\}. \quad . \quad . \quad (11)$$

By analogy with (8), we should expect $[uw]$ to be a function of $\partial\mathrm{W}/\partial x$ as well as of $\partial\mathrm{U}/\partial z$. In this case, however, $\partial\mathrm{W}/\partial x=\partial\mathrm{U}/\partial z$ so that we may take

$$[uw]=-l_4^2\left|\frac{\partial\mathrm{U}}{\partial z}\right|\frac{\partial\mathrm{U}}{\partial z},$$

where

$$l_4=\mathrm{K}_4\left|\frac{\partial\mathrm{U}}{\partial z}\right|\Big/\left|\frac{\partial^2\mathrm{U}}{dz^2}\right|=\frac{\mathrm{K}_4}{k},$$

where K_4 is a constant.

Similarly, we will take

$$[u^2]=\mathrm{C}_2+\frac{\mathrm{K}_2^2}{k^2}\left|\frac{\partial\mathrm{U}}{\partial x}\right|\frac{\partial\mathrm{U}}{\partial x}$$

and

$$[w^2]=\mathrm{C}_3+\frac{\mathrm{K}_3^2}{k^2}\left|\frac{\partial\mathrm{W}}{\partial z}\right|\frac{\partial\mathrm{W}}{\partial z},$$

where K_2 and K_3 are constants, introduced in a way similar to K_4. C_2 and C_3 are positive constants, since $[u^2]$ and $[w^2]$ are essentially positive. From (11), we have, for the mean rate of loss of energy per volume from the wave motion at a given depth z,

$$\begin{aligned} d\mathrm{W} &= \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\psi\,dt \\ &=\rho k\omega^3a^3e^{3kz}\{(\mathrm{K}_2^2-\mathrm{K}_3^2)\overline{|\sin(\omega t+kx)|\sin^2(\omega t+kx)} \\ &\qquad +2\mathrm{K}_4^2\overline{|\cos(\omega t+kx)|\cos^2(\omega t+kx)\}} \\ &=\frac{4}{3\pi}\rho k\omega^3a^3e^{3kz}(\mathrm{K}_2^2-\mathrm{K}_3^2+2\mathrm{K}_4^2). \end{aligned}$$

For the rate of dissipation per unit area of the sea surface,

$$\mathrm{W}=\int_{-\infty}^{0} d\mathrm{W}\, dz=\frac{4}{9\pi}\,\rho\omega^3a^3(\mathrm{K}_2^2-\mathrm{K}_3^2+2\mathrm{K}_4^2). \quad . \quad . \quad . \quad (12)$$

Hence

$$\mathrm{W}=2\rho\mathrm{K}k^3c^3a^3$$

as in equation (5) if

$$\mathrm{K}=\frac{2}{9\pi}\,(\mathrm{K}_2^2-\mathrm{K}_3^2+2\mathrm{K}_4^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The equations of wave motion in the xz-plane, in the presence of turbulence are (from Lamb, *loc. cit.*, p. 678)

$$\frac{\partial \mathrm{U}}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\,[u^2]+\frac{\partial}{\partial z}\,[uw]=0\,,$$

$$\frac{\partial \mathrm{W}}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\,[w^2]+\frac{\partial}{\partial x}\,[uw]=g\,,$$

where p is the wave pressure. To the first approximation

$$p=\rho gae^{kz}\cos\,(\omega t+kx)\,.$$

The condition that equation (1) should still be valid, in the presence of turbulence, to a first approximation, is that terms such as $\partial[uw]/\partial z$ should be small compared with terms such as $(1/\rho)\partial p/\partial x$.

We have $\left|\frac{\partial}{\partial z}\,[uw]\right|=\mathrm{K}_4^2k\omega^2a^2e^{2kz}$ and $\left|\frac{1}{\rho}\frac{\partial p}{\partial x}\right|=gkae^{kz}$. Since K_4^2 is of the same order of magnitude as K, we may write the condition as

$$\frac{\mathrm{K}k\omega^2a^2e^{2kz}}{gkae^{kz}}\ll 1$$

and since $kc^2=g$, this becomes

$$\mathrm{K}kae^{kz}\ll 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Apart from the factor $\exp kz$, condition (14) is the same as that given by (6).

We may therefore derive the equation for the dissipation of wave motion by turbulence from v. Karman's similarity hypothesis without introducing the concept of eddy viscosity or the analogy with molecular viscosity.

§ 5. The Decay of Ocean Swell.

5.1. *Waves of a Single Period.*

We will consider waves of constant wave velocity c, travelling in the direction Ox and assume a quasi-stationary state, *i. e.* the wave energy varying with distance but constant in time at a given point. Then if E is the energy per area of the waves, the rate of loss of energy with distance, due to eddy viscosity, would be given by

$$\frac{c}{2}\frac{d\mathrm{E}}{dx}=-\mathrm{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

assuming the energy to be propagated with the group velocity $c/2$. Since $E=\frac{1}{2}\rho g a^2$ and taking W from equation (5), we have

$$\frac{da}{dx}=-\frac{4Kg^2a^2}{c^4}$$

using the relation $kc^2=g$. The solution is

$$\frac{1}{a}=\frac{1}{a_0}+\frac{4Kg^2}{c^4}x, \quad (16)$$

where a_0 is the amplitude at $x=0$. In terms of the period T, since $gT=2\pi c$,

$$\frac{1}{a}=\frac{1}{a_0}+\frac{64\pi^4K}{g^2T^4}x. \quad (17)$$

Sverdrup and Munk (1947), in their treatment of the growth and decay of waves, assumed that the effect of eddy viscosity was negligible and attributed the decay of swell to air resistance, the rate of dissipation per area being of the form

$$R=\tfrac{1}{2}s\rho' k^2c^3a^2, \quad (18)$$

where s is a coefficient and ρ' the density of the air. If we replace W by R in equation (15), we are led to a solution

$$a=a_0 \exp\{-(s\rho' g/\rho c^2)x\} \quad (19)$$

or in terms of T

$$a=a_0 \exp\{-(4\pi^2 s\rho'/\rho g T^2)x\}. \quad (20)$$

A comparison of equations (17) and (20) shows the differences to be expected if the decay is due mainly to eddy viscosity or to air resistance. Both equations show that the decay is more rapid for waves of shorter period, the discrimination being greater in the case of eddy viscosity. In either case, therefore, if a spectrum of periods is present in the swell at $x=0$, the maximum amplitude occurring for a period T_0, then at a distance x, owing to the selective dissipation, the component of maximum amplitude will occur at a period T where $T>T_0$. An increase of effective period with distance usually occurs with ocean swell, but Munk (1947) has shown that such an increase may be explained, in some cases at least, on kinematical grounds without any selective dissipation of energy.

5.2. *Effective Period increasing with distance.*

Let us consider the general case of the period T, and thus the velocity c, increasing with x, without assigning the cause, and follow Sverdrup and Munk by assuming our dissipation equation to apply to the " significant waves ". If c varies with x, equation (15) is replaced by

$$\frac{1}{2}\left(c\frac{dE}{dx}+E\frac{dc}{dx}\right)=-W. \quad (21)$$

Hence

$$\frac{da}{dx}=-\frac{1}{2}\frac{a}{c}\frac{dc}{dx}-\frac{4Kg^2a^2}{c^4}. \quad (22)$$

Little is known of the way in which c may be expected to increase with x, but to obtain a solution we will assume that

$$\frac{dc}{dx}=\sigma \text{ (a constant), so that } c=c_0+\sigma x.$$

Then the solution of (22) may be shown to be

$$\frac{1}{ac^{1/2}}=\frac{1}{a_0c_0^{\,1/2}}+\frac{8\mathrm{K}g^2x}{7(c-c_0)}\left(\frac{1}{c_0^{\,7/2}}-\frac{1}{c^{7/2}}\right), \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where a_0, c_0 are values at $x=0$ and a, c are values at x. Putting $g\mathrm{T}=2\pi c$,

$$\frac{1}{a\mathrm{T}^{1/2}}=\frac{1}{a_0\mathrm{T}_0^{\,1/2}}+\frac{128\pi^4\mathrm{K}x}{7g^2(\mathrm{T}-\mathrm{T}_0)}\left(\frac{1}{\mathrm{T}_0^{\,7/2}}-\frac{1}{\mathrm{T}^{7/2}}\right). \quad . \quad . \quad (24)$$

For dissipation by air resistance, it may be shown that the corresponding solution is

$$a\mathrm{T}^{1/2}=a_0\mathrm{T}_0^{\,1/2}\exp\left\{-(4\pi^2s\rho'/\rho g\mathrm{T}_0\mathrm{T})x\right\}. \quad . \quad . \quad . \quad . \quad (25)$$

5.3. *Application to Observations of Swell.*

The above equations have been applied to the examples of swell given by Sverdrup and Munk (*loc. cit.*, Table IV.). s and K have been computed from the recorded values of x, a_0, T_0, a and T, and are shown in Table I. For the examples of swell Nos. 1, 2, 3 and 5, equations (24) and (25) respectively were used. For Nos. 4 and 6, equations (17) and (20) were used, since T differs little from T_0. The scatter of the values of s and K is about the same and one cannot decide on these data whether the decay law for air resistance or eddy viscosity fits the observations better. It is possible, of course, that both processes operate and are of the same order of magnitude.

Taking the mean value of K, we would have

$$\mathrm{N}=5{\cdot}6\times10^{-5}ca. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

As an example, for swell of wave velocity 10 m./sec. and amplitude 2 m., N=11·2 cm.²/sec., which does not appear unreasonable. Referring to the condition given by (6), we see that, with this value of K, the condition is easily satisfied. The influence of eddy viscosity on the rate of attenuation of the waves with depth is, therefore, negligible.

§ 6. The Initial Formation of Waves.

Jeffreys (1925) showed that the early stages of the formation of waves could be explained if a transfer of energy from wind to waves took place through normal pressures on the wave profile. The rate of transfer of energy was taken to be of the form

$$\mathrm{A}=\tfrac{1}{2}s\rho'(ka)^2(\mathrm{V}-c)^2c \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where $\mathrm{V}>c$, s being the "sheltering coefficient" and V the wind speed. Comparing A with D, the rate of dissipation by molecular viscosity, Jeffreys showed the existence of a critical value of V, below which waves could not be formed. From observations he found $V_{min}=110$ cm./sec. (approximately), which corresponds to $s=0{\cdot}27$. If $V>V_{min}$ this treatment sets no limit to the height the waves may attain and they would grow, presumably, until they reached the breaking steepness of $2a/\lambda=1/7$.

From wind tunnel measurements on wave models of various ratios of $2a/\lambda$, Motzfeld (1937) deduced that A was of the form

$$A=\tfrac{1}{2}s'\rho'(ka)^{3/2}(V-c)^2c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

TABLE I.

Decrease in Height of Swell.

1	2	3	4	5	6	7	8
No.	Decay distance x	At end of fetch		At distance x		Air resist-ance s	Eddy visco-sity K
		Ampl. a_0	Period T_0	Ampl. a	Period T		
	km.	m.	sec.	m.	sec.	$\times 10^{-2}$	$\times 10^{-5}$
1	2,320	3·4	7·9	0·7	11·5	1·04	4·3
2	1,480	2·4	8·1	1·3	17·0	0·46	2·2
3	1,950	3·45	8·7	1·0	17·0	1·39	7·1
4	1,110	2·35	10·5	1·35	11·2	1·20	6·1
5	1,480	2·55	12·2	1·5	9·1	1·04	4·6
6	185	1·25	9·0	1·05	9·5	1·84	9·3
					Mean	1·16	5·6

The data in columns 1 to 6 are taken from Table IV. of Sverdrup and Munk's paper (1947). Column 7 gives the value of s, computed from equation (20) or (25), assuming the decay to be due to air resistance only. Column 8 gives the value of K, computed from equation (17) or (24), assuming the decay to be due to eddy viscosity only.

with $s'=0{\cdot}014$. Applying this result to the formation of waves, he found no critical wind speed in the region of 100 cm./sec. but for any value of V, ka is a definite function of c, which is a maximum when $c=V/3$. For $V<100$ cm./sec., the maximum ka would be negligibly small, but would increase rapidly in the region just above 100 cm./sec.

Neumann (1949), after considering observations of wind stress and wave steepness for various wind speeds, has proposed an equation for A of the form

$$A=\tfrac{1}{2}s''\rho' ka(V-c)^2c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

with $s''=0{\cdot}095$. Like Motzfeld, he found ka_{max} to be a definite function of V and to increase rapidly when V was about 100 cm./sec., but his values of ka were appreciably higher than Motzfeld's.

While differing in detail, therefore, all investigators agree that easily observable waves should first appear when $V=100$ cm./sec. approximately, and have a velocity $c=V/3$, *i. e.* 33 cm./sec. approximately. If we take $a=0{\cdot}1$ cm., equation (26) would give $N=1{\cdot}85\times10^{-4}$ cm.^{2}/sec., *i. e.* much less than the kinematic molecular viscosity ν, which is $1{\cdot}8\times10^{-2}$ cm.^{2}/sec. We may conclude that eddy viscosity can play no part in the initial formation of waves and that only dissipation by molecular viscosity need be considered.

It seems unlikely that the relation $N=Kca$ would hold down to values of N comparable with ν. We might, in fact, define a " Reynolds' number for wave motion " by $R_w=ca/\nu$, and expect that turbulence would affect the wave motion only when R_w was above a certain value.

§ 7. The Growth of Waves under the Influence of Wind.

Although Sverdrup and Munk have used an expression for A similar to Jeffreys' (equation (27) for the transfer of energy from wind to waves by normal stresses for $V>6$ m./sec., they found it necessary to take a much lower value of s, i. e. $s=0{\cdot}013$. They showed that transfer of energy may also take place by tangential stresses, according to the equation

$$A'=\gamma^2\rho'(ka)^2V^2c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

where γ^2 is the resistance coefficient and was taken as $2{\cdot}6\times10^{-3}$. Unlike A, A′ remains finite when $c=V$. Dissipation by molecular viscosity is negligible for waves for which c is several m./sec. If eddy viscosity is also negligible (as assumed by Sverdrup and Munk), the height attainable by waves under the influence of moderate and strong winds is limited only by the breaking steepness, provided the duration and fetch are both sufficiently great. In practice, it is observed that, while the shorter waves may be continually breaking, waves travelling with speeds approaching that of the wind seldom reach the breaking steepness, and usually do not exceed about 1/3 of that steepness. On Sverdrup and Munk's theory, the explanation given is that sufficiently large durations and fetches do not, in fact, occur to enable the long waves to reach a greater steepness.

If eddy viscosity is the limiting factor, we see that the wave steepness cannot exceed an equilibrium value given by

$$A+A'=W,$$

Taking A from equation (27) and A′ from equation (30), we find

$$ka_{\max}=\frac{\rho'[s(1-\beta)^2+2\gamma^2]}{4\rho K\beta^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where $\beta=c/V$, and has been termed the " wage age ". Thus if s and γ^2 are constant, or are functions of β only, the equilibrium steepness is a function of the wave age, β, only. This general result is still true if A is given by either of the alternative equations (28) and (29).

From equation (31) the equilibrium steepness given by $ka_{\max}$ will be less than the breaking steepness, for which $ka=\pi/7$, if

$$\frac{\beta^2 K}{s(1-\beta)^2+2\gamma^2}>\frac{7\rho'}{4\pi\rho} \quad i.e. \quad >6\cdot 8\times 10^{-4}$$

inserting the numerical values of ρ' and ρ. We will consider two special cases of this condition :

(*a*) If $\beta=\frac{1}{3}$ and γ^2 is negligible,

$$K>2\cdot 72\times 10^{-3}s.$$

With Jeffreys' value, $s=0\cdot 27$, $K>7\cdot 3\times 10^{-4}$. With Sverdrup and Munk's value, $s=0\cdot 013$, $K>3\cdot 5\times 10^{-5}$, but this is hardly valid, since the value $s=0\cdot 013$ was derived empirically on the assumption that eddy viscosity was negligible.

(*b*) If $\beta=1$, then

$$K>1\cdot 36\times 10^{-3}\gamma^2.$$

Taking $\gamma^2=2\cdot 6\times 10^{-3}$, $K>3\cdot 5\times 10^{-6}$.

Comparing these inequalities with the value of K deduced in § 5.3, it seems possible that eddy viscosity may limit the steepness of waves travelling with a speed approaching that of the wind, but unlikely that it will limit the steepness of waves travelling at about one-third of that speed.

A relation between wave steepness and wave age was, in fact, found empirically by Sverdrup and Munk. Until we have more data on the rate of transfer of energy from wind to waves, however, we cannot say whether the steepness of the longer waves is limited only by the duration and fetch, or by an eddy viscosity of the form we have been considering.

§ 8 The Effect on Waves of Turbulence due to other Causes.

It has sometimes been assumed that the turbulence affecting waves should be attributed to the effects of wind driven or other currents. Alternatively, it has been suggested that the turbulence affecting currents is derived from wave motion. Neither assumption is made in the present treatment, the turbulence being regarded as inherent in the wave motion itself. The wave motion generates the turbulence and the turbulence reacts on the wave motion. The question still arises, however, whether, in the presence of a shearing current, the increased turbulence would not greatly increase the effective eddy viscosity for waves. A clue to the answer may be found in equation (10), from which it is seen that for dissipation of energy by turbulence to occur, the mean values of terms such as $[uw]\partial U/\partial z$ must be finite. This implies not only a correlation between u and w, but also a correlation between $[uw]$ and $\partial U/\partial z$. It is assumed that, in the case of turbulence associated with the waves themselves, such correlations exist.

If we now consider the turbulence associated with, for example, a wind-driven current, there will undoubtedly be some components in the spectrum of turbulence of scale comparable with, or smaller than, the wavelength of the waves. These components will contribute to $[u^2]$ and $[w^2]$ in equation (10) and may possibly have a finite value of $[uw]$. It is only, however, if the contributions to $[u^2]-[w^2]$ and $[uw]$ have components which vary in phase with $\partial U/\partial x$ and $\partial U/\partial z$ respectively, that they will add anything to the dissipation of the waves. There is, therefore, no real discrepancy between the simultaneous existence of a large degree of turbulence due to other causes and a comparatively small eddy viscosity applicable to wave motion.

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