Local properties of sea waves derived from a wave record

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It is demonstrated that for linear deep sea waves with small directional scattering the particle motion at the sea surface and energy transmission may be retrieved from a wave record by means of the Hilbert transform. A physical interpretation of the envelope of the two-dimensional deep sea waves as well as a new method for wave group analysis is presented.

Key Words: Sea waves, wave record, wave energy, wave groupiness, Hilbert transform.

INTRODUCTION

In 1961 Phillips[1] derived a number of expressions for properties of the three dimensional random motion in terms of quantities measured on the free surface at a fixed horizontal location. As an illustration of the use of these expressions in finding lowest-order approximations he related the mean energy-flux vector to the spectrum of surface displacements.

Some expressions derived in this paper are of a type somewhat similar to those obtained by Phillips. The objective of this study is however to show that in the first approximation properties of the deep sea waves with small directional scattering can be derived directly from a wave record by means of the Hilbert transform without calculating the power spectrum. By a complex representation of the wave field, which has been adopted from the communication theory,[2] and by means of the Hilbert transform a physical interpretation is related to the envelope of the two-dimensional deep sea waves. The analysis shows that for the two-dimensional and narrow-band motion the local kinetic energy fluctuates in time as a sea surface envelope squared multiplied by a constant factor and can be obtained directly from a wave record. It is demonstrated that the local fluctuations of the potential energy for the two-dimensional motion as well as the local fluctuations of the total energy for the two-dimensional and narrow band motion can be separated into a slowly varying part and a more rapid oscillating part. Both parts can be evaluated by means of the Hilbert transform.

The paper shows that the Hilbert transform may also be used to analyse wave groupiness.

The derived formulae are illustrated by Wave Wider Bouy data from the Norwegian Continental Shelf.

ASSUMPTIONS

The analysis carried out in this paper is limited to a linear statistical model of the sea waves[2] and to deep water. It is also assumed that the deep sea waves are characterized by small directional scattering so they can be described by the two-dimensional motion. According to the linear statistical model, which treats the sea surface oscillations as a random process stationary in time and homogeneous in space, the sea surface displacement can be represented by an ordinary sum of finite number of waves

$$\zeta(x,t) = \sum_{n=1}^{N} a_n \cos(k_n x - \sigma_n t + \epsilon_n) \tag{1}$$

where a_n = amplitude, k_n = horizontal wave number, x = horizontal Cartesian co-ordinate, σ_n = frequency, ϵ = phase.

For deep water σ_n is related to k_n by

$$\sigma_n^2 = g k_n \tag{2}$$

Corresponding to the free surface elevation (1) is a velocity potential

$$\phi(x,z,t) = \sum_{n=1}^{N} \frac{a_n \sigma_n}{k_n} \exp(k_n z) \sin(k_n x - \sigma_n t + \epsilon_n) \tag{3}$$

where z = vertical co-ordinate.

COMPLEX REPRESENTATION OF THE SEA SURFACE

Within the first approximation most wave recorders of the bouy and staff type register the vertical motion of the sea surface. For the two-dimensional motion the output signal is the vertical displacement (1).

Knowing the vertical displacement (1) we can find the envelope of the sea surface (as defined by Longuet-Higgins[3]) by a complex representation of the wave field called an analytical signal. This complex representation of real polychromatic fields, which was introduced in 1946 by Gabor, is used frequently in communication theory and is described for example by Born and Wolf.[4]

If we have the local vertical sea surface displacement $\zeta(x,t)$ (x is treated as a parameter), then we can uniquely specify a conjugate signal $\xi(x,t)$ being obtained from $\zeta(x,t)$ by shifting the phase of each elementary harmonic of $\zeta(x,t)$ by $\pi/2$. The $\zeta(x,t)$ and $\xi(x,t)$ may be shown to be Hilbert transform of each other

$$\xi(x,t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\zeta(x,\tau)}{\tau - t} \, d\tau \tag{4}$$

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$$\zeta(x,t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\xi(x,\tau)}{\tau - t} \,\mathrm{d}\tau \tag{5}$$

where P denotes the Cauchy principal value at $\tau = t$. Numerically this corresponds to a convolution with a linear filter with the time response function[5] as shown in Fig. 1.

With $\zeta(x,t)$ we may associate the complex function

$$z(x,t) = \zeta(x,t) + i\xi(x,t) \tag{6}$$

Complex functions may generally be expressed on polar form and the phase function may formally be written as $\psi(x,t) - \sigma_0 t$, where σ_0 is quite arbitrary frequency. Hence

$$z(x,t) = A(x,t)\exp\{i[\psi(x,t) - \sigma_0 t]\} \tag{7}$$

where the envelope

$$A(x,t) = \sqrt{\zeta^2(x,t) + \xi^2(x,t)} = |z(x,t)| \tag{8}$$

and the associated phase

$$\psi(x,t) = \sigma_0 t + \mathrm{Arctg}\,\frac{\zeta(x,t)}{\xi(x,t)} \tag{9}$$

Conversely

$$\zeta(x,t) = A(x,t)\cos(\psi(x,t) - \sigma_0 t) \tag{10}$$

$$\xi(x,t) = A(x,t)\sin(\psi(x,t) - \sigma_0 t) \tag{11}$$

Applying then the Hilbert transform to the sea surface (1) and treating x as a parameter one obtains

$$\zeta^2(x,t) = \sum_m \sum_n a_m a_n \cos(k_m x - \sigma_m t + \epsilon_m) \times \cos(k_n x - \sigma_n t + \epsilon_n) \tag{12}$$

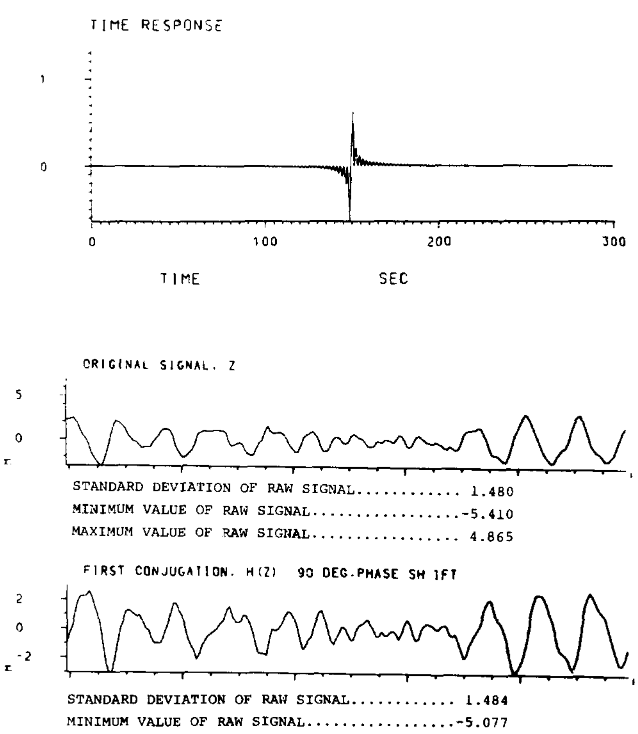

Figure 1. Signal conjugation by the Hilbert transform (Wave Rider record)

$$\xi^2(x,t) = \sum_m \sum_n a_m a_n \sin(k_m x - \sigma_m t + \epsilon_m) \times \sin(k_n x - \sigma_n t + \epsilon_n) \tag{13}$$

According to (8)

$$A^2(x,t) = \sum_m \sum_n a_m a_n \cos[(k_n - k_m)x + (\sigma_n - \sigma_m)t + (\epsilon_n - \epsilon_m)] \tag{14}$$

$A^2(x,t)$ propagates with an average of velocities

$$c_{nm} = \frac{\sigma_n - \sigma_m}{|k_n - k_m|} \tag{15}$$

weighted with $a_n a_m$.

SEA SURFACE PARTICLE MOTION

We can easily note that for the two-dimensional wave motion $\xi(x,t)$ represents a horizontal displacement of the surface particle

$$X(x,t) = \xi(x,t) = -\sum_n a_n \sin(k_n x - \sigma_n t + \epsilon_n) \tag{16}$$

The vertical velocity of the sea surface particle is thus

$$\dot{\zeta}(x,t) = \sum_n a_n \sigma_n \sin(k_n x - \sigma_n t + \epsilon_n) \tag{17}$$

and the horizontal velocity

$$\dot{X}(x,t) = \sum_n a_n \sigma_n \cos(k_n x - \sigma_n x + \epsilon_n) \tag{18}$$

We are free to transform surface motion from Cartesian coordinates to polar coordinates. The squared radial displacement R^2 is hence obtained from (16) and (1)

$$R^2(x,t) = X^2(x,t) + \zeta^2(x,t) = \sum_m \sum_n a_m a_n \cos[(k_n - k_m)x + (\sigma_n - \sigma_m)t + (\epsilon_n - \epsilon_m)] \tag{19}$$

The expression (19) is identical with the envelope squared (14) for two-dimensional motion and it propagates with the same velocity as $A^2(t)$.

Similarly we may find the absolute particle velocity squared

$$U^2(x,t) = \dot{X}^2(x,t) + \dot{\zeta}^2(x,t) = \sum_m \sum_n \sigma_m \sigma_n a_m a_n \cos[(k_n - k_m)x + (\sigma_n - \sigma_m)t + (\epsilon_n - \epsilon_m)] \tag{20}$$

Comparing with (19) shows that the squared absolute velocity propagates with the same speed and in the same phase as the radial particle displacement squared.

At any time at a fixed point x there will exist a local mean frequency

$$\bar{\sigma}^2(x,t) = U^2(x,t)/R^2(x,t) \tag{21}$$

where $U^2(x,t)$ as in (20), and a local mean period

$$\bar{T}(t) = 2\pi/\bar{\sigma}(x,t) \tag{22}$$

Thus the squared radial displacement R^2 and the local frequency $\bar{\sigma}$ may to some extent be used as alternative

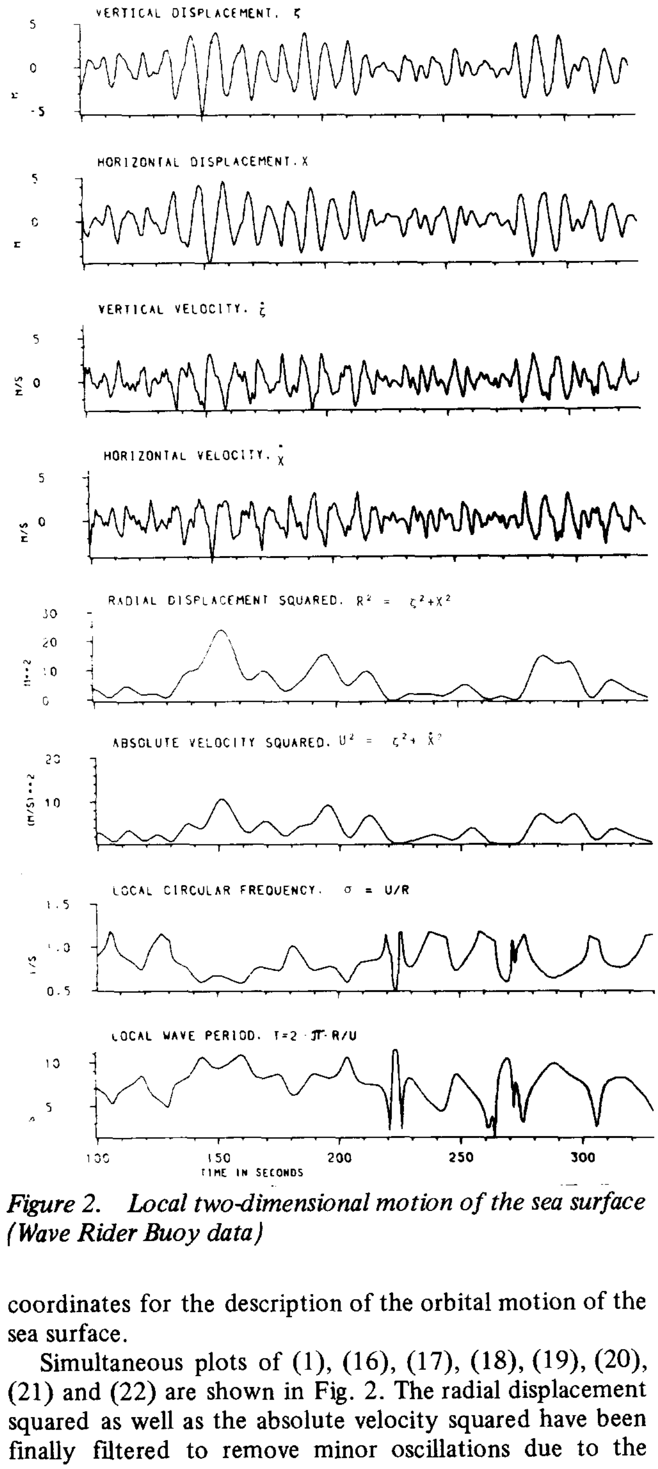

Figure 2. Local two-dimensional motion of the sea surface (Wave Rider Buoy data)

coordinates for the description of the orbital motion of the sea surface.

Simultaneous plots of (1), (16), (17), (18), (19), (20), (21) and (22) are shown in Fig. 2. The radial displacement squared as well as the absolute velocity squared have been finally filtered to remove minor oscillations due to the individual waves.

PULSE AND GROUP TRAINS

We can now introduce a wave group concept which is independent of the width of the spectrum. The squared vertical displacement (12) may be formally rewritten[6]

$$\begin{aligned}\zeta^2(x,t) &= \frac{1}{2}\sum_m \sum_n a_m a_n \cos\left[(k_m+k_n)x-(\sigma_m+\sigma_n)t\right.\\ &\quad \left.+(\epsilon_m+\epsilon_n)\right] + \frac{1}{2}\sum_m\sum_n a_m a_n \cos\left[(k_n-k_m)x\right.\\ &\quad \left.-(\sigma_n-\sigma_m)t+(\epsilon_n-\epsilon_m)\right]\\ &= Y(x,t)+\tfrac{1}{2}R^2(x,t) \qquad (23)\end{aligned}$$

(see equation (19)).

The first term $Y(x, t)$ has zero mean value. It propagates with an average of velocities

$$(\sigma_n+\sigma_m)/(k_n+k_m) = g(\sigma_n+\sigma_m)/(\sigma_n^2+\sigma_m^2) \qquad (24)$$

weighted with $a_n a_m$. The expression (24) is of order equal to the phase velocity.

The last term in (23) is one half of the square radial displacement/the squared envelope and it is varying more slowly than the first one.

Similarly the horizontal squared displacement (16) may be rewritten to

$$X^2(x,t) = -Y(x,t)+\tfrac{1}{2}R^2(x,t) \qquad (25)$$

The pulse train Y may numerically be evaluated as

$$Y(x,t) = \tfrac{1}{2}[\zeta^2(x,t)-X^2(x,t)] \qquad (26)$$

The expression (23) can be used to analyse wave groupiness and it can be compared with the Funke and Mansard method.[6] Funke and Mansard have used the Naess[7] expansion of $\zeta^2(t)$ in four terms in which the last one is identical with the second term of (23). To evaluate it they have, after subtracting $\bar{\zeta}^2$, used the Bartlett filtering. The Bartlett filtering allows to eliminate the oscillations caused by the sum frequencies but in this process the exact zero level of the envelope is usually lost. The present method seems to be more convenient and it does not require the narrow-band spectrum assumption. The disadvantage of this method is however the assumption that the motion is two-dimesional. The method can be easily extended to three-dimensional motion but a physical interpretation of the more slowly varying part must then be revised.

LOCAL FLUCTUATIONS OF SEA WAVE ENERGY

The potential energy per unit area at a fixed point x fluctuates rapidly in time according to a function

$$E_p(x,t) = \tfrac{1}{2}\rho g\zeta^2(x,t) = \tfrac{1}{2}\rho g \sum_m \sum_n a_m a_n \times \cos(k_m x-\sigma_n t+\epsilon_m)\cos(k_n x-\sigma_n t+\epsilon_n) \qquad (27)$$

Following (23) this may be rewritten

$$E_p(x,t) = \tfrac{1}{2}\rho g Y(x,t)+\tfrac{1}{4}\rho g R^2(x,t) \qquad (28)$$

The second term of (28) is always positive while the first has zero mean value.

The local kinetic energy per unit area is

$$E_k(x,t) = \tfrac{1}{2}\rho g \int_{-\infty}^{0} (\mathbf{u})^2\, dz \qquad (29)$$

where $\mathbf{u} = \mathbf{u}(u, w)$ – two-dimensional velocity vector

$$u(x,z,t) = \frac{\partial\phi}{\partial x} = \sum_{n=1}^{N} a_n\sigma_n \exp(k_n z)\cos(k_n x-\sigma_n t+\epsilon_n) \qquad (30)$$

$$w(x,z,t) = \frac{\partial\phi}{\partial z} = \sum_{n=1}^{N} a_n\sigma_n \exp(k_n z)\sin(k_n x-\sigma_n t+\epsilon_n) \qquad (31)$$

Thus

$$E_K(x,t) = \tfrac{1}{2}\rho \sum_m \sum_n a_m a_n \frac{\sigma_m \sigma_n}{k_m + k_n} \cos[(k_n - k_m)x + (\sigma_n - \sigma_m)t + (\epsilon_n - \epsilon_m)] \quad (32)$$

For the narrow-band process

$$\frac{\sigma_m \cdot \sigma_n}{k_m + k_n} \simeq \tfrac{1}{2}g \quad (33)$$

and

$$E_k(x,t) \simeq \tfrac{1}{4}\rho g A^2(x,t) \quad (34)$$

(see equation (14)). The kinetic energy propagates then, for the narrow-band and two-dimensional motion, with the same velocity as the envelope squared/the radial displacement squared and is equal to the convective part of the potential energy. For the two-dimensional motion the difference between $E_k(x,t)$ and $A^2(x,t)$ depends on the width of the spectrum. The local kinetic energy will however always fluctuates more slowly than the potential one.

The total local energy per unit area for the narrow-band and two-dimensional motion

$$E(x,t) = E_p(x,t) + E_k(x,t) = \tfrac{1}{2}\rho g Y(x,t) + \tfrac{1}{2}\rho g A^2(x,t) \quad (35)$$

$E(x,t)$ may thus be separated in a rapid varying part which we may call a pulse term (the term which describes local pulsation of energy) and a more slowly oscillating part which we may call a group term (the term which describes the energy packets transported across the considered area). Both terms can be derived by means of the Hilbert transform from a wave record.

The fluctuations of sea wave energy are illustrated in Fig. 3. The minor oscillations in the group term has been finally filtered the same as in Fig. 2.

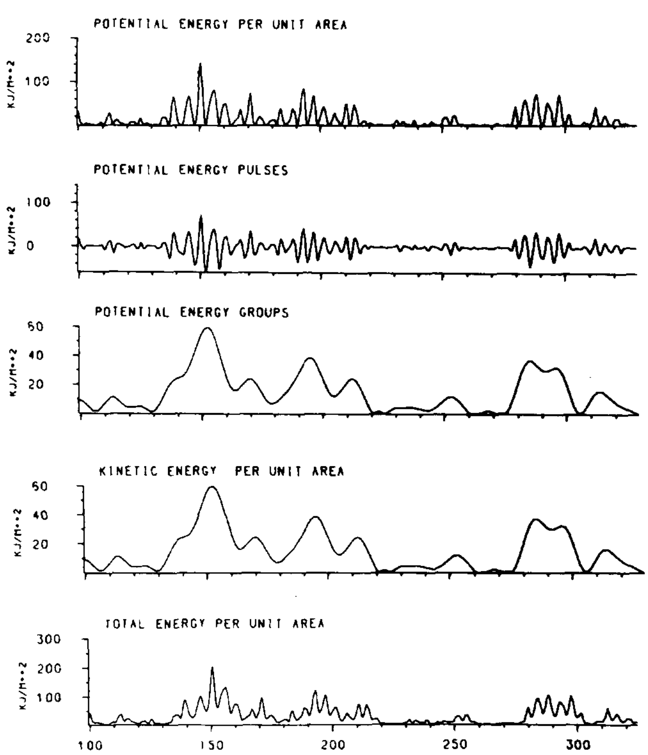

Fig. 3 Local sea wave energy fluctuations for the two-dimensional and narrow-band motion (Wave Rider Buoy data).

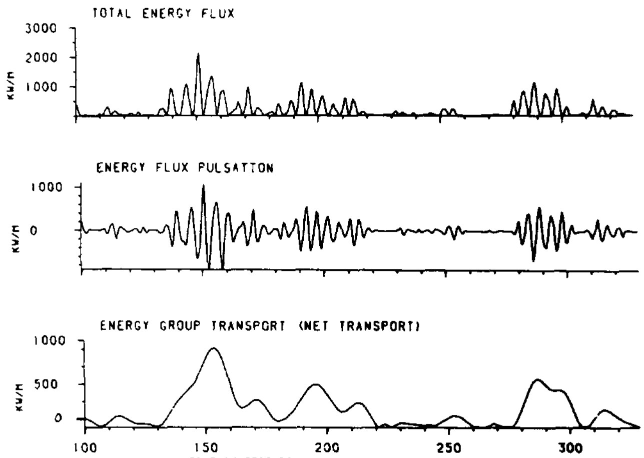

Figure 4. Local energy flux fluctuations for the two-dimensional wave motion (Wave Rider Buoy data)

ENERGY FLUX

The flux of energy of the long crested waves carried across a vertical plane x = constant in direction of the wave motion, per unit length of crest, is[9]

$$F(x,t) = \int_{-\infty}^{0} -\rho \frac{\partial \phi}{\partial t} u \, dz$$

$$= \int_{-\infty}^{0} \sum_m \sum_n \rho g a_m a_n \cos(k_m x - \sigma_m t + \epsilon_m) \times \cos(k_n x - \sigma_n t + \epsilon_n) \cdot \exp(k_m z)\exp(k_n z)\, dz$$

$$= \rho g^2 \sum_m \sum_n a_m a_n \frac{\sigma_n}{\sigma_m^2 + \sigma_n^2} \cos(k_m x - \sigma_m t + \epsilon_m) \times \cos(k_n x - \sigma_n t + \epsilon_n) \quad (36)$$

If $\sigma_m \simeq \sigma_n = \sigma$(narrow-band process), then

$$F \simeq \frac{g}{2\sigma} \cdot 2E_p = \tfrac{1}{2}c \cdot 2E_p \quad (37)$$

where c = phase velocity related to the peak frequency.

As was done for the potential energy the expression (37) may be separated in two terms by (23). Introducing additionally in (37) the local mean frequency (21) we can derive the energy flux directly from a wave record without calculating the energy spectrum

$$F(x,t) \simeq \frac{1}{2}\frac{\rho g^2}{\bar{\sigma}} Y(x,t) + \frac{1}{4}\frac{\rho g^2}{\bar{\sigma}} R^2(x,t) \quad (38)$$

Figure 4 illustrates the formula (38). The first term of (38) may be called an energy flux pulsation. It has zero mean. The second term of (38) may be called an energy group transport (net transport).

CONCLUSION

The carried out analysis has shown that by applying the Hilbert transform to the sea waves and assuming the two-dimensional motion we can derive a number of local sea wave properties without calculating the power spectrum. This can be convenient in many engineering applications specially where the instantaneous properties are more important than the mean ones. We avoid also error due to calculation of the spectrum.

By means of the Hilbert transform wave groupiness can be analysed. The method described in this paper seems to be more convenient than the one presented by Funke and Mansard.

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