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Technical Note

On two solutions of fifth order Stokes waves

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It has been demonstrated numerically that there are two real solutions of the dispersion relation of fifth order Stokes waves given by Skjelbreia and Hendrickson (Fifth order gravity wave theory, *Proc. 7th Coastal Engng Conf.*, The Hague, 1960, pp. 184–196) which is widely used and recommended in offshore codes of practice. The additional solution predicts a triple crested wave and it always seems to coexist with the conventionally accepted solution. Some properties of this solution are given.

INTRODUCTION

The dispersion of a fifth order Stokes wave is governed by two coupled nonlinear equations in two variables, namely the wavelength (L) and the amplitude parameter (a). These are[1,2]:

$$f_1(L,a) = a + \beta^2 a^3 B_{33} + \beta^4 a^6 (B_{35} + B_{55}) - H/2 = 0 \quad (1a)$$

$$f_2(L,a) = L_0(\beta + \beta^3 a^2 C_1 + \beta^5 a^4 C_2) \times \tanh \beta d - 2\pi = 0 \quad (1b)$$

Table 1. Typical waves with results

Wave	d	T	H	$\bar{d}$	$\bar{H}$	L	a	Ref.
1.	30	7·72	18·6667	0·0156	0·0097	250·0902[a]	7·4994[a]	1, 2
						478·9518[*a]	7·5334[*a]	
						241·0070[b]	7·5603[b]	
						395·0810[*b]	13·8372[*b]	
2.	107	16·30	45	0·0125	0·00526	948·7933[a]	19·748[a]	4†
						1962·2000[*a]	22·5032[*a]	
						933·3915[b]	19·8114[b]	
						1487·7230[*b]	44·0268[*b]	
3.	107	16·30	70	0·0125	0·00818	1083·0704[a]	26·26[a]	4†
						1666·8256[*a]	25·1016[*a]	
						993·7100[b]	26·9239[b]	
						1462·0100[*b]	43·811[*b]	
4.	36	15·00	11	0·005	0·0015	550·6992[a]	4·4141[a]	
						840·1729[*a]	4·5271[*a]	
						521·1700[b]	4·507[b]	
						656·9810[*b]	8·2758[*b]	

Tolerance for both L and a is 1·0E − 05. All calculations are in single precision and therefore tolerance should not be smaller than the value used. Lower tolerances are possible by changing the code to double precision mode.
Dimensions of d, H, L and a are in ft. T in seconds. Value of g used is 32·2 ft/s^2.
*Roots for triple crested wave.
†See Figs 4.19 and 4.20.
[a]Theory of Ref. 1, [b]Correction to theory of Ref. 1 as per Ref. 5.

Table 2. Progress of iteration for wave 1

Step	L	a	$f_1(L,a)$	$f_2(L,a)$
I.				
1	0·30000000E+03	0·90000000E+01	0·41386580E+01	0·11768480E+00
2	0·22466266E+03	0·82176971E+01	0·10105621E+01	0·13474387E+01
3	0·24913908E+03	0·75167871E+01	0·19797657E−01	0·39182104E−01
4	0·25009428E+03	0·74993067E+01	−0·14867897E−03	−0·18461708E−03
5	0·25009023E+03	0·74994164E+01	0·16474404E−06	0·27733114E−06
II.				
1	0·40000000E+03	0·90000000E+01	0·45504274E+01	0·11730301E+01
2	0·48975275E+03	0·75094042E+01	−0·74037480E+00	0·22888534E+00
3	0·48631381E+03	0·74709845E+01	−0·44006577E+00	0·41744698E−01
4	0·47926929E+03	0·75311723E+01	−0·18799199E−01	0·28942653E−02
5	0·47895404E+03	0·75333772E+01	−0·13210154E−03	0·19282403E−04
6	0·47895181E+03	0·75333934E+01	0·31340882E−04	0·17237437E−06
III.				
1	0·80000000E+03	0·15000000E+02	−0·11547495E+05	0·45950992E+03
2	0·89264557E+03	0·97570829E+01	−0·29330955E+04	0·12294261E+03
3	0·11514143E+04	0·61505713E+01	−0·19731820E+04	0·49751232E+02
4	0·11911885E+04	0·48916378E+01	−0·78974774E+03	0·19589855E+02
5	0·12237657E+04	0·38882191E+01	−0·29602289E+03	0·53997388E+01
6	0·10898373E+04	0·38933456E+01	−0·11571520E+03	0·13454098E+01
7	0·93136792E+03	0·43877244E+01	−0·59594875E+02	0·54406816E+00
8	0·80764349E+03	0·49890189E+01	−0·35749664E+02	0·45165846E+00
9	0·69722882E+03	0·56787181E+01	−0·19951878E+02	0·38832307E+00
10	0·61415405E+03	0·63212147E+01	−0·10794580E+02	0·34084073E+00
11	0·54938562E+03	0·68927603E+01	−0·49670000E+01	0·24989213E+00
12	0·50766107E+03	0·72810454E+01	−0·18344603E+01	0·13969357E+00
13	0·48582925E+03	0·74770756E+01	−0·41231838E+00	0·43770008E−01
14	0·47951111E+03	0·75290976E+01	−0·32856513E−01	0·42042434E−02
15	0·47895660E+03	0·75333557E+01	−0·28262264E−03	0·35991929E−04
16	0·47895178E+03	0·75333929E+01	0·31198568E−04	0·98639714E−06
IV.				
1	0·30000000E+03	0·12000000E+02	0·13795814E+02	0·47116175E+01
2	0·69234967E+03	0·77340479E+01	−0·11030473E+03	0·13742968E+02
3	0·69241577E+03	0·66458740E+01	−0·46244450E+02	0·51208296E+01
4	0·64510583E+03	0·62828703E+01	−0·17105137E+02	0·11893206E+01
5	0·56727777E+03	0·67567921E+01	−0·66301684E+01	0·36687851E+00
6	0·52190472E+03	0·71515555E+01	−0·28551941E+01	0·19314869E+00
7	0·49142126E+03	0·74290094E+01	−0·76141453E+00	0·74126817E−01
8	0·48069061E+03	0·75197163E+01	−0·10253778E+00	0·12332404E−01
9	0·47899042E+03	0·75331049E+01	−0·22780236E−02	0·30782892E−03
10	0·47895166E+03	0·75333953E+01	0·10088090E−04	0·25180550E−05

(Step 1 in each case gives trial values of L and a)

where

$$L_0 = gT^2/(2\pi);\ \beta = 2\pi/L$$

and the coefficients B_{33}, B_{35}, B_{55}, C_1 and C_2 are functions of β whose forms are given in Ref. 1. In the above, H is the crest to trough height, d is the water depth, T is the wave period and g is the acceleration due to gravity.

Solution of eqn (1a,b), given d, H and T, will yield the unknowns L and a, both of which are required to calculate any quantity of interest, e.g. wave profile, kinematics, etc. A computer program to compute the roots of eqn (1) by the Newton–Raphson iterative algorithm is described in Ref. 2.

A survey of the literature on fifth order Stokes waves reveals that this theory has found widespread use in engineering applications, its domain [in d/gT^2 (or $\bar{d}$) $-H/gT^2$ (or $\bar{H}$) plane] of applicability is well studied, its experimental verification extensive and its peculiarities outside the domain of application as well as near its boundary underscored.

However, no fundamental mathematical analysis of this theory seems to exist and the question of the existence of the solution as well as its uniqueness remains unanswered. This is particularly in contrast to the case of linear wave theory, where all these questions are fully answered. An attempt to answer the question of the existence of the solution was made in Ref. 3, albeit numerically. It was claimed that the region in the $\bar{d}$–$\bar{H}$ plane bounded by the line indicating the breaking wave limit and the axes can be partitioned into three subregions, say (a), (b) and (c) such that in region (a) the solution yields a smooth wave profile, in region (b) the solution yields wave profiles with bumps and in region (c) either no solution exists or the profile is triple

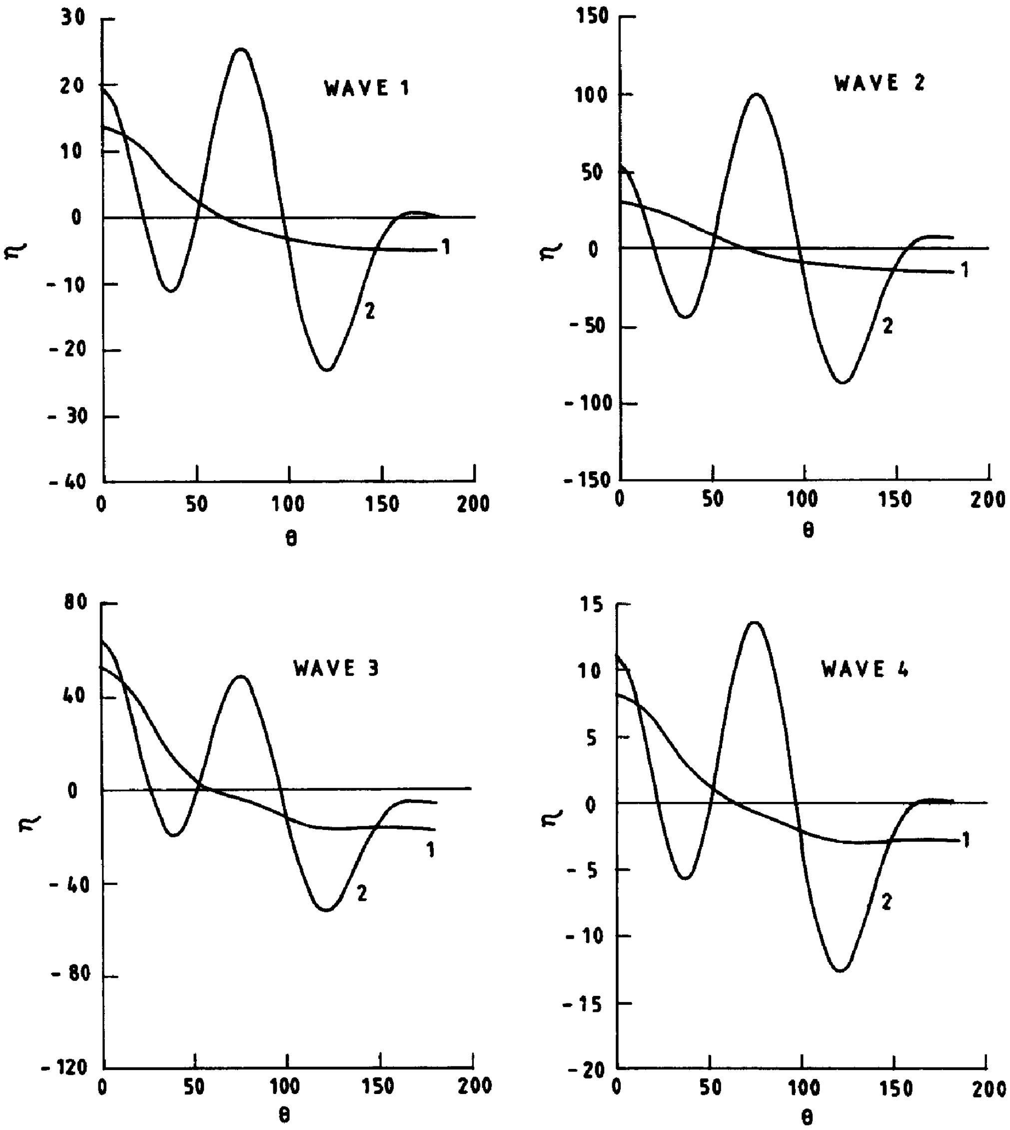


Fig. 1. Wave profiles for waves 1–4 [η in ft, θ in degrees; profiles 1 and 2 correspond to (L_1, a_1) and (L_2, a_2) respectively].

crested. Nothing however was said about the uniqueness of the solution. In this paper, we numerically demonstrate that eqn (1) has two real solutions in its entire domain of applicability and the additional solution is always a triple crested wave. The regions (a), (b) and (c) of Ref. 3 (also see Ref. 4) are reviewed and some weaknesses pointed out.

RESULTS AND DISCUSSION

To obtain the roots L and a of eqn (1), we have used the program given in Ref. 2. For specific illustration we consider four waves as shown in Table 1. Waves 1 and 3 belong to region (b), wave 2 to region (a) and wave 4 to region (c). We have obtained two sets of real roots for each of these cases which are shown in Table 1. For convenience we designate them as (L_1, a_1) and (L_2, a_2) where the latter set is the one which is not considered in the literature. To illustrate the convergence of the Newton–Raphson iteration scheme we choose wave 1 and reproduce the progress of iteration with various starting points especially for (L_2, a_2) in Table 2. It is amply clear from this table that convergence for this set of roots is authentic. Similar convergence was obtained for all other cases. Using both the sets of roots, we computed the wave profiles (phase angle θ vs surface elevation η) and horizontal particle velocities (u) under the wave crest vs water depth (s) for all cases by using the expressions of Ref. 1 and these are shown in Figs 1 and 2 respectively. It can be seen that (L_2, a_2) gives a triple crested wave in each case and in all cases $L_1 < L_2$. Wave 1, despite being in region (b), does not show any bump, which may be because it is very close to the boundary of region (b). Wave 4, well within region (c), contrary to the claim in Ref. 3, has both solutions, i.e. the usual one as well as a triple crested profile. The plots of Fig. 2 show that (L_1, a_1) always gives a larger velocity

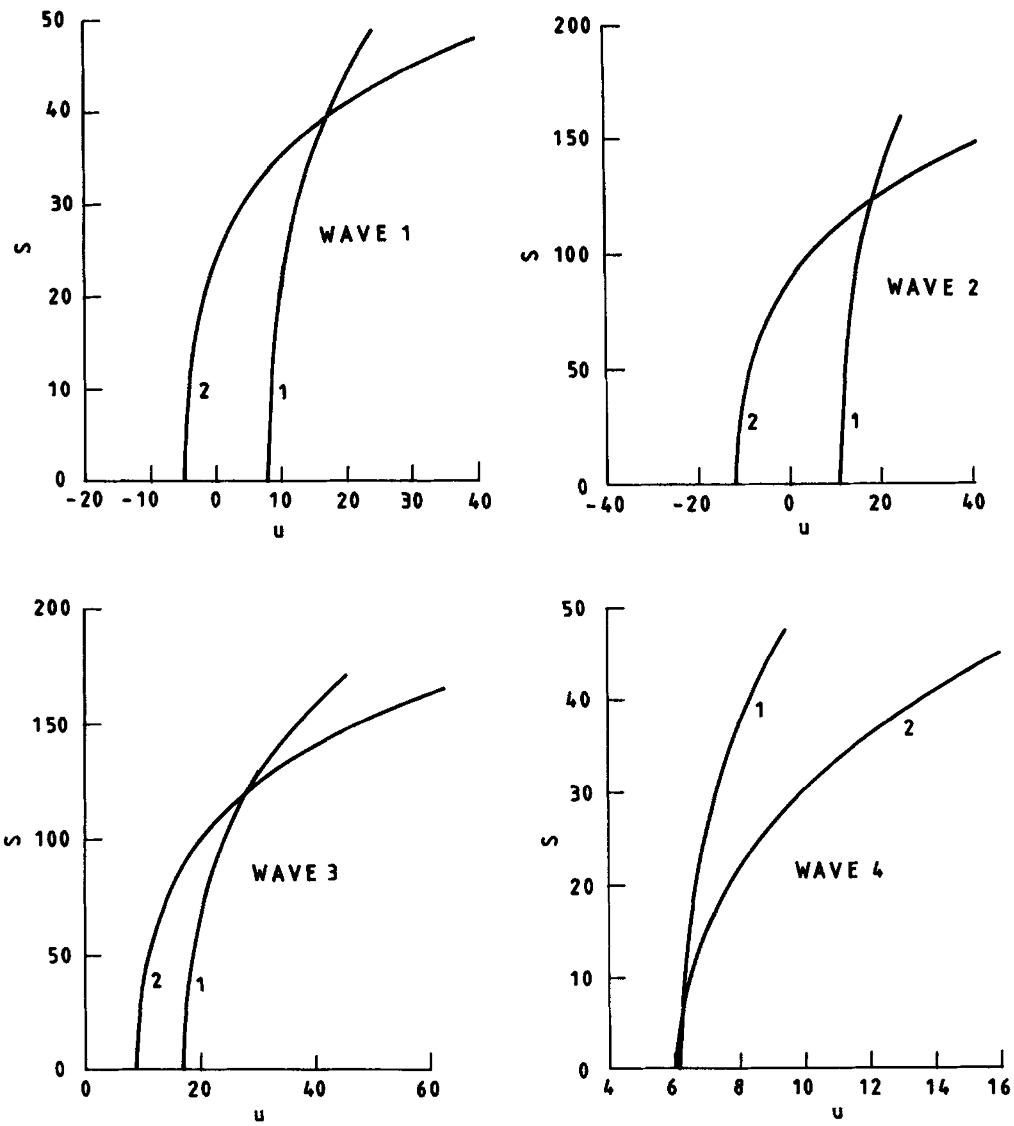
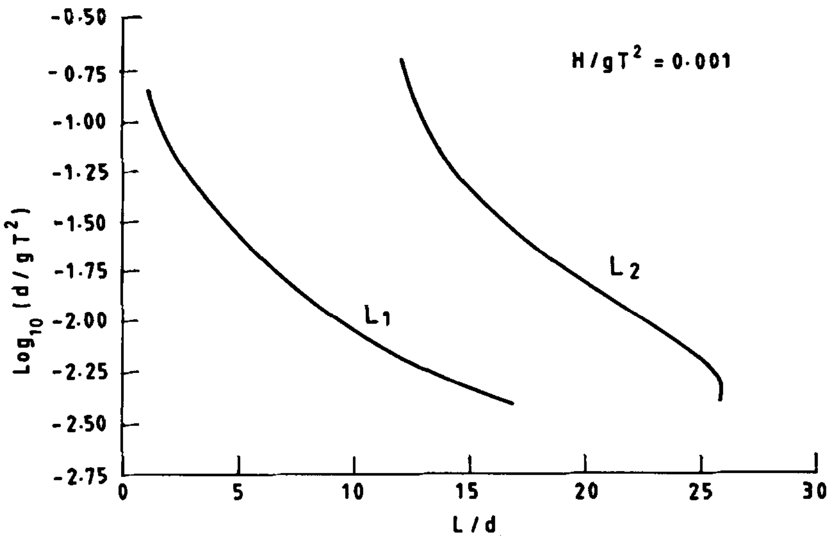


Fig. 2. Horizontal water particle velocities under wave crest ($\theta = 0$) over water depth for waves 1–4 [s in ft and u in ft/s, curves 1 and 2 correspond to (L_1, a_1) and (L_2, a_2) respectively].

at seabed and a smaller velocity at a certain height above seabed as compared to (L_2, a_2) and in the latter case u may even change sign over depth. To find how the roots vary with d, we plotted both sets of roots for a constant value of $\bar{H}$. This is shown in Fig. 3. It can be observed that whereas both L_1 and L_2 increase with decreasing d, an increase in a_1 with decreasing d is associated with a decrease in a_2. The significance of the second solution of the dispersion relation is certainly not clear. Contrary to belief, a solution with bumps or triple crested profile does not indicate that the theory is being used outside the domain of applicability, since this second solution stands on its own right everywhere and at least the wavelengths it predicts are certainly not unbelievable. We found no case where both these solutions do not coexist.

At this stage, it should be mentioned that Fenton[5] has developed a new theory of fifth order Stokes waves using one parameter expansion instead of two (L and a) as in Ref. 1. In this work, the theory of Ref. 1 was shown to be 'wrong in fifth order' and traced it to a wrong coefficient (2592 instead of the correct value of −2592) in the expression for C_2 (see Ref. 1). However, it is not clear that if this change is made the theory becomes 'correct in fifth order'. The results of this correction on the four example waves considered are shown in Table 2 and Fig. 4. From these results it is clear that the change in the regular solution (both wavelength and kinematics) is moderate whereas the change in the triple crested solution is so drastic that the results are plainly untenable.

In another trail of investigation of gravity waves, based on Hamiltonian theory, non-uniqueness of wave form had been demonstrated by Chen & Saffman[6]; their stability and symmetry breaking bifurcation had been studied by Saffman,[7] Zufiria & Saffman[8] and the



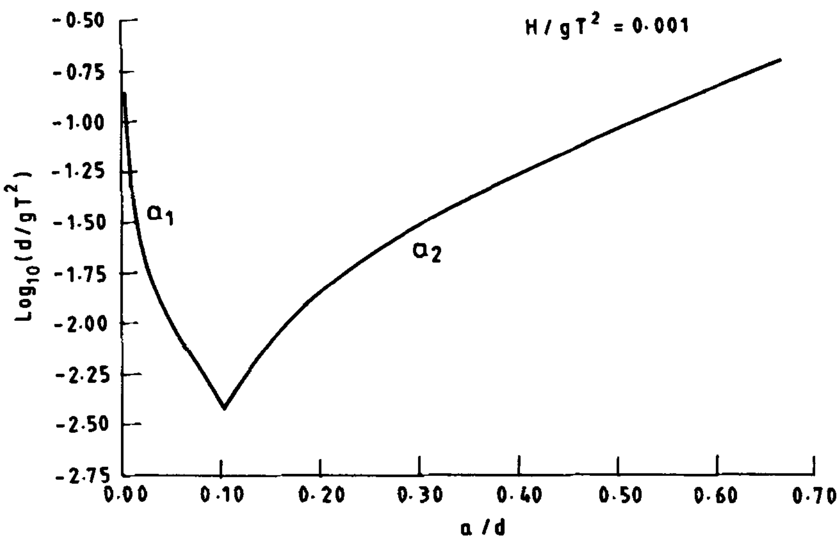


Fig. 3(a). Plot of L/d vs d/gT^2 for a constant H/gT^2. **(b).** Plot of a/d vs d/gT^2 for a constant H/gT^2.

existence of nonsymmetric waves proved by Zufiria.[9] How these results can be reconciled with the theories of Refs 1 and 5 is not apparent, and yet multiple solutions or non-uniqueness seem possible as borne out by these works.

Finally the correction to this theory pointed out in Ref. 5 seems crucial. The offshore codes of practice (e.g. Ref. 10), while recommending its use does not mention this correction, thereby creating a situation that the triple crested solution may inadvertently be chosen in cases where wavelength and amplitude parameters are not too different from the regular solution. In any case, it seems that the theory of Ref. 5 should be recommended in codes of practice and the significance of the triple crested solution should be established.

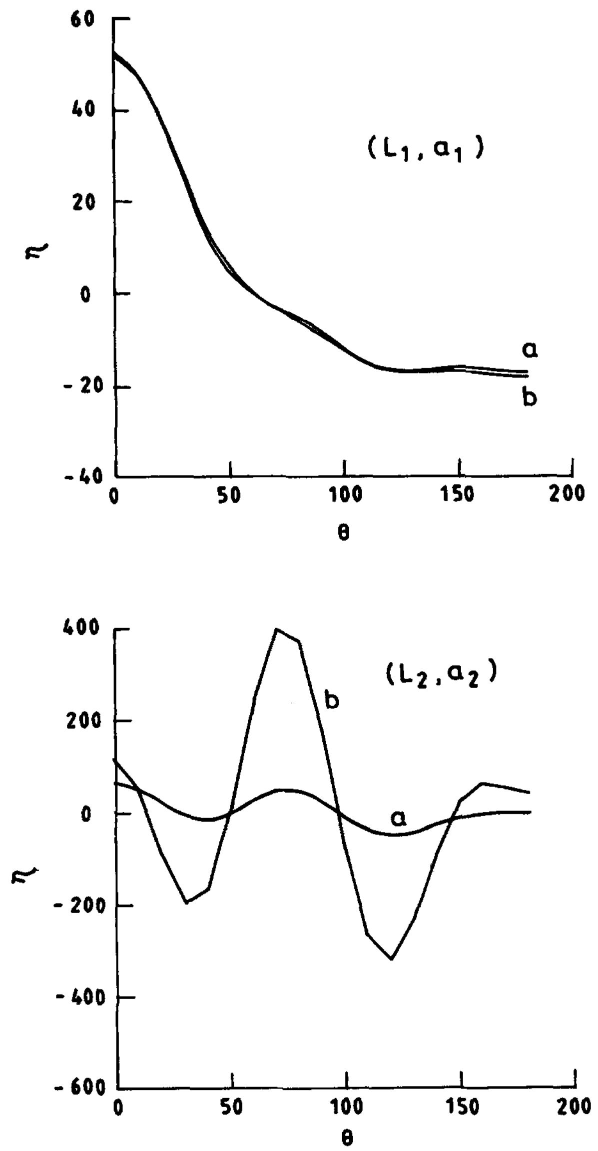


Fig. 4. Effect of correction on wave profiles of example wave 3 (a: without correction, b: with correction).

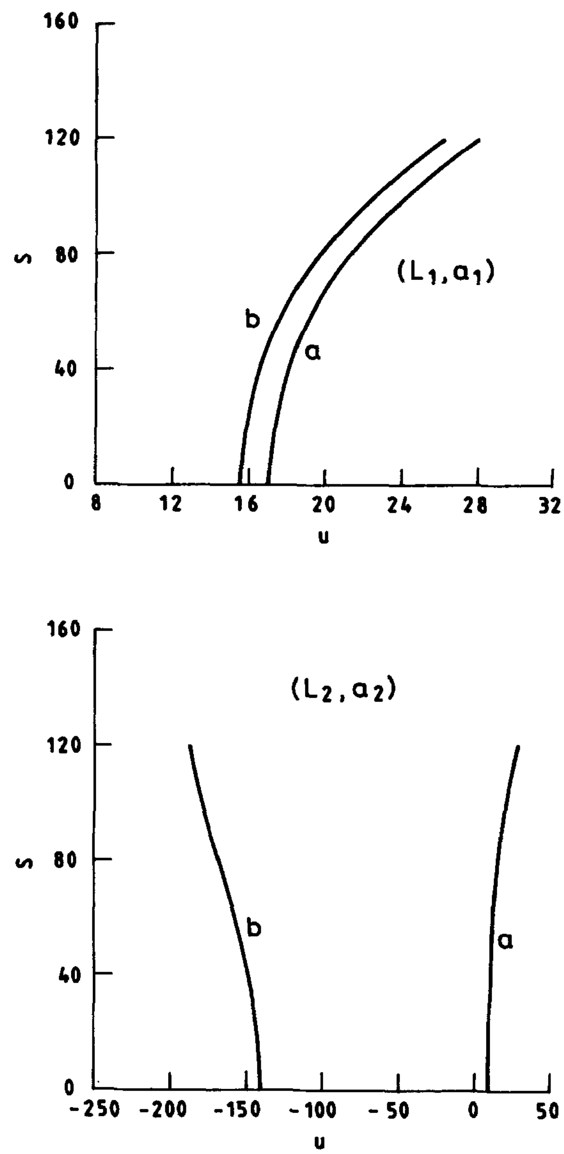


Fig. 5. Effect of correction on horizontal water particle velocity under crest of example wave 3 (a: without correction, b: with correction).

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