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A NOTE ON THE ACCURACY OF THE MILD-SLOPE EQUATION

N. BOOIJ

Department of Civil Engineering, Delft University of Technology, Stevinweg 1, 2628 CN Delft (The Netherlands)

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ABSTRACT

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The mild-slope equation is a vertically integrated refraction-diffraction equation, used to predict wave propagation in a region with uneven bottom. As its name indicates, it is based on the assumption of a mild bottom slope. The purpose of this paper is to examine the accuracy of this equation as a function of the bottom slope. To this end a number of numerical experiments is carried out comparing solutions of the three-dimensional wave equation with solutions of the mild-slope equation.

For waves propagating parallel to the depth contours it turns out that the mild-slope equation produces accurate results even if the bottom slope is of order 1. For waves propagating normal to the depth contours the mild-slope equation is less accurate. The equation can be used for a bottom inclination up to 1:3.

1. INTRODUCTION

The engineer who needs to predict the wave conditions at the coast, has at his disposal the widely used refraction or ray method (see Skovgaard et al., 1975). According to this method the direction of the wave propagation is influenced by the slope of the bottom (currents are ignored). This method is very economical, but it has some disadvantages. Among these are the following: it does not consider diffraction effects, and it requires a very small bottom slope.

In order to overcome these drawbacks, Berkhoff (1972, 1976) developed a combined refraction and diffraction equation, which is now known as the mild-slope equation. Like the refraction equation, it is a vertically integrated model for periodic wave motion. It also assumes a small bottom slope, but this restriction is less strict than for the refraction equation. This restriction is formulated in qualitative terms; the present paper is concerned with a more quantitative determination of the maximum allowable slope.

To this end some calculations have been performed, both with the mild-slope equation and with the three-dimensional model, of which it forms

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a reduction. The three-dimensional model will accurately predict (apart from numerical errors) the wave motion, also for steep bottom slopes. The physical situation is chosen such that both refraction and diffraction terms come into play, and such that the three-dimensional computation can be carried out with reasonable computational effort. The selected situation involves periodic waves propagating over a prismatic slope connecting two regions of constant depth. The incident wave has a direction making a given angle with the depth contours. In this class of problems the dimensionality of the problem is reduced considerably. Time disappears as a result of the periodicity of the waves, and the spatial coordinate parallel to the depth contours disappears because it is assumed that the wave system is independent of this coordinate. The remaining coordinates are z, the vertical coordinate, and x, the coordinate perpendicular to the depth contours. In the mild-slope equation, the coordinate z is eliminated, since it is a vertically integrated model.

Section 2 of this note deals with the equations used in the computation; it can be skipped by those who are interested mainly in the results. Section 3 presents the numerical experiments and section 4 the conclusions.

2. THE EQUATIONS FOR THE WAVE PROPAGATION

Three-dimensional equations

The refraction-diffraction model is derived from the three-dimensional equations for irrotational linear wave motion (Berkhoff, 1972, 1976). Berkhoff's equation will be compared with this equation. Since the motion is assumed irrotational, a velocity potential φ exists. A derivation of the equations used in the sequel is found in Lamb (1963). The fluid is considered incompressible, so that the divergence of the velocity field vanishes. It follows that the potential obeys the Laplace equation:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \tag{1}$$

The bottom, which is defined by $z=-h(x,y)$, is assumed rigid and impermeable. The velocity component normal to the bottom surface vanishes, and thus the normal derivative of φ vanishes on the bottom:

$$\boldsymbol{n}\cdot\nabla\varphi = 0 \qquad \text{on} \quad z = -h(x,y) \tag{2}$$

$\boldsymbol{n}$ being the outward normal to the bottom surface.

Furthermore, the model is restricted to linear approximations. On the mean free surface, located at $z=0$, the following boundary condition holds:

$$\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial z} = 0 \qquad \text{on} \quad z = 0 \tag{3}$$

For the equations mentioned above, a solution exists which is purely harmonic in time. This is expressed by:

$$\varphi = \mathrm{Re}\,[e^{-i\omega t}\tilde{\varphi}(x,y,z)] \tag{4}$$

For the (complex) function $\tilde{\varphi}$ the condition at the surface transforms into:

$$-\omega^2\tilde{\varphi} + g\,\frac{\partial\tilde{\varphi}}{\partial z} = 0 \qquad \text{on } z = 0 \tag{5}$$

The equation for the interior of the fluid (eq. 1), and the boundary condition on the bottom (eq. 2) are not affected by the transition to the complex potential $\tilde{\varphi}$.

Vertically integrated equations

The vertically integrated model is derived from the equations for $\tilde{\varphi}$. It is assumed that the vertical distribution of the wave potential is as if the bottom were entirely horizontal. This is expressed by:

$$\tilde{\varphi} = \frac{\cosh\{k(h+z)\}}{\cosh(kh)}\,\Phi(x,y) \tag{6}$$

Berkhoff's procedure consists of the following steps: eq. 6 is substituted into the equations for $\tilde{\varphi}$. This expression is multiplied by an appropriate weighting function. This product is integrated over the depth to obtain the mild-slope equation. Due to the use of eq. 6 the resulting equation will be only approximately valid if the bottom is not horizontal. The function Φ depends only on x and y. It obeys the following partial differential equation, which is known as the mild-slope equation:

$$\frac{\partial}{\partial x}\left(G\,\frac{\partial\Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(G\,\frac{\partial\Phi}{\partial y}\right) + k^2 G\Phi = 0 \tag{7}$$

The conditions at the bottom and at the surface are incorporated into this equation. Frequency and depth enter the equation through the coefficient k, which is equal to the wave number, and G, the product of phase velocity and group velocity. k and G are calculated by means of the following relations:

$$\omega^2 = gk\tanh(kh) \tag{8}$$

$$G = g\,\frac{\frac{1}{4}\sinh(2kh) + \frac{1}{2}kh}{k\,\{\cosh(kh)\}^2} \tag{9}$$

It is noted that the vertical displacement of the free surface can be obtained from the wave potential at $z=0$ by:

$$Z(x,y,t) = \mathrm{Re}\left\{\frac{\mathrm{i}\omega}{g}\ \mathrm{e}^{-\mathrm{i}\omega t}\tilde{\varphi}(x,y,0)\right\}$$

Reduction to parallel depth contours

In the special case for which the numerical computations will be carried out, the depth is dependent on only one coordinate, so that $h = h(x)$. In the other direction y, which is parallel to the depth contours, the wave system is assumed periodic, so that:

$$\tilde{\varphi} = \mathrm{e}^{\mathrm{i}my}\ P(x,z) \tag{10}$$

for the three-dimensional model. For the vertically integrated model:

$$\Phi = \mathrm{e}^{\mathrm{i}my}\ Q(x) \tag{11}$$

From eq. 1 the following partial differential equation for the function $P(x,z)$ results:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} - m^2 P = 0 \tag{12}$$

with boundary conditions on bottom and surface as eqs. 2 and 5. The equation for $Q(x)$ follows from the mild-slope equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(G\,\frac{\mathrm{d}Q}{\mathrm{d}x}\right) - m^2 GQ + k^2 GQ = 0 \tag{13}$$

In addition, boundary conditions are needed to account for the incident wave and the waves radiated away. At this moment it is necessary to distinguish between the case of waves propagating parallel to the depth contours, and waves incident under some angle with these contours.

Oblique or normal incidence

The geometry of the model used in this section is shown in Fig. 1. It consists of two regions of constant depth, both extending to infinity, separated by a prismatic transition zone. Both boundaries of the computational model $x = 0$ and $x = W$ are located in regions with constant depth. The incident wave is coming from the constant depth region $x \leqslant 0$. The value of m is determined by the angle of incidence, which is equal to $\arcsin(m/k)$, where k is the wave number in the region $x \leqslant 0$.

In the regions of constant depth the wave motion consists of a progressive wave to the right and one to the left. For each progressive wave the wave number is found from eq. 8. The y-component of the vectorial wave number

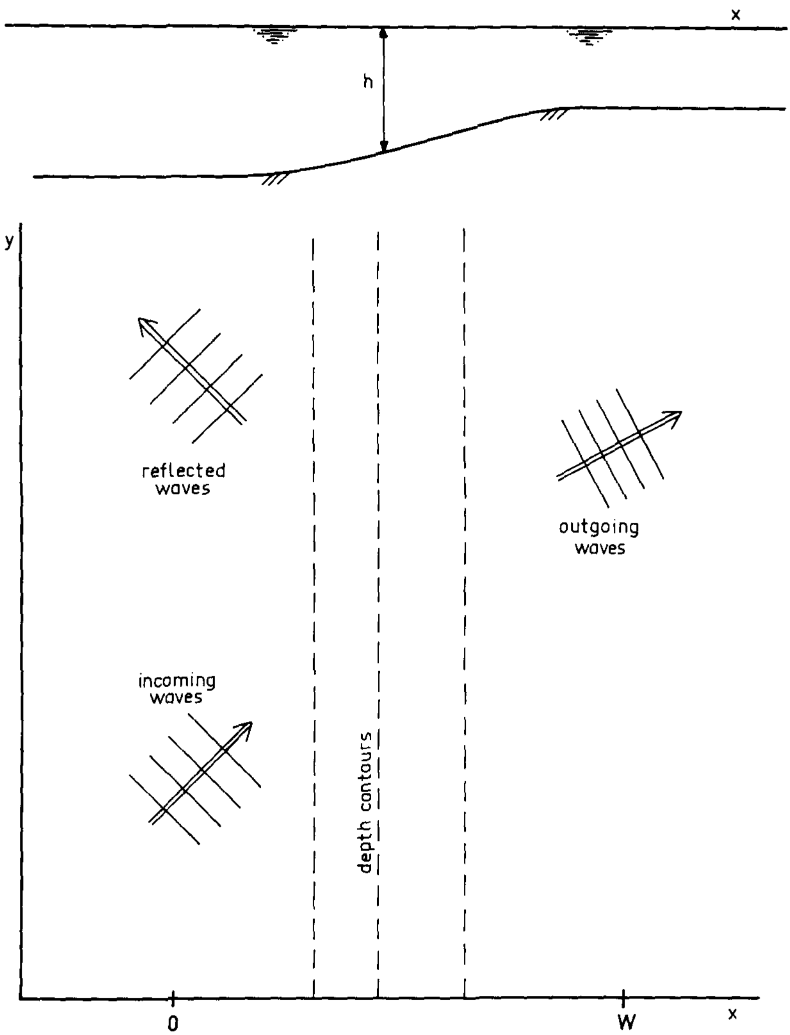

Fig. 1. Sketch of the wave system in the case of oblique incidence.

is known, equal to m. The x-component, which is called l, is found from:

$$l^2 + m^2 = k^2 \tag{14}$$

It is assumed that in the constant-depth region $x \geqslant W$ there is no reflected wave. So there exists only a wave propagating in positive direction, which obeys:

$$\frac{\partial P}{\partial x} = \mathrm{i}lP \qquad \text{at } x = W \tag{15}$$

The same boundary condition is used for the function Q.

At $x=0$ there is an incoming wave as well as a reflected wave. The incoming wave is assumed to have amplitude 1, without loss of generality due to the linearity of the equations. Since the boundary is located in a region with horizontal bottom the potential of the incident wave reads:

$$P_i = e^{ilx} \frac{\cosh\{k(z+h)\}}{\cosh(kh)}$$

The function P is the sum of the potentials of the incoming and the reflected waves. The relation for the reflected wave P_r is similar to eq. 15:

$$\frac{\partial P_r}{\partial x} = -ilP_r$$

Since $P=P_i+P_r$, the boundary condition for P reads:

$$\frac{\partial P}{\partial x} = -ilP + 2ilP_i \qquad \text{at } x = 0 \tag{16}$$

For Q the corresponding boundary condition is:

$$\frac{dQ}{dx} = -ilQ + 2ilQ_i \qquad \text{at } x = 0 \tag{17}$$

Waves parallel to depth contours

In the case of wave propagation parallel to the depth contours, a different geometry must be assumed. The model is now a prismatic wave channel bounded by fully reflecting walls (see Fig. 5). These walls are located at $x=0$ and $x=W$. The boundary conditions for this example are as follows:

$$\frac{\partial P}{\partial x} = 0 \qquad \text{at } x = 0 \text{ and } x = W \tag{18}$$

$$\frac{dQ}{dx} = 0 \qquad \text{at } x = 0 \text{ and } x = W \tag{19}$$

This time the number m cannot be determined beforehand. It is an unknown quantity that follows from the computation. Mathematically the set of equations forms an eigenvalue problem, with m as eigenvalue.

In the case of a channel with constant depth the main eigenvalue would be equal to k, the wave number associated with the depth in the channel. If the depth is not constant in lateral direction, m cannot be determined so easily. Its physical meaning is illustrated by the property that the computed wave pattern propagates (without deformation) through the channel with velocity ω/m.

3. THE NUMERICAL EXPERIMENTS

A number of numerical experiments are carried out in order to check the mild-slope equation against the three-dimensional model. The differential equations eqs. 12 and 13 are discretized by means of the Finite Element Method (see e.g. Zienkiewicz, 1977, or Connor and Brebbia, 1977), because of its flexibility regarding the representation of curved boundaries. Triangles are chosen as finite elements in the (x,z)-domain with shape functions linear over these elements. Figure 2 shows an example of this finite-element distribution. For the mild-slope equation the elements are intervals of the x-axis, but otherwise the method employed is the same.

In the case of propagation normal to the slope, the set of equations can also be solved by means of a boundary element method (see e.g. Salmon et al., 1980), which is more efficient as far as computational effort is concerned. This method is not as easily applicable to the case of oblique or normal incidence, due to the presence of the term m^2P (see eq. 12). Since finding a computationally efficient method for three-dimensional wave motion is not the primary aim of this study, the finite element method is considered adequate.

The numerical accuracy of both models was determined by varying the mesh size. In the case of the three-dimensional model it was also verified, by means of a variation of the location of the boundaries $x=0$ and $x=W$, whether the boundaries were chosen sufficiently far away from the slope. The numerical accuracy of the mild-slope equation was the best of the two. Due to its lower number of dimensions, a smaller mesh size could be used, leading to an accuracy about 10 times as high as in the three-dimensional model. The accuracy of the latter was about 0.5%. As far as the displacement of the free surface was concerned, this was satisfactory. For the reflection coefficients, calculated from the displacement, it meant a rather large error. Reflection coefficients of 2% or lower were unreliable.

The length measures in the model are all multiplied by the wave number in deep water $k_0=\omega^2/g$, in order to obtain non-dimensional quantities. In the test cases presented, the depths are chosen such that there is a transition from almost shallow to almost deep water, viz. from $k_0h=0.2$ to $k_0h=0.6$ (see Fig. 2).

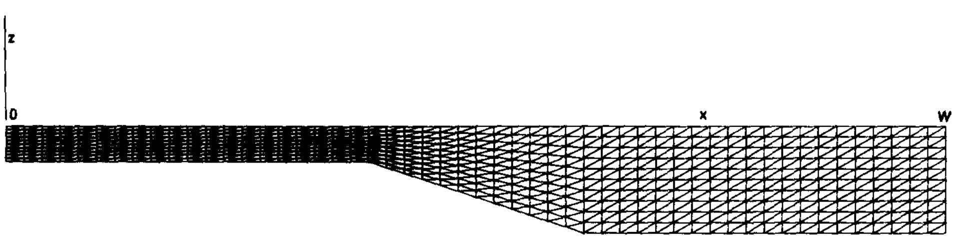

Fig. 2. Finite-element distribution used in a case of normal wave incidence.

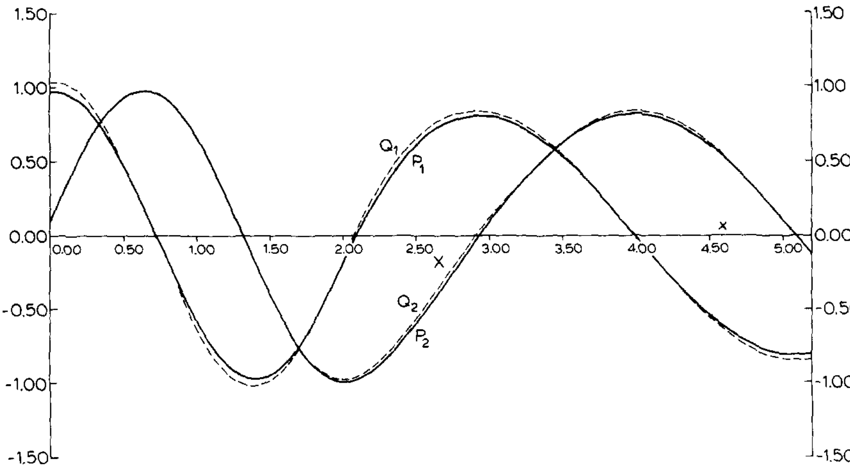

Fig. 3. Free-surface elevation computed by the mild-slope equation (broken line) and by the three-dimensional equation (solid line). $P_1=\text{Re}(P)$, $P_2=\text{Im}(P)$, three-dimensional model; $Q_1=\text{Re}(Q)$, $Q_2=\text{Im}(Q)$, mild-slope equation.

In the first series of tests, waves propagating normal to the depth contours ($m=0$) are considered. As an example, Fig. 3 shows the surface displacement for a 1:3-slope steepness. It is noted that the surface displacement is proportional to the potential. The quantities shown are the real and imaginary parts of the complex expression for the displacement. These can be interpreted as positions of the free surface at two times, a quarter of a period apart. The full line indicates the three-dimensional model and the dashed line the vertically integrated model. Both the amplitude and the phase differ slightly.

An important quantity related with the surface displacement is the reflection coefficient. Figure 4 shows how the reflection depends on the length of the slope, W_s, both with the mild-slope equation (full line) and with the three-dimensional model (crosses). It appears that the reflection coefficients calculated by both models are in good agreement for slopes with $\tan\alpha < 1/3$. Furthermore, the mild-slope equation predicts a reflection coefficient of the right order of magnitude, even for nearly vertical slopes. A third conclusion is that the refraction method, which would predict zero reflection, can only be used for very mild slopes. Figure 4 gives an indication of what slopes are acceptable when using the refraction method.

The second series of experiments pertains to wave propagation along the axis of a channel with uneven bottom. The depth contours are parallel to the channel axis. The depths at both sides of the transition are again $k_0h=0.6$ and $k_0h=0.2$, and the width of the channel is $k_0W=2$ in all cases (see Fig. 5). Results for different transition lengths k_0W_s are shown. This time the value of m is unknown, so that it can be used as a means to compare both models (see Table I). The number displayed in the table is

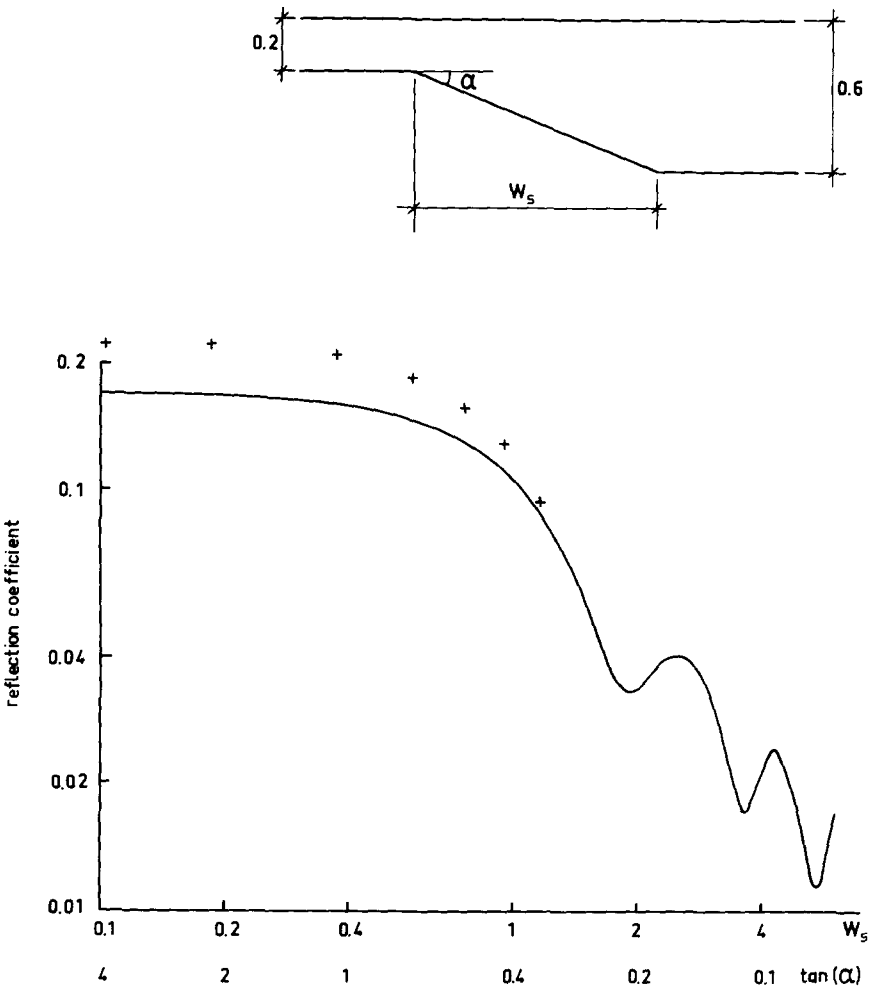

Fig. 4. Reflection coefficient as function of bottom inclination (normal incidence). The cross-section is shown in the upper part of the figure. Curve: refraction-diffraction model. Crosses: three-dimensional model.

m/k_0, a non-dimensional quantity which is equal to the reciprocal of the non-dimensionalized propagation velocity.

Figures 6, 7 and 8 show the elevation of the free surface for the same set of examples. It is curious to note that the correspondence of the three-dimensional model and the mild-slope equation is slightly better for the steeper slope, also with respect to the eigenvalue m. An explanation for this phenomenon was not found. In Figs. 6 to 8 no vertical scale is given. This is a consequence of the linearity of the wave equations used.

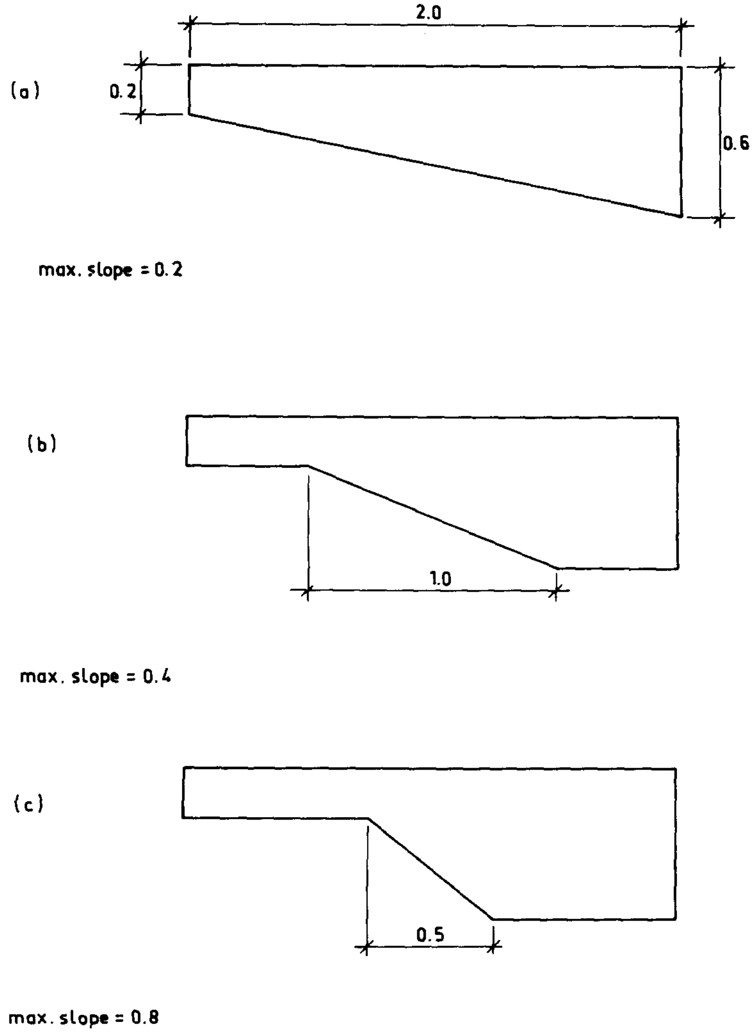

Fig. 5. Cross-sections of three wave channels, on non-distorted scale. Measures are non-dimensional.

TABLE I

(Non-dimensional) eigenvalue as function of lateral slope

Slope	Eigenvalue m/k_0	
	Vertical integrated model	Three-dimensional model
0.2	1.785	1.777
0.4	1.906	1.894
0.8	1.959	1.955

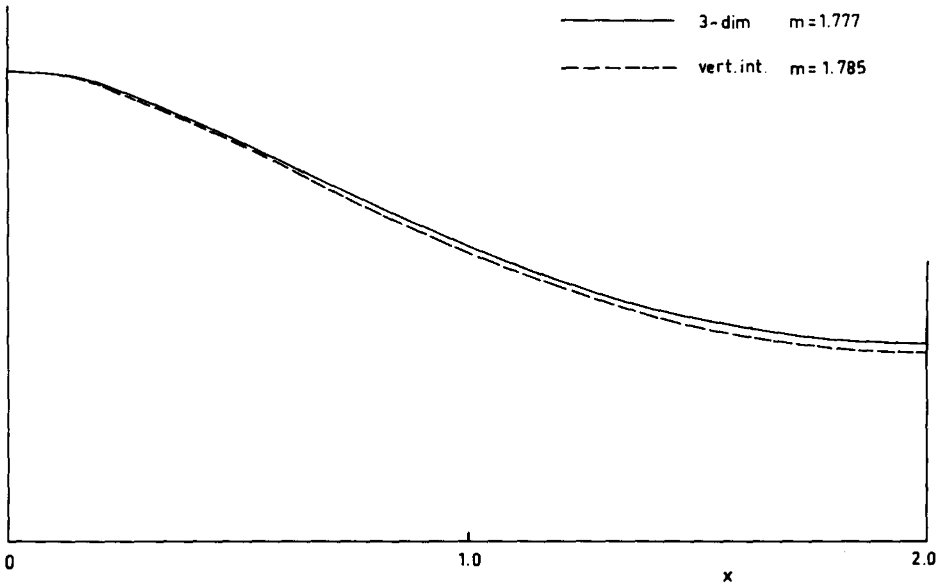

Fig. 6. Wave system in channel with maximum bottom steepness 0.2 (see cross-section (a) in Fig. 5).

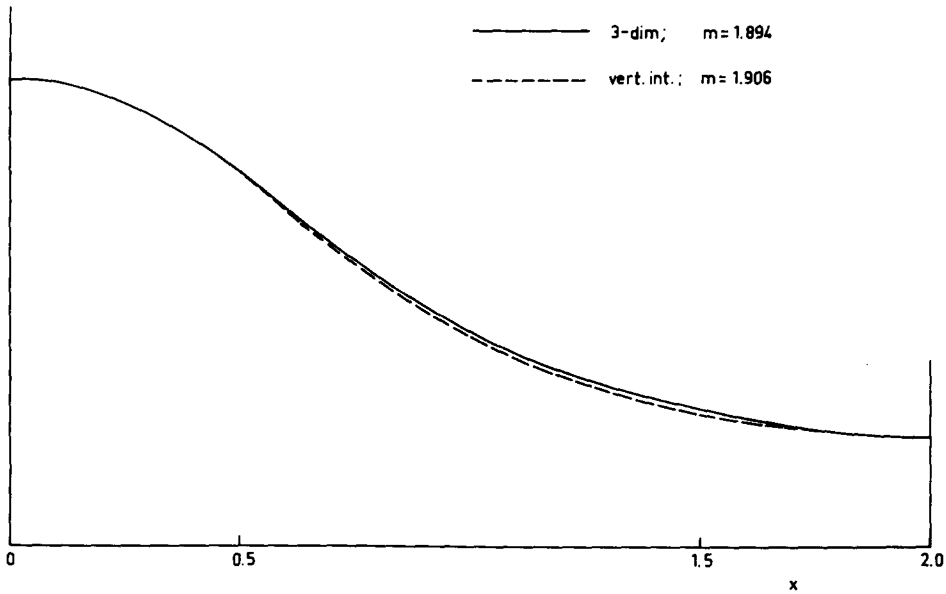

Fig. 7. Wave system in channel with maximum bottom steepness 0.4 (see cross-section (b) in Fig. 5).

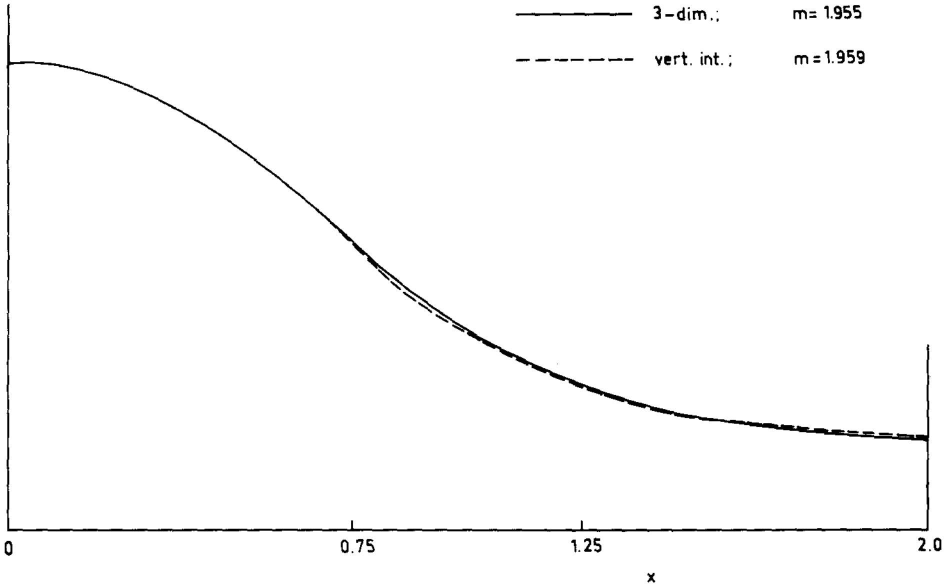

Fig. 8. Wave system in channel with maximum bottom steepness 0.8 (see cross-section (c) in Fig. 5).

4. CONCLUSIONS

A number of numerical experiments have been carried out to investigate the range of slopes for which the mild-slope equation can be used. One series of experiments is related to waves perpendicular to an undersea slope, the other to waves parallel to a slope. It is concluded from these experiments, that the mild-slope equation gives good results for slopes up to 1:3. In this respect the mild-slope equation is clearly superior to the refraction method, which would predict zero reflection in the first series of examples, and which cannot produce at all a solution of the type presented in the wave channel problem. The refraction method can only be used for very mild slopes.

Another point in favour of the mild-slope equation is that even for slopes steeper than 1:3 it gives a qualitatively correct representation of the wave field.

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