

A Multiscale Discontinuous Galerkin Method

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Abstract. We propose a new class of Discontinuous Galerkin (DG) methods based on variational multiscale ideas. Our approach begins with an additive decomposition of the discontinuous finite element space into continuous (coarse) and discontinuous (fine) components. Then, variational multiscale analysis is used to define an interscale transfer operator that associates coarse and fine scale functions. Composition of this operator with a donor DG method yields a new formulation that combines the advantages of DG methods with the attractive and more efficient computational structure of a continuous Galerkin method. The key to the success of the new approach is efficient computation of the interscale operator. Variational Multiscale Analysis leads to a natural definition of local, elementwise problems that mimic the structure of the donor DG formulation. The new class of DG methods is illustrated for a model scalar advection-diffusion boundary value problem.

1 Introduction

Discontinuous Galerkin (DG) methods offer several important and valuable computational advantages over their conforming Galerkin counterparts. The finite element spaces in DG methods are not subject to inter-element continuity conditions and local element spaces can be defined independently from each other. This makes DG methods particularly well-suited for application of h and p -adaptivity strategies. DG methods are also felt to have advantages of robustness over conventional Galerkin methods for first-order differential operators associated with hyperbolic equations [9–11]. It is also possible to link DG with the numerical fluxes (i.e., solutions of the one-dimensional Riemann problem) used

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in finite volume methods and develop higher-order accurate procedures for wave-propagation. There has also been recent interest in applying DG to elliptic problems so that advective-diffusive phenomena can be modeled; see Brezzi *et al.* [2], Dawson [6], and Hughes, Masud and Wan [8]. For a review of work in this area, see Arnold *et al.* [1]. For a valuable summary of the current state-of-the-art and introduction to the literature we refer to the collection [5].

Despite the increased interest in DG methods, there are shortcomings that limit their practical utility. Foremost among these is the size of the DG equation system for interpolations of linear and higher order. By virtue of the fact that the trial functions are completely discontinuous, there is no sharing of degrees-of-freedom at element boundaries. Consequently, the size of the solution space “explodes.” Storage and solution cost are, obviously, adversely affected, which seems the main reason for the small industrial impact the DG method has had so far.

In [7] we proposed a new multiscale DG method that has the computational structure of a standard continuous Galerkin method. This method was developed by starting from a given continuous finite element space and then associating it with a completely discontinuous space by releasing all continuity requirements. Using locally defined element problems we obtained parameterization of discontinuous degrees-of-freedom by their continuous counterparts and a DG method with a much improved computational efficiency.

In this paper we extend this idea to a general multiscale framework for Discontinuous Galerkin methods that leads to formulations with more efficient and flexible computational structures. Our approach starts with an additive decomposition of a given discontinuous finite element space into continuous (coarse) and discontinuous (fine) components. Then, variational multiscale analysis is used to define an interscale transfer operator that associates coarse and fine scale functions. Composition of this operator with a donor DG method yields a new formulation that combines the advantages of DG methods with the attractive and more efficient computational structure of a continuous Galerkin method. The key to the success of the new approach is efficient computation of the interscale operator. Variational Multiscale Analysis leads to a natural definition of local, elementwise problems that mimic the structure of the donor DG formulation. The new class of DG methods is illustrated for a model scalar advection-diffusion boundary value problem.

The paper is organized as follows. Section 2 introduces the relevant notation. The new multiscale DG approach is formulated and presented in Section 3, while Section 4 shows application of the new method to a model scalar advection-diffusion equation. Conclusions and future directions are outlined in Section 5.

2 Notation

Throughout this paper Ω will denote an open bounded region in $\mathbb{R}^n$, $n = 2, 3$ with a polyhedral boundary $\partial\Omega$. We recall the space $L^2(\Omega)$ of all square integrable functions and the space $H^1(\Omega)$ of all functions in $L^2(\Omega)$ that have square

integrable derivatives. To define approximation spaces, we consider a uniformly regular partition $\mathcal{T}_h$ of Ω into finite elements K . We assume that $\mathcal{T}_h$ contains only regular nodes [3], that is, each element vertex is also a vertex to all adjacent elements and there are no "hanging" nodes. For simplicity, we limit our discussion to two space dimensions where K can be a triangle T or a quadrilateral Q . Extension to three dimensions where the possible choices are tetrahedra, hexahedra, prisms or pyramids is straightforward.

Every element $K \in \mathcal{T}_h$ is an image of a reference element $\hat{K}$ that can be a triangle $\hat{T}$ or a square $\hat{Q}$. The vertices $\mathbf{v}$ and the edges $\mathbf{e}$ of K form the sets $V(K)$ and $E(K)$, with dimensions $n_K(V)$ and $n_K(E)$, respectively. The sets $V(\mathcal{T}_h)$ and $E(\mathcal{T}_h)$ contain the vertices and the edges in the partition $\mathcal{T}_h$, Γ_h^0 is the set of all internal edges and Γ_h is the set of all edges on $\partial\Omega$.

The local space. Let $\hat{K}$ denote a reference element. The reference space $S^{p(\hat{K})}(\hat{K})$ on $\hat{K}$ depends on the type of the element and is defined as follows:

$$S^{p(\hat{K})}(\hat{K}) = \begin{cases} \varphi = \sum_{i,j} a_{ij} \xi_1^i \xi_2^j, & 0 \leq i, j \leq p(\hat{K}); i+j \leq p(\hat{K}) & \text{if } \hat{K} = \hat{T} \\ \varphi = \sum_{i,j} a_{ij} \xi_1^i \xi_2^j, & 0 \leq i, j \leq p(\hat{K}) & \text{if } \hat{K} = \hat{Q} \end{cases} \quad (1)$$

The local element spaces $S^{p(K)}(K)$ are defined in the usual manner by a mapping of the reference space (1) to the physical coordinate frame. While the reference space is always polynomial, the local space $S^{p(K)}(K)$ may consist of non-polynomial functions. This is the case when $\mathcal{T}_h$ is a non-affine family of finite elements.

We assume that $S^{p(K)}(\hat{K})$ is endowed with a hierarchical basis [3] consisting of $n_{\hat{K}}(V)$ *vertex* shape functions $\{\hat{V}_{\mathbf{v}}(\boldsymbol{\xi})\}$, $(p(K)-1)n_{\hat{K}}(E)$ *edge* shape functions $\{\hat{E}_{\mathbf{e}}(\boldsymbol{\xi})\}$ and $n_{\hat{K}}(B)$ *bubble* shape functions $\{\hat{B}_i(\boldsymbol{\xi})\}$. The vertex and edge shape functions have non-zero trace on the element boundary $\partial\hat{K}$ and are associated with the *external* degrees-of-freedom on $\hat{K}$. The bubble shape functions have vanishing traces on $\partial\hat{K}$ and define the *internal* degrees-of-freedom on $\hat{K}$. The reference hierarchical basis defines a local hierarchical basis $\{V_i(\mathbf{x})\} \cup \{E_i(\mathbf{x})\} \cup \{B_i(\mathbf{x})\}$ for $S^{p(K)}(K)$ so that every $\varphi_h \in S^{p(K)}(K)$ can be written as

$$\varphi_h(\mathbf{x}) = \sum_{\mathbf{v} \in V(\hat{K})} v_{\mathbf{v}} V_{\mathbf{v}}(\mathbf{x}) + \sum_{\mathbf{e} \in E(\hat{K})} e_{\mathbf{e}} E_{\mathbf{e}}(\mathbf{x}) + \sum_k b_k B_k(\mathbf{x}) \quad (2)$$

We recall that the vertex basis functions are interpolatory and so $v_{\mathbf{v}}$ are nodal degrees-of-freedom. The other basis functions may be associated with generalized momenta of the function and so $e_{\mathbf{e}}$ and b_k are not necessarily nodal values.

The discontinuous finite element space. We consider the following finite element subspace of $L^2(\Omega)$

$$\Phi_h(\Omega) = \left\{ \varphi_h \in L^2(\Omega) \mid \varphi_h|_K \in S^{p(K)}(K); \forall K \in \mathcal{T}_h \right\}. \quad (3)$$

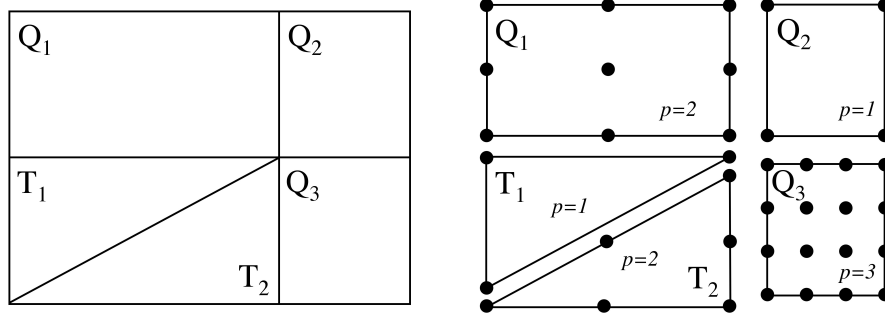


Fig. 1. Partition $\mathcal{T}_h$ of Ω into finite elements (left) and the space $\Phi_h(\Omega)$ (right)

The functions in $\Phi_h(\Omega)$ are not constrained by inter-element continuity and their polynomial degrees can be chosen independently for each element K ; see Fig. 2. The integers $p_{\max} = \max_{K \in \mathcal{T}_h} p(K)$ and $p_{\min} = \min_{K \in \mathcal{T}_h} p(K)$ will denote the maximal and minimal degree of the local spaces in $\Phi_h(\Omega)$. We will assume that $p_{\min} \geq 1$, i.e., the local spaces are defined by at least linear or bilinear reference polynomials.

Note that $\Phi_h(\Omega)$ is a formal union of the local spaces $S^{p(K)}(K)$. The union of the local hierarchical basis functions forms a hierarchical basis of $\Phi_h(\Omega)$ and so, every $\varphi_h \in \Phi_h(\Omega)$ can be expanded as

$$\varphi_h(\mathbf{x}) = \sum_{K \in \mathcal{T}_h} \left(\sum_{\mathbf{v} \in V(K)} a_{\mathbf{v}} V_{\mathbf{v}}(\mathbf{x}) + \sum_{\mathbf{e} \in E(K)} e_{\mathbf{e}} E_{\mathbf{e}}(\mathbf{x}) + \sum_i b_i B_i(\mathbf{x}) \right) \quad (4)$$

The continuous finite element space. The additive⁴ decomposition of $\Phi_h(\Omega)$, required in the application of variational multiscale analysis, is induced by a finite element subspace $\bar{\Phi}_h(\Omega)$ of $H^1(\Omega)$, defined with respect to the same partition $\mathcal{T}_h$ of Ω into finite elements. Given a $\mathcal{T}_h$, the space $\bar{\Phi}_h(\Omega)$ can be defined in many possible ways. However, to ensure H^1 conformity, functions in this space are constrained to be continuous across element interfaces; see [4]. Consequently, polynomial degrees on adjacent elements may not be selected independently of each other. Here, for simplicity we consider a minimal choice of $\bar{\Phi}_h(\Omega)$ given by (see Fig. 2)

$$\bar{\Phi}_h(\Omega) = \{\bar{\varphi}_h \in H^1(\Omega) \mid \bar{\varphi}_h|_K \in S^1(K)\}. \quad (5)$$

In $\bar{\Phi}_h(\Omega)$ we consider a standard nodal basis $\{\bar{V}_{\bar{\mathbf{v}}}\}$; $\bar{\mathbf{v}} \in V(\mathcal{T}_h)$ of shape functions such that

$$\bar{V}_{\bar{\mathbf{v}}_i}(\bar{\mathbf{v}}_j) = \delta_{ij}.$$

⁴ Variational multiscale analysis can be applied to multiplicative decompositions as well, however, this is not necessary for the purposes of this paper.

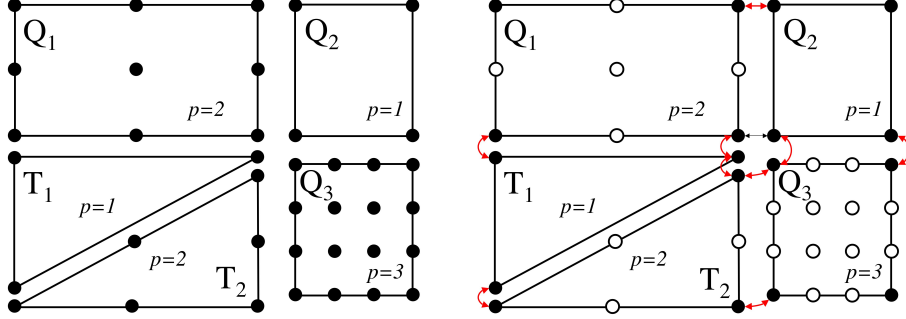


Fig. 2. The space $\Phi_h(\Omega)$ (left) and the corresponding minimal C^0 space $\bar{\Phi}_h(\Omega)$ (right).

These basis functions have local supports given by $\text{supp}(\bar{V}_{\bar{\mathbf{v}}}) = \cup_{\mathbf{v} \in V(K)} K$. It is easy to see that for $K \in \text{supp}(\bar{V}_{\bar{\mathbf{v}}})$

$$\bar{V}_{\bar{\mathbf{v}}}|_K = V_{\mathbf{v}}$$

where $\mathbf{v} \in V(K)$ is the local vertex that corresponds to the global vertex $\bar{\mathbf{v}} \in V(\mathcal{T}_h)$. Owing to the assumption $p_{\min} \geq 1$ the space $\bar{\Phi}_h(\Omega)$ is contained in $\Phi_h(\Omega)$. This is not essential to the application of variational multiscale analysis where we only require the availability of an additive decomposition for $\Phi_h(\Omega)$. Thus, while the actual choice of $\bar{\Phi}_h(\Omega)$ and the resulting decomposition will have an impact on the accuracy of the multiscale DG, it will not affect formulation of the overall framework.

Orientations, jumps and averages. In Discontinuous Galerkin methods solution values on adjacent elements are coupled by virtue of generalized flux functions that include jump and average terms. We briefly review the relevant notation following the Brezzi conventions. We assume that all edges in $E(\mathcal{T}_h)$ are endowed by orientation. A convenient way to orient an edge is to pick a normal direction to that edge; see Fig. 3. An element can be oriented by selecting one of the two possible normal directions to its boundary ∂K as a source (outward normal) or as a sink (inward normal). Without loss of generality, we agree to orient all elements as sources.

An internal edge $\mathbf{e} \in \Gamma_h^0$ is shared by exactly two elements. The outward normal on one of these elements will coincide with the normal used to orient $\mathbf{e}$; we call this element K^- . The outward normal on the other element will have the opposite direction to the normal on $\mathbf{e}$; we call this element K^+ ; see Fig. 3. Edge orientation also induces partition of the boundary of an internal element into ∂K^+ , consisting of all edges whose normal direction coincides with the outer normal on ∂K and ∂K^- , consisting of all edges $\mathbf{e}$ whose normal direction is opposite to the outer normal on ∂K . These conventions for labeling elements

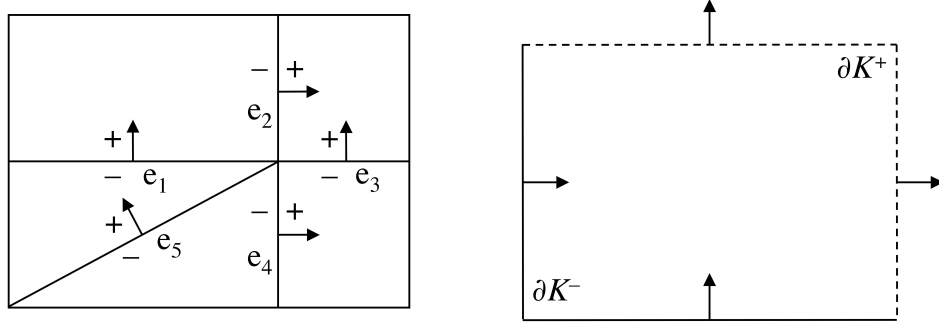


Fig. 3. Orientation of internal edges in $\mathcal{T}_h$ and $+/-$ elements with respect to an edge (left). Partition of element boundary into ∂K^+ and ∂K^- (right).

and their boundaries can be explained by viewing edge normals as advective directions. Then, K^- is the *upwind* element relative to the edge and K^+ is the *downwind* element relative to that edge. Likewise, ∂K^- is the *inflow* part of ∂K and ∂K^+ is the outflow element boundary. This notation can be extended in an obvious manner to external edges $\mathbf{e} \in \Gamma_h$ and elements K that have such edges.

Let φ be a scalar field, and $\varphi^\pm := \varphi|_{K^\pm}$. For $\mathbf{e} \in \Gamma_h^0$ we define the *average* and the *jump* as

$$\langle \varphi \rangle := \frac{1}{2}(\varphi^+ + \varphi^-) \quad \text{and} \quad \llbracket \varphi \rrbracket := \varphi^+ \mathbf{n}^+ + \varphi^- \mathbf{n}^-, \quad (6)$$

respectively. Analogously, if $\mathbf{u}$ is a vector field,

$$\langle \mathbf{u} \rangle := \frac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-) \quad \text{and} \quad \llbracket \mathbf{u} \rrbracket := \mathbf{u}^+ \cdot \mathbf{n}^+ + \mathbf{u}^- \cdot \mathbf{n}^-. \quad (7)$$

Note that, by definition of “ $\llbracket \cdot \rrbracket$ ”, the jump of a scalar quantity is a vector and the jump of a vector quantity is a scalar. Jump definitions do not depend on the ordering of the elements. For edges belonging to Γ_h we also have the specializations

$$\llbracket \varphi \rrbracket = \varphi \mathbf{n}, \quad \langle \mathbf{u} \rangle = \mathbf{u}, \quad \forall \mathbf{e} \in \Gamma_h. \quad (8)$$

It will not be necessary to define $\langle \varphi \rangle$ and $\llbracket \mathbf{u} \rrbracket$ on the boundary Γ , because they are never utilized.

3 Multiscale Discontinuous Galerkin Method

We consider an abstract linear boundary value problem

$$\mathcal{L}(\mathbf{x}, D)\varphi = f \text{ in } \Omega \quad \text{and} \quad \mathcal{R}(\mathbf{x}, D)\varphi = g \text{ on } \Gamma. \quad (9)$$

where $\mathcal{L}(\mathbf{x}, D)$ and $\mathcal{R}(\mathbf{x}, D)$ are a differential and a boundary operators, respectively.

The multiscale DG framework for the approximate numerical solution of (9) has two basic components. First, we assume that there is a well-posed *donor* Discontinuous Galerkin formulation for (9) that reads: *find* $\varphi_h \in \Phi_h(\Omega)$ *such that*

$$B_{DG}(\varphi_h; \psi_h) = F_{DG}(\psi_h) \quad \forall \psi_h \in \Phi_h(\Omega). \quad (10)$$

In (10), $B_{DG}(\cdot; \cdot)$ is a continuous bilinear form $\Phi_h(\Omega) \times \Phi_h(\Omega) \mapsto \mathbb{R}$ and $F_{DG}(\cdot)$ is a bounded linear functional $\Phi_h(\Omega) \mapsto \mathbb{R}$. We assume that (10) has a unique solution φ_h that depends continuously on the data and converges (in a suitable norm) to all sufficiently smooth solutions φ of (9).

The second component in our framework is an interscale transfer (or expansion) operator

$$T : \bar{\Phi}_h(\Omega) \mapsto \Phi_h(\Omega). \quad (11)$$

We assume that T is a bounded linear operator, however, it is not required to be surjective, or invertible. Thus, in general $T(\bar{\Phi}_h(\Omega))$ will be a proper subspace of the discontinuous space $\Phi_h(\Omega)$.

We define the Multiscale Discontinuous Galerkin (MDG) method by a composition of the donor DG scheme with the interscale transfer operator T . The MDG variational problem is: *find* $\bar{\varphi}_h \in \bar{\Phi}_h(\Omega)$ *such that*

$$B_{DG}(T\bar{\varphi}_h; T\bar{\psi}_h) = F_{DG}(T\bar{\psi}_h) \quad \forall \bar{\psi}_h \in \bar{\Phi}_h(\Omega). \quad (12)$$

Substitution of discontinuous test and trial functions in the donor DG method by images $T\bar{\psi}_h$ and $T\bar{\varphi}_h$ of continuous finite element functions has the effect of reducing the number of degrees-of-freedom in the MDG formulation to that of a standard Galerkin method posed on the C^0 space $\bar{\Phi}_h(\Omega)$. Because T is not surjective, its range does not reproduce completely the discontinuous space. However, this operator acts to expand the continuous space in a way that admits a wider range of possible solutions. Thus, the new MDG formulation occupies a middle ground between a DG and a conventional Galerkin method for (9).

It is also clear that the key to a robust, efficient and accurate MDG method is the definition of the interscale operator T . For instance, it is desirable to compute T in a local, elementwise manner, instead of by a global solution procedure that would require inversion of a large matrix. In the next section we consider a variational multiscale approach to the definition of this operator.

3.1 Definition of the interscale operator

We assume that the bilinear form in the donor DG method is given by

$$B_{DG}(\varphi_h; \psi_h) = \sum_{K \in \mathcal{T}_h} B_K(\varphi_h; \psi_h) + \sum_{\mathbf{e} \in \Gamma_h} B_\Gamma(\varphi_h; \psi_h) + \sum_{\mathbf{e} \in \Gamma_h^0} B_{\mathbf{e}}(\{\varphi_h^-, \varphi_h^+\}; \{\psi_h^-, \psi_h^+\}) \quad (13)$$

where $B_K(\cdot; \cdot)$ is a bilinear *local* element form defined for every $K \in \mathcal{T}_h$, $B_\Gamma(\cdot; \cdot)$ is a bilinear boundary form defined on the boundary edges, and $B_{\mathbf{e}}(\{\cdot\}; \{\cdot\})$ is

an *edge* bilinear form that depends only on the values from the two elements that share edge $\mathbf{e}$. The local element form serves to define the weak form of (9) on each element. The boundary form $B_\Gamma(\cdot; \cdot)$ may be used, for example, to weakly enforce certain types of boundary conditions. The purpose of the numerical flux form, on the other hand, is to couple the local element problems into a single DG variational equation.

To define the interscale operator T we proceed to formally split functions $\varphi_h \in \Phi_h(\Omega)$ into a continuous (“coarse” scale) part $\bar{\varphi}_h \in \bar{\Phi}_h(\Omega)$ and a discontinuous (“fine” scale) component $\varphi'_h \in \Phi_h(\Omega)$, viz.

$$\varphi_h = \bar{\varphi}_h + \varphi'_h. \quad (14)$$

The additive decomposition in (14) is formal in the sense that neither one of the two components is fixed. Our goal will be to define a problem that relates the fine scale component φ'_h to the coarse scale function $\bar{\varphi}_h$. The operator T will then be defined as the solution operator for this problem.

Presently, we proceed to substitute test and trial functions in the donor DG problem (10) by their split representations (14). Then, (10) takes the following form:

$$\begin{aligned} B_{DG}(\bar{\varphi}_h; \bar{\psi}_h) + B_{DG}(\varphi'_h; \bar{\psi}_h) &= F_{DG}(\bar{\psi}_h) \quad \forall \bar{\varphi}_h \in \bar{\Phi}_h(\Omega) \\ B_{DG}(\varphi'_h; \psi'_h) + B_{DG}(\bar{\varphi}_h; \psi'_h) &= F_{DG}(\psi'_h) \quad \forall \psi'_h \in \Phi_h(\Omega) \end{aligned} \quad (15)$$

The first line in (15) is the coarse scale equation. The second line is the fine scale equation that will be used to define T . Treating the coarse scale function as data we write this equation as: *find* $\varphi'_h \in \Phi_h(\Omega)$ *such that*

$$B_{DG}(\varphi'_h; \psi'_h) = F_{DG}(\psi'_h) - B_{DG}(\bar{\varphi}_h; \psi'_h) \quad \forall \psi'_h \in \Phi_h(\Omega). \quad (16)$$

We restrict (16) to an element K by choosing test functions $\psi'_h \in S^{p(K)}(K)$ that vanish outside of this element. To state the fine scale problem in terms of the forms and fluxes in (13) note that with the above selection of a test function,

$$(\psi'_h)^+ = \begin{cases} \psi'_h & \text{for } \mathbf{e} \in \partial K^- \\ 0 & \text{for } \mathbf{e} \in \partial K^+ \end{cases} \quad \text{and} \quad (\psi'_h)^- = \begin{cases} 0 & \text{for } \mathbf{e} \in \partial K^- \\ \psi'_h & \text{for } \mathbf{e} \in \partial K^+ \end{cases}$$

Succinctly,

$$(\psi'_h)^+ = \chi(\partial K^-) \psi'_h \quad \text{and} \quad (\psi'_h)^- = \chi(\partial K^+) \psi'_h \quad (17)$$

where $\chi(\cdot)$ is the characteristic function. Using (17) and that

$$(\bar{\varphi}_h)^+ = (\bar{\varphi}_h)^- = \bar{\varphi}_h,$$

for a C^0 function, the restricted fine scale problem can be expressed as follows:
find $\varphi'_h \in S^{p(K)}(K)$ such that

$$\begin{aligned}
& B_K(\varphi'_h; \psi'_h) + B_\Gamma(\varphi'_h; \psi'_h) + \\
& \sum_{\mathbf{e} \in E(K)} B_{\mathbf{e}}(\{(\varphi'_h)^-, (\varphi'_h)^+\}; \{\chi(\partial K^+) \psi'_h, \chi(\partial K^-) \psi'_h\}) \\
& = F_{DG}(\psi'_h) - B_K(\bar{\varphi}_h; \psi'_h) - B_\Gamma(\bar{\varphi}_h; \psi'_h) \\
& - \sum_{\mathbf{e} \in E(K)} B_{\mathbf{e}}(\{\bar{\varphi}_h, \bar{\varphi}_h\}; \{\chi(\partial K^+) \psi'_h, \chi(\partial K^-) \psi'_h\}) \quad \forall \psi'_h \in S^{p(K)}(K).
\end{aligned} \tag{18}$$

Problem (18) relates fine scales to the coarse scales, but remains coupled to the contiguous elements through the numerical flux terms in (18). Therefore, it does not meet our criteria for localized computation of the interscale transfer operator T . However, we make the important observation that our goal is not to solve the DG problem (15) but rather use it to define a local computation procedure for T that maps $\bar{\varphi}_h$ into the local space $S^{p(K)}(K)$. We note that this objective is reminiscent of other applications of variational multiscale framework in which the fine scale problem is used for *estimation* rather than *approximation* of the unresolved solution component. Thus, we may consider redefinition of numerical fluxes that will uncouple (18) from the contiguous elements. This process can be accomplished by a modification of the numerical flux inherited from the donor DG formulation, or by using a new flux defined only in terms of the local function $\varphi'_h \in S^{p(K)}(K)$. Let $B'_{\mathbf{e}}(\{\cdot\}; \{\cdot\})$ be the new numerical flux. The local fine scale problem obtained from (18) is: *find $\varphi'_h \in S^{p(K)}(K)$ such that*

$$\begin{aligned}
& B_K(\varphi'_h; \psi'_h) + B_\Gamma(\varphi'_h; \psi'_h) + \sum_{\mathbf{e} \in E(K)} B'_{\mathbf{e}}(\{\varphi'_h\}; \{\psi'_h\}) \\
& = F_{DG}(\psi'_h) - B_K(\bar{\varphi}_h; \psi'_h) - B_\Gamma(\bar{\varphi}_h; \psi'_h) \\
& - \sum_{\mathbf{e} \in E(K)} B_{\mathbf{e}}(\{\bar{\varphi}_h, \bar{\varphi}_h\}; \{\chi(\partial K^+) \psi'_h, \chi(\partial K^-) \psi'_h\}) \quad \forall \psi'_h \in S^{p(K)}(K).
\end{aligned} \tag{19}$$

Problem (19) is a local equation that can be solved on an element by element basis. This problem defines an operator $T_K : \bar{\Phi}_h(\Omega) \mapsto S^{p(K)}(K)$ that maps any given C^0 finite element function $\bar{\varphi}_h$ to a function in the local element space $S^{p(K)}(K)$. Therefore,

$$T : \bar{\Phi}_h(\Omega) \mapsto \Phi_h(\Omega); \quad T|_K = T_K \quad \forall K \in \mathcal{T}_h \tag{20}$$

defines an interscale transfer operator T for the MDG method.

The abstract variational equation (12) and the local problem (19) complete the definition of the MDG framework. In the next section this framework is illustrated by an application to a scalar advection-diffusion equation.

4 Multiscale DG for a scalar advection-diffusion problem

We consider a model advection diffusion problem written in conservative form as

$$\begin{cases} \nabla \cdot (F_a + F_d) = f \text{ in } \Omega \\ \varphi = g \text{ on } \Gamma_g \\ -(F_a + F_d) \cdot \mathbf{n} = h^- \text{ on } \Gamma_h^- \\ -(F_d) \cdot \mathbf{n} = h^+ \text{ on } \Gamma_h^+ \end{cases} \quad (21)$$

where

$$F_d = -\kappa \nabla \varphi \quad \text{and} \quad F_a = \mathbf{a} \varphi$$

denote diffusive and advective flux, respectively. The total flux is $F = F_a + F_d$. The Neumann boundary condition can be written compactly as

$$-(\chi(\Gamma_h^-)F_a + F_d) \cdot \mathbf{n} = h; \quad h = \chi(\Gamma_h^-)h^- + \chi(\Gamma_h^+)h^+.$$

4.1 A donor DG method for the model problem

When dealing with advection-diffusion problems it is profitable to coordinate edge orientations with the advective direction. Given an edge $\mathbf{e}$ we choose the normal $\mathbf{n}_{\mathbf{e}}$ for which $\mathbf{n}_{\mathbf{e}} \cdot \mathbf{a} \geq 0$, that is, the normal that points in the direction of $\mathbf{a}$.

A general weighted residual form of a Discontinuous Galerkin method for (21) is given by: find $\varphi \in \Phi_h(\Omega)$ such that

$$\begin{aligned} & \sum_{i=1}^{n_{el}} - \int_{K_i} (F_i \cdot \nabla \psi_i + f \psi) d\Omega + \int_{\Gamma_h} (\chi(\Gamma_h^+) F_a \cdot \mathbf{n} - h) \psi_i d\mathbf{l} + \int_{\Gamma_g} (F \cdot \mathbf{n}) \psi d\mathbf{l} \\ & + \epsilon \int_{\Gamma_g} (\varphi - g) W(\psi) d\mathbf{l} \quad \rightarrow \quad \text{Weak Dirichlet condition} \\ & + \sum_{\mathbf{e} \in \Gamma_h^0} \int_{\mathbf{e}} \mathcal{F}_b(\varphi^+; \varphi^-) (\psi^+ - \psi^-) d\mathbf{l} \quad \rightarrow \quad \text{Flux balance} \\ & + \sum_{\mathbf{e} \in \Gamma_h^0} \int_{\mathbf{e}} \mathcal{F}_c(\psi^+; \psi^-) (\varphi^+ - \varphi^-) d\mathbf{l} \quad \rightarrow \quad \text{Continuity of } \varphi \\ & + \sum_{\mathbf{e} \in \Gamma_h^0} \int_{\mathbf{e}} \alpha \llbracket \varphi \rrbracket \llbracket \psi \rrbracket d\mathbf{l} \quad \rightarrow \quad \text{LS stabilization} \\ & = 0 \quad \forall \psi \in \Phi_h(\Omega). \end{aligned} \quad (22)$$

Above, $\mathcal{F}_b$ and $\mathcal{F}_c$ are interface functions defined on the internal edges Γ_h^0 , whose purpose is to impose flux balance and solution continuity, respectively. $W(\psi)$ is a weight function that enforces the Dirichlet boundary condition weakly. A general form of the interface functions is

$$\mathcal{F}_b = F_b^h \cdot \mathbf{n}^+ = -F_b^h \cdot \mathbf{n}^- \quad \text{and} \quad \mathcal{F}_c = F_c^h \cdot \mathbf{n}^+ = -F_c^h \cdot \mathbf{n}^- \quad (23)$$

where

$$F_b^h \stackrel{\text{def}}{=} s_{11}F_a^h + s_{12}F_d^h \quad \text{and} \quad F_c^h \stackrel{\text{def}}{=} s_{21}F_a^h + s_{22}F_d^h \quad (24)$$

are numerical models of the total flux across $\mathbf{e} \in \Gamma_h^0$ and

$$F_a^h \stackrel{\text{def}}{=} F_a^h(\varphi^+, \varphi^-) \quad \text{and} \quad F_d^h \stackrel{\text{def}}{=} F_d^h(\varphi^+, \varphi^-) \quad (25)$$

are constitutive relations for the advective and the diffusive fluxes across $\mathbf{e}$ in terms of the solution states φ^+ and φ^- from the two elements that share $\mathbf{e}$. Using (23), the flux balance terms in (22) can be expressed as

$$\begin{aligned} \int_{\mathbf{e}} \mathcal{F}_b(\varphi^+; \varphi^-)(\psi^+ - \psi^-) d\mathbf{l} &= \int_{\mathbf{e}} (F_b^h(\varphi^+; \varphi^-) \cdot \mathbf{n}^+) \psi^+ + (F_b^h(\varphi^+; \varphi^-) \cdot \mathbf{n}^-) \psi^- d\mathbf{l} \\ &= \int_{\mathbf{e}} F_b^h(\varphi^+; \varphi^-) \cdot (\psi^+ \mathbf{n}^+ + \psi^- \mathbf{n}^-) d\mathbf{l} \\ &= \int_{\mathbf{e}} F_b^h(\varphi^+; \varphi^-) \cdot \llbracket \psi \rrbracket d\mathbf{l}. \end{aligned}$$

Similar expression exist for the continuity of φ terms:

$$\int_{\mathbf{e}} \mathcal{F}_c(\psi^+; \psi^-)(\varphi^+ - \varphi^-) d\mathbf{l} = \int_{\mathbf{e}} F_c^h(\psi^+; \psi^-) \cdot \llbracket \varphi \rrbracket d\mathbf{l}.$$

Using the last two identities (22) takes the following form:

$$\begin{aligned} &\sum_{i=1}^{n_{el}} - \int_{K_i} (F_i \cdot \nabla \psi_i + f \psi) d\Omega \\ &+ \int_{\Gamma_h} (\chi(\Gamma_h^+) F_a \cdot \mathbf{n} - h) \psi_i d\mathbf{l} + \int_{\Gamma_g} (F \cdot \mathbf{n}) \psi d\mathbf{l} + \epsilon \int_{\Gamma_g} (\varphi - g) W(\psi) d\mathbf{l} \quad (26) \\ &+ \sum_{\mathbf{e} \in \Gamma_h^0} \int_{\mathbf{e}} (F_b^h(\varphi^+; \varphi^-) \cdot \llbracket \psi \rrbracket + F_c^h(\psi^+; \psi^-) \cdot \llbracket \varphi \rrbracket + \alpha \llbracket \varphi \rrbracket \llbracket \psi \rrbracket) d\mathbf{l} = 0. \end{aligned}$$

The component bilinear forms in (13) can be easily identified from (26):

$$B_K(\varphi; \psi) = \int_K -F_K \cdot \nabla \psi_K d\Omega \quad (27)$$

$$B_\Gamma(\varphi; \psi) = \int_{\Gamma_h} (\chi(\Gamma_h^+) F_a \cdot \mathbf{n}) \psi d\mathbf{l} + \int_{\Gamma_g} (F \cdot \mathbf{n}) \psi d\mathbf{l} + \epsilon \int_{\Gamma_g} \varphi W(\psi) d\mathbf{l} \quad (28)$$

$$\begin{aligned} B_{\mathbf{e}}(\{\varphi^+; \varphi^-\}; \{\psi^+; \psi^-\}) &= \\ &\int_{\mathbf{e}} (F_b^h(\varphi^+; \varphi^-) \cdot \llbracket \psi \rrbracket + F_c^h(\psi^+; \psi^-) \cdot \llbracket \varphi \rrbracket + \alpha \llbracket \varphi \rrbracket \llbracket \psi \rrbracket) d\mathbf{l}. \end{aligned} \quad (29)$$

Remark 1. It is possible to consider modifications of the flux definitions on the Neumann and Dirichlet boundaries of the domain Ω that enter definition of $B_\Gamma(\cdot; \cdot)$. However, for consistency with a continuous weak formulation of (21), on Γ_h and Γ_g we retain the problem defined fluxes.

Table 1. Specialization of fluxes and weight function for the donor DG methods.

Function	DG-A	DG-B
$F_b^h(\varphi^+; \varphi^-)$	$F_a(\varphi^-) + \langle F_d(\varphi) \rangle$	$F_a(\varphi^-) + F_d(\varphi^-)$
$F_c^h(\psi^+; \psi^-)$	$s \langle F_d(\psi) \rangle$	$s F_d(\psi^-)$
$W(\psi)$	$\psi + s F_d(\psi) \cdot \mathbf{n}$	

A particular donor DG method is obtained from (26) by specification of the numerical fluxes in (23)-(25) for the internal edges Γ_h^0 , and the weight function $W(\psi)$. A standard choice for F_a^h is the upwinded advective flux

$$F_a^h(\psi^+; \psi^-) = F_a(\psi^-) = \mathbf{a}\varphi^-.$$

Possible choices for the numerical diffusive flux are the averaged flux

$$F_d^h(\psi^+; \psi^-) = \langle F_d(\psi) \rangle = -\frac{1}{2} (\kappa \nabla \psi^+ + \kappa \nabla \psi^-)$$

or the upwinded flux

$$F_d^h(\psi^+; \psi^-) = F_d(\psi^-) = -\kappa \nabla \psi^-,$$

To define F_b^h and F_c^h we set $s_{11} = s_{12} = 1$, $s_{21} = 0$ and $s_{22} = s \in \{-1, 0, +1\}$ in (24). This leads to two different donor DG methods: DG-A which uses averaged diffusive flux, and DG-B which uses the upwinded version of that flux; see [7]. Flux and weight function definitions for the two methods are summarized in Table 1.

For DG-B the numerical flux F_b^h is simply the upwinded total flux $F(\varphi^-)$. DG-A and DG-B have the same element form $B_K(\cdot; \cdot)$ (given by (28)) and the same boundary form:

$$B_\Gamma(\varphi; \psi) = \int_{\Gamma_h} (\chi(\Gamma_h^+) F_a \cdot \mathbf{n}) \psi \, d\mathbf{l} + \int_{\Gamma_g} (F \cdot \mathbf{n}) \psi \, d\mathbf{l} + \epsilon \int_{\Gamma_g} \underbrace{\varphi (\psi - s \kappa \nabla \psi \cdot \mathbf{n})}_{W(\psi)} \, d\mathbf{l} \quad (30)$$

The internal edge form for DG-A is

$$B_e(\{\varphi^+; \varphi^-\}; \{\psi^+; \psi^-\}) = \int_e \alpha \llbracket \varphi \rrbracket \llbracket \psi \rrbracket \, d\mathbf{l} + \int_e \left(\underbrace{(\mathbf{a}\varphi^- - (\kappa \nabla \varphi^+ + \kappa \nabla \varphi^-)/2)}_{F_b^h} \cdot \llbracket \psi \rrbracket - \underbrace{s(\kappa \nabla \psi^+ + \kappa \nabla \psi^-)/2 \cdot \llbracket \varphi \rrbracket}_{F_c^h} \right) \, d\mathbf{l} \quad (31)$$

while for DG-B this form is given by

$$B_e(\{\varphi^+; \varphi^-\}; \{\psi^+; \psi^-\}) = \int_e \alpha \llbracket \varphi \rrbracket \llbracket \psi \rrbracket d\mathbf{l} + \int_e \left(\underbrace{(\mathbf{a}\varphi^- - \kappa \nabla \varphi^-) \cdot \llbracket \psi \rrbracket}_{F_b^h} - \underbrace{s\kappa \nabla \psi^- \cdot \llbracket \varphi \rrbracket}_{F_c^h} \right) d\mathbf{l}. \quad (32)$$

4.2 The interscale operator

The local problem (19), that defines the interscale transfer operator T , is obtained from the donor DG formulation by a selection of an alternative edge form $B'_e(\{\cdot\}; \{\cdot\})$. While, in principle, this form can be obtained independently, we prefer to develop a consistent approach that reduces the edge form $B_e(\{\cdot\}; \{\cdot\})$ in the donor DG method to a form defined in terms of the local (fine scale) variable φ' and test function ψ' . In doing so we aim to preserve as much as possible from the structure of the donor DG method in the local problem, so as to obtain a reliable estimate of the fine scale solution.

To develop such a reduction process we need to consistently redefine the calculation of the jump, the average and the states $\varphi^\pm$, $\psi^\pm$ in the numerical fluxes. Here we adopt the following convention for the calculation of these quantities in terms of test and trial functions on a *single* element K : given $\psi \in S^{p(K)}(K)$ its states are defined by

$$\psi^+ = \chi(\partial K^-)\psi \quad \text{and} \quad \psi^- = \chi(\partial K^+)\psi \quad (33)$$

its jump is the vector

$$\llbracket \psi \rrbracket = \mathbf{n}_K \psi, \quad (34)$$

and its average is the function itself:

$$\langle \psi \rangle = \psi. \quad (35)$$

The rules in (33)-(35) have the following interpretation. To compute the states and the jump of ψ , extend by zero to a function $\psi_0 \in L^2(\Omega)$. Using (17)

$$\llbracket \psi_0 \rrbracket = \mathbf{n}^+ \chi(\partial K^-)\psi_0 + \mathbf{n}^- \chi(\partial K^+)\psi_0 = \mathbf{n}_K \psi_0,$$

which is precisely (34). Definition (35) can be motivated by noting that for affine elements ψ is a polynomial function that can be trivially extended to a function $\psi_\infty \in C^\infty(\Omega)$ for which

$$\langle \psi_\infty \rangle = \frac{1}{2}(\psi_\infty + \psi_\infty) = \psi_\infty$$

giving (35). The local definitions of the numerical fluxes obtained through (33)-(35) are summarized in Table 2.

Table 2. Specialization of fluxes for the local problem.

Function	DG-A	DG-B
$F_b^h(\varphi)$	$F_a(\chi(\partial K^+)\varphi) + F_d(\varphi)$	$F_a(\chi(\partial K^+)\varphi) + F_d(\chi(\partial K^+)\varphi)$
$F_c^h(\psi)$	$sF_d(\psi)$	$sF_d(\chi(\partial K^+)\psi)$

Local problem for DG-A The localized edge form for DG-A method is

$$B'_e(\{\varphi\}; \{\psi\}) = \int_e \left(\underbrace{(\mathbf{a}\chi(\partial K^+)\varphi - \kappa\nabla\varphi) \cdot \mathbf{n}_K \psi}_{F_b^h} - \underbrace{s\kappa\nabla\psi \cdot \mathbf{n}_K \varphi}_{F_c^h} + \alpha\varphi\psi \right) dl. \quad (36)$$

The last two terms can be combined into a single weight function

$$W_\alpha(\psi) = \alpha\psi - s\kappa\nabla\psi \cdot \mathbf{n}_K,$$

similar to the weight function $W(\psi)$ used in the weak Dirichlet boundary condition. Thus, the local problem obtained from DG-A is: given a $\bar{\varphi} \in \bar{\Phi}_h(\Omega)$ find $\varphi' \in S^{p(K)}(K)$ such that

$$\begin{aligned} & B_K(\varphi'; \psi') + B_\Gamma(\varphi'; \psi') + \\ & \sum_{e \in \partial K} \int_e \left((\mathbf{a}\chi(\partial K^+)\varphi' - \kappa\nabla\varphi') \cdot \mathbf{n}_K \psi' + \varphi' W_\alpha(\psi') \right) dl \\ & = F_{DG}(\psi') - B_K(\bar{\varphi}; \psi') - B_\Gamma(\bar{\varphi}; \psi') \\ & - \sum_{e \in \partial K} B_e(\{\bar{\varphi}, \bar{\varphi}\}; \{\chi(\partial K^+)\psi', \chi(\partial K^-)\psi'\}) \quad \forall \psi' \in S^{p(K)}(K). \end{aligned} \quad (37)$$

Remark 2. This local problem is identical to the one used in [7].

Local problem for DG-B For DG-B we have the following localized edge form:

$$B'_e(\{\varphi\}; \{\psi\}) = \int_e \left(\underbrace{\chi(\partial K^+)(\mathbf{a}\varphi - \kappa\nabla\varphi) \cdot \mathbf{n}_K \psi}_{F_b^h} - \underbrace{s\chi(\partial K^+)\kappa\nabla\psi \cdot \mathbf{n}_K \varphi}_{F_c^h} + \alpha\varphi\psi \right) dl. \quad (38)$$

The last two terms can be combined into the weight function

$$W_\alpha^-(\psi) = \alpha\psi - s\chi(\partial K^+)\nabla\psi \cdot \mathbf{n}_K,$$

which can be viewed as an "upwinded" version of $W_\alpha(\psi)$ defined earlier. The local problem associated with DG-B is: given a $\bar{\varphi} \in \bar{\Phi}_h(\Omega)$ find $\varphi' \in S^{p(K)}(K)$ such that

$$\begin{aligned}
& B_K(\varphi'; \psi') + B_\Gamma(\varphi'; \psi') + \\
& \sum_{\mathbf{e} \in \partial K} \int_{\mathbf{e}} \left(\chi(\partial K^-) (\mathbf{a}\varphi' - \kappa \nabla \varphi') \cdot \mathbf{n}_K \psi' + \varphi' W_\alpha(\psi') \right) d\mathbf{l} \\
& = F_{DG}(\psi') - B_K(\bar{\varphi}; \psi') - B_\Gamma(\bar{\varphi}; \psi') \\
& - \sum_{\mathbf{e} \in \partial K} B_{\mathbf{e}}(\{\bar{\varphi}, \bar{\varphi}\}; \{\chi(\partial K^-)\psi', \chi(\partial K^+)\psi'\}) \quad \forall \psi' \in S^{p(K)}(K).
\end{aligned} \tag{39}$$

5 Conclusions

In this work we extended the DG method developed in [7] to a general framework for multiscale DG methods that have the computational structure of continuous Galerkin methods. This represents a solution to a fundamental and long-standing problem in discontinuous-Galerkin technology, namely, restraining the proliferation of degrees-of-freedom. Numerical results reported in [7] indicate that for a scalar advection-diffusion equation the new method at least attains, and even somewhat improves upon, the performance of the associated continuous Galerkin method. Within the framework of the multiscale discontinuous Galerkin method, the local problem provides a vehicle for incorporating the necessary stabilization features such as discontinuity capturing and upwinding. There seems to be a potential connection here with ideas from wave propagation methods based on solutions of the Riemann problem, which is worth exploring in more detail.

The MDG formulation can be also viewed as an approach that enables uncoupling of *storage* locations of the data from the *computational* locations where this data is used. For example, one can envision a situation where information is stored at the nodes and then mapped to flux and circulation degrees-of-freedom by the operator T . Such an extension of MDG appears to be a fruitful direction for further research.

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