The Pressure Term in the Anelastic Model: A Symmetric Elliptic Solver for an Arakawa C Grid in Generalized Coordinates

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ABSTRACT

For the anelastic or pseudoincompressible system, the diagnostic continuity equation is the constraint filtering sound waves. Hamiltonian fluid dynamics considers the pressure force as the reaction force to this constraint. The author emphasizes the notion of an adjoint operator, as it provides the link between the constraint and the reaction. The elliptic equation for pressure is self-adjoint.

Applied to a discretized model, the author discusses the possibility to maintain this symmetry in the pressure equation. Its discretization is deduced from one of the anelastic constraints. The author takes the example of a 2D model with orography, discretized on an Arakawa C grid in generalized coordinates. A specific treatment of boundaries is necessary to prevent Gibbs-like errors in the pressure term.

It is possible to solve the pressure equation by a plain conjugate gradient method. Preconditioning is achieved by the Laplacian with no orography solved by a fast direct method. Criteria for efficiency depending upon the domain geometry are given.

1. Introduction

The anelastic equations filter out sound waves by replacing density, where appropriate, by a reference state value. Neglecting its temporal variation in the continuity equation leads to the anelastic constraint, and pressure becomes a diagnostic variable.

The form of the pressure term in the anelastic equations is known to be important. Wilhemson and Ogura (1972) noticed that the original term is not adequate as it does not lead to conservation of energy when the reference density varies in the vertical, and as a consequence, there is also no conservation of the vertical flux of horizontal momentum. Lipps and Hemler (1982) have shown by a scale analysis how the pressure term had to be modified to cure this problem.

Lorenz (1960) emphasized energy conservation to derive simplified sets of equations. Energy conservation is important not only for long-term integrations but also for the behavior of propagating waves. This approach was generalized by Hamiltonian methods of fluid dynamics (Shepherd 1990). Total energy is approximated, and from there the corresponding equations of motion are deduced. Other conservations (momentum, potential vorticity) are corollary.

The anelastic system (like other systems of meteorological interest) differs from the general equations for a compressible fluid by the presence of a diagnostic equation. This is linked to the presence of a constraint. (For the hydrostatic primitive system, the hydrostatic constraint leads to a diagnostic equation for vertical velocity.) For the anelastic system in two dimensions, the explicit use of a constraint can be avoided (Scinocca and Shepherd 1992) by a streamfunction formulation.

In this paper we explicitly deal (using Hamiltonian methods) with the constraint of the anelastic system. Pressure appears as the Lagrange multiplier for the constraint, and the pressure force appears as a reaction force.

However, as Hamiltonian methods are rather technical, the derivation is made in the appendixes, and we use as a starting point the property that a reaction force produces no work. The notion of an adjoint operator is necessary to make clear the relationship between the constraint and the reaction force. This is applied to the anelastic and pseudoelastic equations (Durran 1989). The approach is general enough to incorporate boundary conditions with orography in the constraints.

These considerations pertain also to the discretized equations. We will show that an energetically consistent model is obtained only when the pressure equation is self-adjoint. However, as we shall see, adjoint discretized operators generally have a pathological behavior when, as in the Arakawa C grid, extrapolations are necessary to express boundary conditions and deriva-

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tives near the boundary. We will study the adjoint of derivatives, averaging, and extrapolations that compose the divergence operator so as to avoid these problems.

The common practice in anelastic models (Clark 1977; Bernard and Kapitza 1992) does not lead to self-adjoint formulations for the pressure equation, and energy is not exactly conserved. In these models, a conservative form of the pressure gradient is used. It is thus desirable to show whether a formulation for the pressure problem can be accurate and symmetric and what its properties are for conservation of total momentum.

With gridpoint models in general curvilinear coordinates, the stencil (number of neighboring points involved in the calculation of the elliptic operator) becomes enormous, leading in three dimensions to complex block-diagonal matrices. As in Viviand (1974) and Vinokur (1974), the present model simultaneously uses two sets of wind components: Cartesian and contravariant (i.e., normal to coordinate surfaces). Extrapolations to evaluate derivatives and convert Cartesian components to contravariant ones near the boundary are bound to be intricate and, although carefully designed, to lack a firm foundation. Direct methods of inversion of the Laplacian are accordingly complex, and expensive in computer time and memory. We want to examine whether considerations of symmetry may clarify the choice of the extrapolations.

The equation is usually nonseparable between the horizontal and vertical due to the presence of metric terms. The approach common to spectral and gridpoint methods is to push these terms from the left- to the right-hand side of the equation: an approximate problem is solved, which is bound to create defects with steep orography in mesoscale models. Another possibility is to use an iterative method. The drawback is that most of these methods become less and less efficient when model truncation or aspect ratio between horizontal and vertical scales increases, and again, the solution is bound to be approximate.

The pressure equation without orography is separable and can be solved efficiently by direct methods (Clark 1977). With moderate orography, Clark uses the "flat Laplacian" as a basis of an iterative method. We propose to extend this approach and to study the influence of domain geometry (dynamics of scale factor and bottom slope) upon the number of iterations necessary for the elliptic solver.

The pressure equation is a central concern for the anelastic model, since an overwhelming resource is devoted to the sole determination of pressure when orography is present. Although a variety of iterative methods has been designed to cope with nonsymmetric problems, it is still a challenge to get a simple, accurate, and efficient determination of the pressure term.

Numerical examples pertain to steady orographic flows using a 2D periodic anelastic model with a Gal-Chen coordinate transformation (Gal-Chen and Somerville 1975) and lateral and top sponge layers. The model is a stripped-down adiabatic version of the Redelsperger (1986) model used for convection and squall-line studies (Redelsperger 1988) and modified for generalized coordinates.

2. Pressure in the anelastic equations

Following Bonnet and Luneau (1989), the original Euler equations in conservative form are[1]

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{u} = 0$$

$$\frac{\partial \rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\times\mathbf{u} + p\mathbb{I}) = \rho\mathbf{g}$$

$$\frac{\partial \rho\theta}{\partial t} + \nabla\cdot\rho\mathbf{u}\theta = 0.$$

The fluid is in a closed domain Ω with a rigid wall boundary condition $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$. The equation of state is $p = \rho RT$. Multiplying the second equation by $\mathbf{u}$, the third by $c_p(T/\theta)$, and integrating, the energy conserved by the elastic system is

$$\mathcal{H} = \int_\Omega \left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u} + \rho c_v T + \rho g z\right)dxdz. \quad (2.1)$$

It is the sum of kinetic, internal, and potential energy of the fluid.

a. The anelastic system

The point of view of Hamiltonian fluid dynamics is to approximate total energy and to derive the equations of motion from the simplified Hamiltonian. The anelastic equations are cast in terms of $\mathbf{u}$, θ only; so has to be total energy; pressure, as we shall see later, appears as a Lagrange multiplier. We sketch here the series of approximations leading to the total energy [Eq. (2.2)] as in Scinocca and Shepherd (1992), based on the scale analysis of Lipps and Hemler (1982).

We introduce reference values $\bar{\rho}(z)$, $\bar{\theta}(z)$, $\bar{p}(z)$ in hydrostatic balance: $d\bar{p} = \bar{\rho}gdz$. Deviations are the primed quantities: $\rho(x, z) = \bar{\rho}(z) + \rho'(x, z)$, etc.

Potential energy is first-order hydrostatic. Integrating by parts,[2] (2.1) becomes

[1] The symbols have their usual meaning: $\mathbf{u}$ is velocity; ρ is density; p is pressure; $\mathbf{g}$ is gravity; T is temperature; potential temperature θ is defined by $\theta = T(p_0/p)^{R/c_p}$, where p_0 is a reference pressure; R is the perfect gas constant; c_p is the calorific coefficient at constant pressure; and $\mathbf{n}$ is the outward vector normal to the domain boundary $\partial\Omega$. Advection of $\mathbf{v}$ by a nondivergent velocity field $\mathbf{u}$ is expressed in conservative form as $\nabla\cdot\mathbf{u}\times\mathbf{v}$. In a Cartesian coordinate frame, the components are $(\nabla\cdot\mathbf{u}\times\mathbf{v})_i = \Sigma_j\, \partial u_j v_i/\partial x_j$.

[2] Potential energy is integrated by parts with the hydrostatic relation $dp = -\rho g dz$: $\int_\Omega \rho g z dxdz = -\int_\Omega z dx dp = \int_{\partial\Omega} pzdx + \int_\Omega \rho RT dxdz$.

$$\mathcal{H} = \int_\Omega \left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u} + \rho c_p T\right) dxdz + \int_{\partial\Omega} zpdx.$$

We drop the boundary term. It contains the Lagrange multiplier p and as such pertains to the boundary constraints. We assume $\rho' \ll \bar{\rho}$, so that we replace the density ρ by $\bar{\rho}$. Exner pressure is defined as $\pi = T/\theta$, and at first order, according to Lipps, for convection scale motions, $T' = \bar{\pi}\theta'$. Finally, the expression for the energy of the anelastic system is

$$\mathcal{H} = \int_\Omega \bar{\rho}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + c_p\bar{\pi}\theta\right) dxdz, \qquad (2.2)$$

where $\bar{\pi}$, the reference Exner pressure, is a function of z.

Methods of Hamiltonian fluid dynamics (Salmon 1983) were used in appendix A to take the Hamiltonian (2.2) and the anelastic constraint $\rho = \bar{\rho}$ as the sufficient starting point to derive the anelastic equations[3]:

$$\frac{\partial\bar{\rho}\mathbf{u}}{\partial t} + \nabla\cdot(\bar{\rho}\mathbf{u}\times\mathbf{u}) = -\mathbf{g}\frac{\bar{\rho}\,\theta}{\bar{\theta}} - \mathbf{P}$$

$$\frac{\partial\bar{\rho}\,\theta}{\partial t} + \nabla\cdot(\bar{\rho}\,\theta\mathbf{u}) = 0.$$

The anelastic constraint is independent of time and acts upon the relative positions of the particle parcels, so it is categorized in classical dynamics as an holonomous constraint (Goldstein 1980, p. 377). The pressure force $\mathbf{P}$ is the reaction force to this constraint.

Multiplying the first equation by $\mathbf{u}$, the second by $c_p\bar{\pi}$, and integrating in the whole domain with the boundary conditions, we get

$$\frac{d\mathcal{H}}{dt} = -\int_\Omega \mathbf{u}\cdot\mathbf{P}dxdz.$$

In appendix A the form of the pressure term, its relation to the anelastic constraint, and the following properties, which might seem physically obvious, have been derived.

- The pressure term appears in the momentum equation only.
- It produces no work, because it is orthogonal to all velocities satisfying the anelastic constraint.

We take these properties as a starting point. It will be simpler to deal with the boundary conditions than with the more classical derivation of appendix A. These properties are easy to apply to a discretized model, so it will highlight the subsequent discretization. The relation between the constrained velocities and the pressure force is a simple orthogonality condition. To derive the pressure force from this condition we will have to introduce the notion of an adjoint operator. In the following we take some care making explicit the various vector spaces and metrics used to define the adjoint.

The anelastic constraint and boundary conditions can be grouped into one with the aid of the linear operator $\mathcal{D}$:

$$\mathcal{D}\bar{\rho}\mathbf{u} \equiv \begin{pmatrix} \nabla\cdot\bar{\rho}\mathbf{u} \\ -\bar{\rho}\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} \end{pmatrix} = 0; \qquad (2.3)$$

$\mathcal{D}$ is a linear operator $\mathbb{E}\to\mathbb{F}$ acting upon $\mathbb{E}$, vector space of model velocities $\mathbf{u}$, and taking its values in $\mathbb{F}$, space of scalars p defined upon $\Omega\times\partial\Omega$.

The adjoint of $\mathcal{D}$, $\mathcal{D}^*$, is a linear operator from $\mathbb{F}$ into $\mathbb{E}$. For it to be defined, we need a scalar product $\langle ; \rangle$ in $\mathbb{E}$ and another $\langle\langle ; \rangle\rangle$ in $\mathbb{F}$. Then, for any p, the relation

$$\langle\langle \mathcal{D}\bar{\rho}\mathbf{u}; p\rangle\rangle = \langle \mathbf{u}; \bar{\rho}\mathcal{D}^*p\rangle, \qquad (2.4)$$

valid for all $\mathbf{u}$, defines $\mathcal{D}^*p$.

This relation shows that provided $\mathbf{P}$ is of the form $\mathbf{P} = \bar{\rho}\mathcal{D}^*p$ it will produce no work. We need the converse: provided that $\langle\mathbf{u}; \mathbf{P}\rangle = 0$ whenever $\mathcal{D}\bar{\rho}\mathbf{u} = 0$, then p exists such that $\mathbf{P} = \bar{\rho}\mathcal{D}^*p$. It happens to be a general property of adjoint operators.[4]

The scalar product in $\mathbb{E}$

$$\langle\mathbf{u}; \mathbf{u}'\rangle = \int_\Omega \mathbf{u}\cdot\mathbf{u}'dxdz$$

is important as it defines kinetic energy

$$E_c = \frac{1}{2}\langle\bar{\rho}\mathbf{u}; \mathbf{u}\rangle.$$

The scalar product in the image $\mathbb{F}$ of $\mathcal{D}$ is taken as

$$\langle\langle p; p'\rangle\rangle = \int_\Omega pp' + \int_{\partial\Omega} pp';$$

it is arbitrary. Changing this scalar product will change the value of the Lagrange multiplier p but not the pressure term.

With these scalar products the adjoint of $\mathcal{D}$ is defined. It is immediate to verify that it is the gradient operator $\mathcal{D}^*p = -\nabla p$. Thus, from the above-mentioned prop-

[3] The relation $\nabla\bar{\pi} = \mathbf{g}/\bar{\theta}$ is necessary to derive the buoyancy term. The buoyancy is expressed in terms of θ, instead of θ'; it is equivalent. The difference will be absorbed in the pressure term.

[4] When the model is discretized, $\mathbb{E}$ has a finite dimension N, and $\mathcal{D}$ is expressed on a basis of vectors as a matrix $\mathbf{D}$. Upon an orthonormal basis of $\mathbb{E}$ and $\mathbb{F}$, the adjoint is just the transpose of $\mathbf{D}$; taking $\mathbf{u}$ in the null space of $\mathbf{D}$ of dimension n, (2.4) shows that Ker($\mathbf{D}$), null space of $\mathbf{D}$, and Im($\mathbf{D}^*$), image of $\mathbf{D}^*$, are orthogonal: Ker($\mathbf{D}$) $\perp$ Im($\mathbf{D}^*$). It is well known that a matrix and its transpose have the same rank. For $\mathbf{D}$ it is $N - n$. Thus, $[\mathrm{Ker}(\mathcal{D})]^\perp = \mathrm{Im}(\mathcal{D}^*)$. In the continuous case, the above property holds provided that the image of $\mathcal{D}^*$ is a closed subspace. This is the case here for the gradient. For mathematical aspects, the reader is referred to Girault (1979).

erty of adjoint operators, the pressure term $\mathbf{P}$, orthogonal to all velocities satisfying (2.3), is of the form

$$\mathbf{P} = \bar{\rho}\nabla p.$$

This form of the pressure term is identical to the ones derived by Lipps and Hemler (1982) and Lipps (1990) through a scale analysis. The modified anelastic equations as defined by Wilhemson (1972) do not ensure conservation of energy. Their pressure term $\mathbf{P} = \nabla p$ is not in the range of $\bar{\rho}\mathcal{D}^*$.

b. *Elliptic problem for pressure*

Let us gather in s buoyancy and advection terms, so that the momentum equation is

$$\frac{\partial \bar{\rho}\mathbf{u}}{\partial t} = \bar{\rho}\mathbf{s} - \bar{\rho}\nabla p. \qquad (2.5)$$

We have two prognostic equations: one for $\mathbf{u}$ and one for θ and none for p. Applying the anelastic and boundary constraints (2.3) to the momentum tendency, we obtain the elliptic equation for p:

$$\nabla\cdot\bar{\rho}\nabla p = \nabla\cdot\mathbf{s}$$
$$\nabla p\cdot\mathbf{n}|_{\partial\Omega} = \mathbf{s}\cdot\mathbf{n}. \qquad (2.6)$$

Its symmetry is best shown when these two equations are written in the equivalent form

$$\mathcal{D}\bar{\rho}\mathcal{D}^* p = -\mathcal{D}\bar{\rho}\mathbf{s}. \qquad (2.7)$$

For lateral open boundaries the Orlanski radiative condition

$$\frac{\partial \mathbf{u}\cdot\mathbf{n}}{\partial t} + c\frac{\partial \mathbf{u}\cdot\mathbf{n}}{\partial x} = 0$$

still leads to the specification of $\partial\mathbf{u}/\partial t\cdot\mathbf{n}$ and to a Neumann problem for p.

3. Pressure term in the pseudoincompressible system

The pressure term used by Durran (1989) in his pseudoincompressible system can be guessed in the same way from his continuity equation and approximation to the kinetic energy. One uses the pseudodensity

$$\rho^* = \frac{\bar{\rho}\bar{\theta}}{\theta},$$

and the constraints have the form

$$\nabla\cdot\bar{\rho}\bar{\theta}\mathbf{u} = 0$$
$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

In contrast with the energy (2.1), no combination is made of potential and internal energy involving the hydrostatic approximation. However, the pseudodensity approximation is used in (2.1) leading to

$$\mathcal{H} = \int \rho^*\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + c_v\frac{\bar{T}}{\bar{\theta}}\theta\right)dxdz.$$

The pressure term can be found from the expression of kinetic energy and the anelastic constraint only. When (3.1) is satisfied, then

$$\langle\mathbf{u};\mathbf{P}\rangle = \int \rho^*\mathbf{u}\cdot\mathbf{P} = \int (\bar{\rho}\bar{\theta}\mathbf{u})\cdot\frac{\mathbf{P}}{\theta} = 0. \qquad (3.1)$$

The adjoint of (3.1) for the above scalar product is $\cdot \rightarrow \theta\nabla\cdot$; so a scalar π exists such that

$$\mathbf{P} = c_p\theta\nabla\pi,$$

which is precisely the form appearing in Durran [1989, Eqs. (13) and (14)]. Thus, our derivation of the pressure term is not limited to the Lipps–Hemler anelastic system.

4. Convergence for the discretized pressure problem

Before dealing with a particular discretization of the pressure problem, we review some mathematical results that might be applied to prove convergence of the pressure term.

As in the incompressible case, the pressure term has a geometric interpretation as a projector. It has been shown to be orthogonal to all "anelastic velocities." Let s be a velocity field with $\mathcal{D}\bar{\rho}\mathbf{s} \neq 0$; *if* $\mathbf{u}$ is defined by $\mathbf{u} = \mathbf{s} - \nabla p$ and $\mathcal{D}\bar{\rho}\mathbf{u} = 0$, then $\mathbf{u}$ is the projection $\mathcal{P}$ of s on the subspace defined by the anelastic constraint and boundary conditions. It is orthogonal for the norm $|\mathbf{u}|^2 = \langle\bar{\rho}\mathbf{u};\mathbf{u}\rangle$ defining kinetic energy, so it is characterized by stationarity of

$$\mathcal{L}(\mathbf{u},p) = |\mathbf{u}-\mathbf{s}|^2 + \langle\langle p;\mathcal{D}\mathbf{u}\rangle\rangle$$
$$= \langle\bar{\rho}(\mathbf{u}-\mathbf{s});\mathbf{u}-\mathbf{s}\rangle - \langle\mathcal{D}^*p;\bar{\rho}\mathbf{u}\rangle \qquad (4.1)$$

or, in conventional notations,

$$\mathcal{L}(\mathbf{u},p) = \int_\Omega \bar{\rho}(\mathbf{u}-\mathbf{s})^2 + \int_\Omega \bar{\rho}\mathbf{u}\cdot\nabla p. \qquad (4.2)$$

a. *Discretized variational problem*

Discretization of (4.2) is made up of discretization of the gradient operator, denoted by G, and discretization of the integral denoted by brackets $\langle;\rangle_\mathbf{u}$ so that $\langle\mathbf{u};\mathbf{u}'\rangle_\mathbf{u} \approx \int_\Omega \mathbf{u}\cdot\mathbf{u}'$.

The discretized variational problem for a mesh size h is thus

$$\mathcal{L}_h(\mathbf{u}_h,p_h) = \langle\bar{\rho}(\mathbf{u}_h-\mathbf{s}_h);\mathbf{u}_h-\mathbf{s}_h\rangle_\mathbf{u} + \langle Gp_h;\bar{\rho}\mathbf{u}_h\rangle. \qquad (4.3)$$

When examining the minimization problem (4.1), one would expect that convergence of the projection $\mathbf{u}$

of s would depend solely on a sound discretization of $\mathcal{L}$. General convergence theorems (see Temam 1984, p. 45) do not seem to apply in this case, as the quadratic form $\langle \bar{\rho}\mathbf{u}_h;\ \mathbf{u}_h \rangle_{\mathbf{u}}$ does not constrain derivatives of $\mathbf{u}_h$.[5] It is confirmed by the Gibbs-like phenomenon we find when we use most approximations for the divergence, which will be displayed in the sequel.

b. *Discretized Euler conditions*

Euler conditions for (4.1) can use two unrelated discretizations: G for the gradient and D for the constraint $\mathcal{D}$. The Euler conditions are

$$\begin{bmatrix} \bar{\rho}\,\cdot & \bar{\rho}G\,\cdot \\ -D(\bar{\rho}\,\cdot) & 0 \end{bmatrix} \begin{pmatrix} \mathbf{u}_h \\ p_h \end{pmatrix} = \begin{pmatrix} \bar{\rho}\mathbf{s}_h \\ 0 \end{pmatrix}. \quad (4.4)$$

Convergence theorems (see Godunov 1977 for references) indicate that velocity $\mathbf{u}_h$ and pressure p_h will be order-m approximations in mesh size h if G and D are order-m approximations of the gradient and of $\mathcal{D}$. These approximations D and G are said to be "consistent."

Given that p and u fields are represented by sufficiently regular finite elements, the discretization of the gradient and divergence operators is automatic and the G and $-D$ operators are adjoint.[6]

On the contrary, in the case of finite differences, if D is a consistent discretization of $\mathcal{D}$, then it does not follow necessarily that D^* is a consistent discretization of the gradient, especially because of the evaluation of the derivatives at the boundary. In this case it is safer to adopt separate, nonadjoint discretizations for D and G. However, the projection $\mathcal{P}: \mathbf{s} \rightarrow \mathbf{u}$ is no longer orthogonal. It is now a projection upon $\mathrm{Ker}(D\bar{\rho})$ parallel to $\mathrm{Im}(\bar{\rho}G)$. The pressure term is then

$$\mathbf{P} = \bar{\rho}G(D\bar{\rho}G)^{-1}D\bar{\rho}\mathbf{s}. \quad (4.5)$$

The elliptic operator $D\bar{\rho}G$ we have to invert is not necessarily symmetric, and accordingly, energy is not conserved.

In Bernard and Kapitza [1992, Eq. (5.20)], a separate consistent discretization of the product $D\bar{\rho}G$ is also designed. No mathematical proof of convergence of the pressure term is given.

5. Discretization with the Arakawa C grid and orography

We need to abandon the Cartesian system of coordinates in order to ease the uneven boundary conditions produced by orography. We found it simpler to consider generalized coordinates instead of limiting ourselves to a modified vertical coordinate.

The momentum equation in (2.3) is projected onto the Cartesian basis vectors $(\mathbf{i}, \mathbf{k})$. The horizontal component of velocity $u = \mathbf{u}\cdot\mathbf{i}$ is a scalar, so no Christoffel symbols are needed for the advection term (Viviand 1974; Vinokar 1974). The evolution equation for u is

$$\frac{\partial \bar{\rho} u}{\partial t} = -\nabla\cdot(\bar{\rho}\mathbf{u}u) - \mathbf{i}\cdot\bar{\rho}\nabla p.$$

The change of coordinates is apparent only in the divergence and gradient operators.

a. *General coordinates*

Dutton (1986) provides an adequate treatment of general coordinates. We summarize here only the relation of the gradient and divergence to the Cartesian components and show that two different expressions can be used as a basis to the conservative and nonconservative (Thompson 1985) discretizations of the gradient.

• *Covariant basis and contravariant components.* The terrain-following coordinates $(\bar{x}^i, i = 1, 2) = (\bar{x}, \bar{z})$ are defined in relation to the Cartesian coordinates $(x^j, j = 1, 2) = (x, z)$ and basis vectors $(\mathbf{i}^j, j = 1, 2) = (\mathbf{i}, \mathbf{k})$. The vectors tangent to the isolines of $\bar{x}^i$ are given by

$$\begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{Bmatrix} = J^x_{\bar{x}} \begin{Bmatrix} \mathbf{i} \\ \mathbf{k} \end{Bmatrix} \quad \text{where} \quad J^x_{\bar{x}} = \begin{pmatrix} \dfrac{\partial x}{\partial \bar{x}} & \dfrac{\partial z}{\partial \bar{x}} \\ \dfrac{\partial x}{\partial \bar{z}} & \dfrac{\partial z}{\partial \bar{z}} \end{pmatrix} \quad (5.1)$$

and are called the covariant basis (the contravariant basis vectors are $\mathbf{e}^i = \nabla \bar{x}^i$). We relate the contravariant components $\bar{u}^i$ to the Cartesian ones u_i. From $\mathbf{u} = \bar{u}^i\mathbf{e}_i = u_j\mathbf{i}^j$, with (5.1), we get

$$\bar{u}^i = C^{ij}u_j \quad \text{where}$$

$$C = (J^x_{\bar{x}})^{-t} = \frac{1}{g^{1/2}} \begin{pmatrix} \dfrac{\partial z}{\partial \bar{z}} & -\dfrac{\partial x}{\partial \bar{z}} \\ -\dfrac{\partial z}{\partial \bar{x}} & \dfrac{\partial x}{\partial \bar{x}} \end{pmatrix}, \quad (5.2)$$

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[5] Let us consider the regularized minimization problem $J_\epsilon(\mathbf{u}) = \int (\mathbf{u} - \mathbf{s}) + \epsilon \int (\nabla\mathbf{u})^2 + \int \lambda\nabla\cdot\mathbf{u}$. This is the problem one would get when adding viscosity in the model with an implicit time stepping. The Euler equation is $\mathbf{u} = \mathbf{s} - \nabla p' + \epsilon\Delta\mathbf{u}$. For this problem, convergence in an integral sense for $\mathbf{u}$ and its first derivative is guaranteed by a proper discretization of the operators appearing in J_ϵ.

[6] Let velocity be represented on the basis $\mathbf{e}_i$, and p on the basis f_i: $\mathbf{u}(x, z) = \Sigma\, u_i\mathbf{e}_i(x, z)$ and $p(x, z) = \Sigma\, p_i f_i(x, z)$. Provided the finite elements $\mathbf{e}_i, f_i$ are sufficiently regular to have a gradient or a divergence, we define the gradient $\nabla p = \Sigma\, g_i\mathbf{e}_i$ by the relation $\int \mathbf{e}_j\nabla p = p_k \int \mathbf{e}_j\nabla f_k = g_i \int \mathbf{e}_i\mathbf{e}_j$ or $g_i = (M_{\mathbf{u}})^{-1}_{ij}(\int \mathbf{e}_j\nabla f_k)p_k$, with the mass matrix given by $(M_{\mathbf{u}})_{ij} = \int \mathbf{e}_i\mathbf{e}_j$. The augmented divergence $\tilde{\nabla}\cdot\mathbf{u} = \Sigma\, d_i f_i$ is defined by $\int f_j\tilde{\nabla}\cdot\mathbf{u} = u_i \int f_j\nabla\cdot\mathbf{e}_i - u_i \int_{\partial\Omega} f_j\mathbf{e}_i\cdot\mathbf{n}$. From $\int f_j\tilde{\nabla}\cdot\mathbf{e}_i + \int \nabla f_j\cdot\mathbf{e}_i = 0$ it is apparent that the two operators ∇ and $\tilde{\nabla}\cdot$ are adjoint.

where $g^{1/2} = \det(J_{\bar{z}}^{x})$ is the volume element used to calculate integrals in the new coordinate system:

$$\int f(x, y)dxdy = \int g^{1/2} f(\bar{x}, \bar{z})d\bar{x}d\bar{z}. \quad (5.3)$$

• *Nonconservative gradient.* The gradient of a scalar is defined as the vector satisfying

$$\mathbf{e}_i \cdot \nabla p = \frac{\partial p}{\partial \bar{x}^i}.$$

Its scalar product with any vector $\mathbf{u}$ is

$$\mathbf{u} \cdot \nabla p = \bar{u}^i e_i \cdot \nabla p = \bar{u}^i \frac{\partial p}{\partial \bar{x}^i},$$

so the Cartesian components are

$$\mathbf{i}^j \cdot \nabla p = C^{ij} \frac{\partial p}{\partial \bar{x}^i}. \quad (5.4)$$

In the case of a Gal-Chen grid (Gal-Chen and Somerville 1975),

$$C = \frac{1}{\left(\dfrac{\partial z}{\partial \bar{z}}\right)} \begin{pmatrix} \dfrac{\partial z}{\partial \bar{z}} & 0 \\ -\dfrac{\partial z}{\partial \bar{x}} & 1 \end{pmatrix} \quad \text{and}$$

$$\frac{\partial z}{\partial \bar{z}} \nabla p = \left(\frac{\partial z}{\partial \bar{z}}\frac{\partial p}{\partial \bar{x}} - \frac{\partial z}{\partial \bar{x}}\frac{\partial p}{\partial \bar{z}}\right)\mathbf{i} + \frac{\partial p}{\partial \bar{z}}\mathbf{k}.$$

• *Divergence.* Using (5.3), the Stokes formula is

$$\int_\Omega g^{1/2}\mathbf{u} \cdot \nabla p d\bar{x}d\bar{z}$$

$$= -\int_\Omega g^{1/2} p \nabla \cdot \mathbf{u} d\bar{x}d\bar{z} + \int_{\partial\Omega} p\mathbf{u} \cdot \mathbf{n}, \quad (5.5)$$

so $\nabla \cdot \mathbf{u}$ in terms of the contravariant components is $\nabla \cdot \mathbf{u} = g^{-1/2}(\partial g^{1/2}\bar{u}^i / \partial \bar{x}^i)$ or, in terms of the Cartesian components, using (5.2), is

$$\nabla \cdot \mathbf{u} = \frac{1}{g^{1/2}} \frac{\partial C^{ij} g^{1/2} u_j}{\partial \bar{x}^i}. \quad (5.6)$$

• *Conservative gradient.* Another expression for the gradient can also be found by applying the divergence formula. As the constant unit vectors $\mathbf{i}^j$ of the Cartesian basis have a null divergence, $\mathbf{i}^j \cdot \nabla p = \nabla \cdot \mathbf{i}^j p$, so

$$\mathbf{i}^j \cdot \nabla p = \frac{1}{g^{1/2}} \frac{\partial C^{ij} g^{1/2} p}{\partial \bar{x}^i}. \quad (5.7)$$

With the Gal-Chen grid, we get

$$\frac{\partial z}{\partial \bar{z}} \nabla p = \left[\frac{\partial}{\partial \bar{x}}\left(\frac{\partial z}{\partial \bar{z}} p\right) - \frac{\partial}{\partial \bar{z}}\left(\frac{\partial z}{\partial \bar{x}} p\right)\right]\mathbf{i} + \frac{\partial p}{\partial \bar{z}}\mathbf{k}.$$

The two expressions of the gradient, (5.4) and (5.7), are equivalent, as can be seen from the derivatives of (5.2). However, discretized derivatives do not commute with averaging operators, so discretizations starting from (5.4) or (5.7) are not necessarily equivalent unless what are coined "metric identities" are satisfied (Thompson 1985).

b. The grid and the basic operators

The Arakawa C grid is used. Grid mesh size in transformed coordinates is assumed to be unity: $\Delta\bar{x} = \Delta\bar{z} = 1$. The domain $\Omega = [0, I] \times [0, K]$ is rectangular.

Cartesian u and contravariant $\bar{u}$ velocity components are defined at the same grid points N_u; N_u nodes have integer abscissa. We distinguish nodes on the boundary ∂N at $\bar{x} = 0, I$. In the same way we define a grid N_w for w (see Fig. 1).

The grid is staggered, so the calculation of $\bar{w}$ in (5.2) necessitates averaging of u at N_w points. On the boundary ∂N, u has to be extrapolated. It is a common practice in finite-difference methods to ease programming by considering fictitious points outside the domain. We extrapolate u to an extended grid $\bar{N}_u$ with points at $\bar{z} = -1/2, K + 1/2$ and then average to ∂N.

We also define the N_p grid, where momentum divergence will be evaluated. Boundary values of pressure are separate variables. They are located at $\bar{z} = 0, K$ so as to form the extended grid $\bar{N}_p$.

The classical Schumann operators are used:

$$\delta_x\theta = \theta\left(\bar{x} + \frac{1}{2}, \bar{z}\right) - \theta\left(\bar{x} - \frac{1}{2}, \bar{z}\right),$$

$$m_x\theta = \frac{1}{2}\left[\theta\left(\bar{x} + \frac{1}{2}, \bar{z}\right) + \theta\left(\bar{x} - \frac{1}{2}, \bar{z}\right)\right].$$

c. Discretization of $\nabla \cdot$ *and* ∇

1) CONTRAVARIANT COMPONENTS

We use the following definition of the metric coefficients: $\partial z/\partial\bar{z} \to \delta_z z$, $\partial z/\partial\bar{x} \to \delta_x z$, $\partial x/\partial\bar{x} \to \delta_x x$, $\partial x/\partial\bar{z} \to \delta_z x$, with x, z being the Cartesian coordinates of

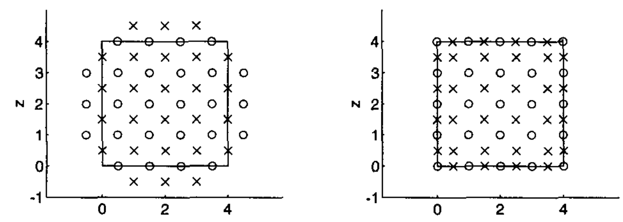

FIG. 1. Arakawa C grid with $I = K = 4$; ∂N is the square box; (a) u nodes are labeled by crosses, w nodes by circles; and (b) p nodes are labeled by crosses, ζ nodes by circles.

the N_ζ grid nodes (see Fig. 1). The relation between the Cartesian components of velocity and the contravariant components of momentum is

$$\begin{pmatrix} U \\ W \end{pmatrix} = \mathbf{C}\begin{pmatrix} g^{1/2}\bar{\rho}u \\ g^{1/2}\bar{\rho}w \end{pmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \delta_z z g^{-1/2}\cdot & -\delta_z x m_{xz} e_w(g^{-1/2}\cdot) \\ -\delta_x z m_{xz} e_u(g^{-1/2}\cdot) & \delta_x x g^{-1/2}\cdot \end{bmatrix}, \quad (5.8)$$

where $\mathbf{C}$ is a matrix of operators and the dot $(\cdot)$ indicates where to enter the argument. On the first and last lines of $\mathbf{C}$, m_{xz} are different operators: $m_{xz} : \bar{N}_w \mapsto N_u$ and $m_{xz} : \bar{N}_u \mapsto N_w$. The extrapolation e_u (resp. e_w) is used to define $u(w)$ on the extended grid $\bar{N}_u(\bar{N}_w)$.

2) Divergence and constraint operator $\mathbf{D}$

It is necessary for momentum and mean potential temperature conservation by the advection terms that the advection be discretized under flux form and, as a consequence that divergence also be under flux form. The discretization for the divergence in accord with (5.6) and (5.8) is

$$\nabla\cdot\mathbf{u} = \frac{1}{g^{1/2}}(\delta_x \quad \delta_z)\mathbf{C}g^{1/2}\begin{pmatrix} u \\ w \end{pmatrix}. \quad (5.9)$$

Let us show that for temperature advection, with the divergence (5.6), the conservation equation for θ is

$$\frac{\partial g^{1/2}\bar{\rho}\theta}{\partial t} = -\delta_x(Um_x\theta) - \delta_z(Wm_z\theta). \quad (5.10)$$

The sum over all grid points is null due to periodicity and the boundary condition $W = 0$, and mean potential temperature is conserved. From the identity $\delta_x(Um_x\theta) = m_x(U\delta_x\theta) + \theta\delta_x U$ (5.10) is equivalent to an advection for θ when

$$\delta_x U + \delta_z W = 0,$$

so (5.9) is the right discretization for divergence.

As stated earlier, it is convenient to consider the constraint operator $\mathbf{D}$ so that divergence and boundary conditions can be merged. Operator $\mathbf{D}$ is an approximation of $g^{1/2}\nabla\cdot\mathbf{g}^{-1/2}$ at interior points. Let us denote by $\bar{\mathbf{D}}$ the $\mathbf{D}$ operator in the Cartesian case $\mathbf{C} = \mathbf{I}$, $g^{1/2} = 1$.

For simplicity let us show the monodimensional case $I = 1$; $\bar{\mathbf{D}}$ is a $(K + 1) \times K$ rectangular matrix with two boundary terms at the first and last line:

$$\bar{\mathbf{D}}\mathbf{u} = \begin{cases} -w_K \\ \delta_z w \\ w_0 \end{cases} \quad \text{or}$$

$$\bar{\mathbf{D}} = \begin{pmatrix} -1 & & & & & \\ 1 & -1 & & & & \\ & \cdots & \cdots & \cdots & & \\ & & & 1 & -1 & \\ & & & & & 1 \end{pmatrix}. \quad (5.11)$$

In the non-Cartesian case, $\mathbf{D}$ obtains from $\bar{\mathbf{D}}$ by

$$\mathbf{D}\mathbf{u} = \bar{\mathbf{D}}\mathbf{C}\begin{pmatrix} u \\ w \end{pmatrix}. \quad (5.12)$$

d. Adjoint of the constraint operator

We want to design a consistent approximation to the gradient from the adjoint of the constraint $\mathbf{D}$. For this we need to study the behavior of the adjoint of the basic extrapolation and Schuman operators, and we will use the property of composition $(AB)^* = B^*A^*$.

The adjoint is defined by the relation (2.4). Discretized velocity and pressure are represented by column arrays of real numbers, and scalar products are represented by positive-definite matrices $\mathbf{M}_p$ and $\mathbf{M}_\mathbf{u}$ such that

$$\langle\langle p; p'\rangle\rangle = p^{\mathrm{T}}\mathbf{M}_p p' \text{ and}$$

$$\langle \mathbf{u}; \mathbf{u}'\rangle = u^{\mathrm{T}}\mathbf{M}_u u' + w^{\mathrm{T}}\mathbf{M}_w w' = \mathbf{u}^{\mathrm{T}}\mathbf{M}_\mathbf{u}\mathbf{u}'.$$

The adjoint of the matrix representing $\mathbf{D}$ from (2.4) relates to the transpose of $\mathbf{D}$ by

$$\mathbf{D}^* = \mathbf{M}_\mathbf{u}^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{M}_p, \quad (5.13)$$

so not any choice of scalar products will lead to a consistent adjoint. In the case where the $\mathbf{M}$ matrices are identity the adjoint is just the transpose.

1) Kinetic energy and scalar products

Several obvious choices, referred to as u_1 and $u_{1/2}$, leading to kinetic energy are possible according to the weight we give at nodes located on ∂N:

$$\langle \mathbf{u}; \mathbf{u}'\rangle_{u_1} = \sum_{N_u} u_{ik}u'_{ik} + \sum_{N_w} w_{ik}w'_{ik}$$

$$\langle \mathbf{u}; \mathbf{u}'\rangle_{u_{1/2}} = \langle \mathbf{u}; \mathbf{u}'\rangle_{u_1} - \frac{1}{2}\sum_{\partial N} u_{ik}u'_{ik} - \frac{1}{2}\sum_{\partial N} w_{ik}w'_{ik}.$$

Discretized kinetic energy is defined by

$$E_c = \frac{1}{2}\langle g^{1/2}\bar{\rho}\mathbf{u}; \mathbf{u}\rangle_{u_1 \text{ or } u_{1/2}},$$

where discretization u_1 gives more symmetric formulations, and discretization $u_{1/2}$ is a better approximation of the integral of kinetic energy, based on the trapezoidal rule.

We need the discretization of (5.5). A scalar product has to be defined upon pressure points of $\bar{N}_p$, not only at inner points. We take

$$\langle\langle p; p'\rangle\rangle = \sum_{\bar{N}_p} p_{ik}p'_{ik}.$$

2) Adjoint of $\mathbf{C}$

With generalized coordinates, we need to consider the adjoint of the extrapolation operators. Let us show that the discretization of $\mathbb{C}^*$ is consistent for the scalar product $u_{1/2}$ and an appropriate extrapolation.

We again show the 1D case $I = 1$. For the scalar product u_1, matrices $\mathbb{M}$ are identity and the adjoint does not differ from the transpose. For $u_{1/2}$, in the 1D case, $\mathbb{M}_w = \mathrm{diag}(1/2, 1, \cdots, 1, 1/2)$.

In the 1D case, the u and w components of wind have a null $\bar{x}$ derivative. We thus need only the second line of $\mathbb{C}$ in (5.8) to calculate the 1D divergence. It is built by the composition of the two following operators:

1) averaging: $m_z : \bar{N}_u \to N_w$;
2) extrapolation (by copy): $e_u : N_u \to \bar{N}_u$.

We verify easily that the transpose (or adjoint for u_1) of $m_z e_u$ is the matrix:

$$(m_z e_u)^{\mathrm{T}} N_w \to N_u : \begin{pmatrix} 2 & 1 & & \cdot\cdot & & \\ & 1 & 1 & \cdot\cdot & & \\ & & \cdot\cdot & \cdot\cdot & & \\ & & & \cdot\cdot & 1 & 1 \\ & & & \cdot\cdot & & 1 & 2 \end{pmatrix}.$$

We deduce the adjoint for $u_{1/2}$ as in (5.13). It is $(m_z e_u)^* = \mathbb{M}_u^{-1}(m_z e_u)^{\mathrm{T}}\mathbb{M}_w$, so

$$(m_z e_u)^* = (m_z e_u)^{\mathrm{T}}\mathbb{M}_w = \begin{pmatrix} 1 & 1 & & \cdot\cdot & & \\ & 1 & 1 & \cdot\cdot & & \\ & & \cdot\cdot & \cdot\cdot & & \\ & & & \cdot\cdot & 1 & 1 \\ & & & \cdot\cdot & & 1 & 1 \end{pmatrix}$$

to be compared with

$$m_z N_w \to N_u : \begin{pmatrix} 1 & 1 & & \cdot\cdot & & \\ & 1 & 1 & \cdot\cdot & & \\ & & \cdot\cdot & \cdot\cdot & & \\ & & & \cdot\cdot & 1 & 1 \\ & & & \cdot\cdot & & 1 & 1 \end{pmatrix}.$$

So we proved

$$(m_z e_u)^* = m_z.$$

Generalized to two dimensions, the result is

$$\mathbb{C}^* = \begin{bmatrix} g^{-1/2}\delta_z z\cdot & -g^{-1/2}m_{xz}e_w(\delta_x z\cdot) \\ -g^{-1/2}m_{xz}e_u(\delta_z x\cdot) & g^{-1/2}\delta_x x\cdot \end{bmatrix} \quad (5.14)$$

for the scalar product $u_{1/2}$ and extrapolation by copy. We check that this is a consistent discretization of $\mathbb{C}^*$.

3) ADJOINT OF $\mathbb{D}$

The gradient is a linear operator from $\bar{N}_p \to N_u$. Again we show only the one-dimensional Cartesian case $I = 1$, $C = I$. The gradient $\bar{\mathbb{G}}_p$ is vertical, and the matrix $\bar{\mathbb{G}}$ is rectangular $K \times (K+1)$:

$$\bar{\mathbb{G}} : \bar{N}_p \to N_w \quad \bar{\mathbb{G}}p = \begin{cases} 2(p_K - p_{K-1/2}) \\ \delta_z p = p_{k+1/2} - p_{k-1/2} \\ 2(p_{1/2} - p_0) \end{cases}$$

$$\text{or } \bar{G} = \begin{pmatrix} 2 & -2 & & & & \\ & 1 & -1 & & & \\ & & \cdots & \cdots & \cdots & \\ & & & 1 & -1 & \\ & & & & 2 & -2 \end{pmatrix}. \quad (5.15)$$

It is easy to check that $\bar{\mathbb{G}}$ in (5.15) and $-\bar{\mathbb{D}}$ from (5.11) are adjoint.

Combining the previous results, the adjoint of $-\mathbb{D}$ is the following consistent discretization in nonconservative form of the gradient:

$$\mathbb{G}p = \begin{bmatrix} g^{-1/2}\delta_z z\cdot & -g^{-1/2}m_{xz}e_w(\delta_x z\cdot) \\ -g^{-1/2}m_{xz}e_u(\delta_z x\cdot) & g^{-1/2}\delta_x x\cdot \end{bmatrix}\bar{\mathbb{G}}p. \quad (5.16)$$

The extrapolations in (5.16) mean we calculate the horizontal component of the gradient at the bottom boundary so that

$$\mathbb{i}\cdot g^{1/2}\mathbb{G}p|_{\bar{z}=0} = \delta_z z \delta_x p_0 - 2m_x[\delta_x z(p_{1/2} - p_0)].$$

We conclude that when extrapolation by copy and scalar product $u_{1/2}$ are chosen, the gradient as in (5.16) is the adjoint of (5.12) used for the divergence. We have two adjoint operators consistently discretized. These operators may be used for a convergent discretization of the pressure term.

However, adjoint discretized operators are not necessarily discretizations of the adjoint operator; for example, with the scalar product u_1, the adjoint divergence $-\bar{\mathbb{D}}^*$ on boundary points is half the gradient $\bar{\mathbb{G}}$.

e. Discussion

1) CONSERVATIVE GRADIENT

The kind of discretization used by Clark (1977) or Kapitza (1992) considers pressure as a physical entity defined only in the inner domain. So two levels of extrapolation are necessary:

1) define the contravariant components of the pressure gradient at the boundary, and
2) go from contravariant to Cartesian components.

Then remains the problem of enforcing boundary conditions for pressure. The next discretization is indicative and does not show all the extrapolations needed.

The adequate discretization of the gradient (5.7) takes into account the localization of p nodes to avoid extra averaging in the calculation of $\nabla\cdot p\mathbb{i}$. The line vector representing components of $\mathbb{G}p$ are given by

$$g^{1/2}\mathbf{G}p = (\delta_x, \delta_z)\begin{bmatrix} (m_x\delta_z z)\cdot & -m_z(\delta_z x m_x\cdot) \\ -m_x(\delta_x z m_z\cdot) & (m_z\delta_x x)\cdot \end{bmatrix} p. \quad (5.17)$$

A variety of extrapolations could be designed for determination of the off-diagonal terms of the above matrix at the boundaries.

2) Metric identities

In this section we show that, outside boundaries, the nonconservative and conservative forms are equivalent.

The horizontal component of the gradient in conservative form is

$$g^{1/2}\nabla\cdot(p\mathbf{i}) = \delta_x[m_x(\delta_z z)p] - \delta_z m_x(\delta_x z m_z p).$$

We use the identity

$$\delta_x(p m_x u) = \check{u}\delta_x p + m_x(p\delta_x u), \quad (5.18)$$

in which $\check{u}$, $\check{z}$ are defined only at inner points of N_u, N_ζ. For the first term it gives

$$\delta_z\check{z}\delta_x p + m_x[p\delta_x(\delta_z z)],$$

and for the second

$$-m_{xz}(\delta_x z\delta_z p) - m_x[p\delta_z(\delta_x z)];$$

thus,

$$g^{1/2}\nabla\cdot(p\mathbf{i}) = \delta_z\check{z}\delta_x p - m_{xz}(\delta_x z\delta_z p),$$

which is the discretization for the nonconservative form. The two forms are thus equivalent outside boundaries. The particular definition of the metric coefficients we took was necessary. The result is what Thompson (1982) coins "metric identities."

3) Conservation of momentum

For an inviscid fluid with orography in an $\bar{x}$ periodic domain, there is no conservation of total horizontal momentum due to the pressure drag. When we use generalized coordinates, $\bar{\rho}$ is a function of $\bar{x}$, so there is really no conservative form for the horizontal part of the pressure gradient. We might require only that the horizontal momentum flux varies only due to boundary contributions.

If the pressure force $\mathbf{P}$ is expressed in nonconservative form, the total momentum $\mathcal{I}$ created by the pressure term is

$$\mathcal{I} = \langle \mathbf{i};\, g^{1/2}\bar{\rho}\mathbf{G}p\rangle = -\langle\langle pj\mathbf{D}g^{1/2}\bar{\rho}i\rangle\rangle,$$

where $\mathbf{D}g^{1/2}\bar{\rho}\mathbf{i}$ at interior points evaluates the divergence of $\bar{\rho}\mathbf{i}$. For $\mathcal{I}$ to present only boundary contributions for any pressure field, we need this divergence to be null. This will occur in generalized coordinates only with a special choice of $\bar{\rho}$ at velocity nodes. If the pressure gradient is discretized in conservative form according to (5.17), from the metric identities, we expect the same condition to occur. There is no decisive advantage from a conservative form of the gradient here.

f. Elliptic equation

The pressure term in (4.5) when taking $\mathbf{D}$ from (5.12) and $\mathbf{G}$ from (5.16) is

$$\mathbf{P} = \bar{\rho}\mathbf{C}^*\bar{\mathbf{D}}^*(\bar{\mathbf{D}}\mathbf{C}g^{1/2}\bar{\rho}\mathbf{C}^*\bar{\mathbf{D}}^*)^{-1}\bar{\mathbf{D}}\mathbf{C}g^{1/2}\bar{\rho}\mathbf{s}. \quad (5.19)$$

The matrix inside the Poisson equation for pressure is (we display only the first column)

$$\mathbf{C}g^{1/2}\bar{\rho}\mathbf{C}^* = \begin{bmatrix} (\delta_z z)^2 g^{-1/2}\bar{\rho} + \delta_z x m_{xz} e_w(g^{-1/2}\bar{\rho} m_{xz} e_u \delta_z x\cdot) & \cdots \\ -\delta_x z m_{xz} e_u(g^{-1/2}\bar{\rho}\,\delta_z z\cdot) - \delta_x x g^{-1/2}\bar{\rho} m_{xz} e_u(\delta_z x\cdot) & \cdots \end{bmatrix},$$

when the same matrix in Bernard and Kapitza [1992, Eq. (3.19)] with our notations is apparently

$$\mathbf{Q} = \begin{Bmatrix} g^{-1/2}\bar{\rho}[(\delta_z z)^2 + (\delta_x z)^2]\cdot & \cdots \\ g^{-1/2}\bar{\rho}[-\delta_z x(m_{xz}e_u\delta_z z)m_{xz}e_u(\cdot) - \delta_x x(m_{xz}e_u\delta_z x)m_{xz}e_u(\cdot)] & \cdots \end{Bmatrix}.$$

The placement of the grid coordinates is the same; the extrapolations of the grid coordinates and velocities is the same. The two averaging operators on the diagonal terms, unnecessary for a consistent discretization of the equations, are removed. Off-diagonal terms are designed differently, so their matrix cannot be self-adjoint due to a lack of symmetry in the placement of the averaging operators and of the term $g^{-1/2}\bar{\rho}$.

In any case, our elliptic equation for p is not symmetric. It is self-adjoint for the kinetic energy, with half weight given to boundary velocities.

In our derivation, the boundary conditions are treated explicitly, without any attempt at eliminating the boundary pressures. This renders the design of a self-adjoint matrix more clear; it is also straightforward to cope with nonhomogeneous boundary conditions.

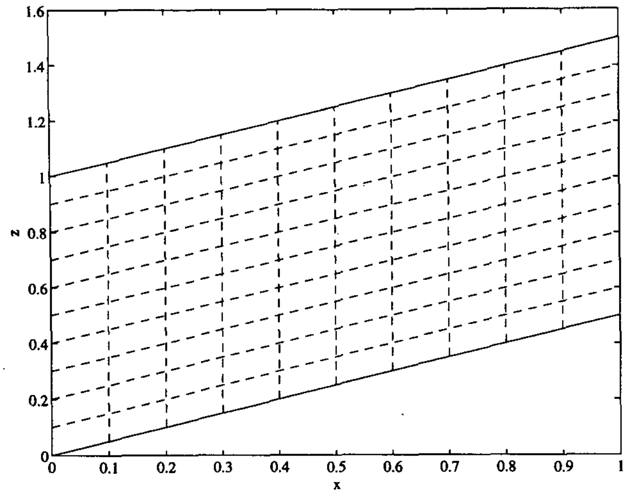

FIG. 2. Geometry of the x-periodic channel used to study convergence.

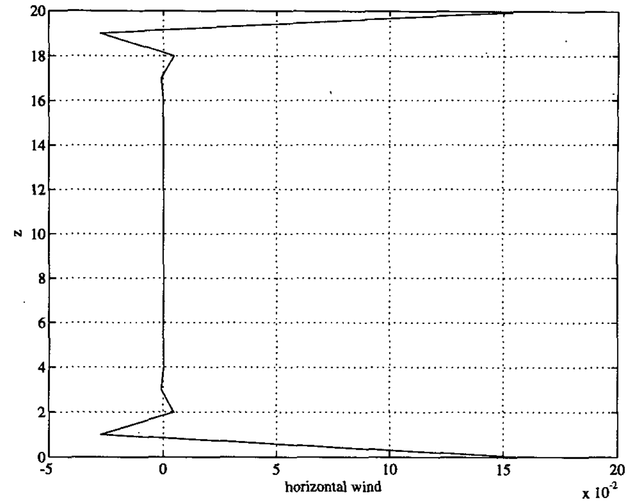

FIG. 3. Relative error of the wind component parallel to the boundary in experiment 1.

The pressure term in Bernard and Kapitza is

$$\mathbf{P} = \bar{\rho}\mathbb{G}(\bar{\mathbb{D}}\mathbb{Q}\bar{\mathbb{D}}^*)\mathbb{G}g^{1/2}\bar{\rho}\mathrm{s}. \qquad (5.20)$$

It necessitates the construction of the unrelated operators **G**, **D**, **Q**. It does not correspond to an orthogonal projection, so it conserves energy only approximately.

6. Numerical study of convergence—Analytical tests

We study convergence by projecting analytical velocity fields deriving from a gradient ($\mathrm{s} = \nabla f$) in a periodic channel of uniform slope α (see Fig. 2) and a Gal-Chen grid whose mesh is $\Delta\bar{x}$, $\Delta\bar{z} \to 0$. Only fields independent of x have been considered: $\mathrm{s} = \mathrm{s}(\bar{z})$.

The two components of the wind s have been checked separately.

1) A wind s parallel to the boundaries should be unchanged by pressure projection: $\mathbb{u} = \mathrm{s}$.

2) A constant wind s orthogonal to the boundary should have a zero projection: $\mathbb{u} = \mathbb{0}$.

This limited test, however, addresses the main problem of extrapolations in the presence of bottom slope.

We will consider three extrapolations for the Cartesian component u at the bottom $k = 0$ boundary, to be employed in (5.8), and determine the approximation $\mathbf{u}_h$ to the analytical projection $\mathbb{u}$:

1) $u_{i,-1/2} = 0$;
2) extrapolation by copy $u_{i,-1/2} = u_{i,1/2}$;
3) linear extrapolation $u_{i,-1/2} = 2u_{i,1/2} - u_{i,3/2}$;

and similar expressions at the top boundary.

The adjoint of the divergence leads to a first-order-accurate pressure term near the boundaries only with the specific choice of scalar product and extrapolations.

In other cases, a residual wind parallel to the boundary oscillates with exponential decay off the boundary. The relative error $e(z) = |\mathrm{s}(z)|^{-1}|\mathbb{u}(z) - \mathbb{u}_h(z)|$ has a maximum amplitude of 15% for a unit slope $\alpha = 1$, independent of grid resolution.

The pressure term has been calculated from (4.5) in the following experiments:

1) $\mathbb{G} = -\mathbb{D}^*$ for the scalar product u_1, projection of the parallel velocity is distorted by extrapolation 1 (Fig. 3) of the orthogonal one by extrapolation 2.

2) $\mathbb{G} = -\mathbb{D}^*$ for $u_{1/2}$ and extrapolation 2 is adequate.

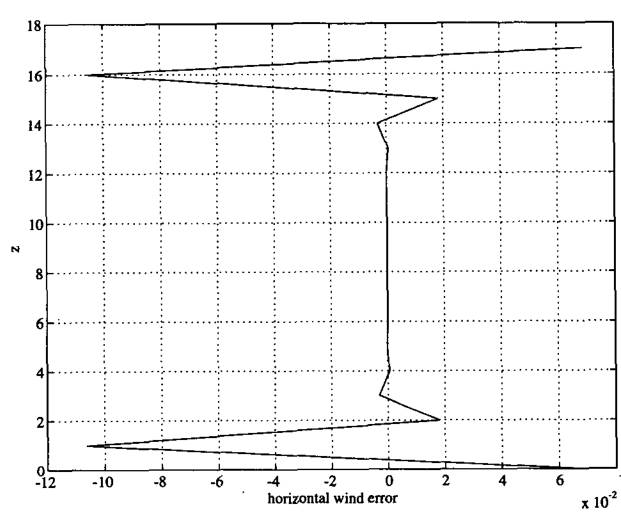

FIG. 4. Relative error of the wind component normal to the boundary in experiment 3.

3) $\mathbf{G} = -\mathbf{D}^*$ for $u_{1/2}$ and extrapolation 3. The pressure term we derive is not convergent (Fig. 4). The only choice is then to give up energy conservation and symmetry.

4) $\mathbf{G}$ from (5.17). A nonsymmetric formulation of the Laplacian is used. Extrapolation 2 (by copy) gives accuracy equivalent to experiment 2. Linear extrapolation gives a better accuracy.

Experiment 2 confirms the analytical results of the previous sections. If a higher accuracy is sought and a linear extrapolation is used, it is necessary to give up the symmetric formulation. However, this advantage might be apparent only for smooth velocity fields.

Integrations of the model to stationary state over a bell-shaped or semi-elliptic orography compare favorably with Long's analytical solutions up to 60% slopes (cf. Figs. 5 and 6).

No noticeable difference was found when the pressure term was calculated along experiment 1 or 2 for slopes up to 30% (Stein 1994). We suspect that some compensation of errors occurs in these integrations. Opposite sign errors for the horizontal wind are generated in the positive and negative slope regions.

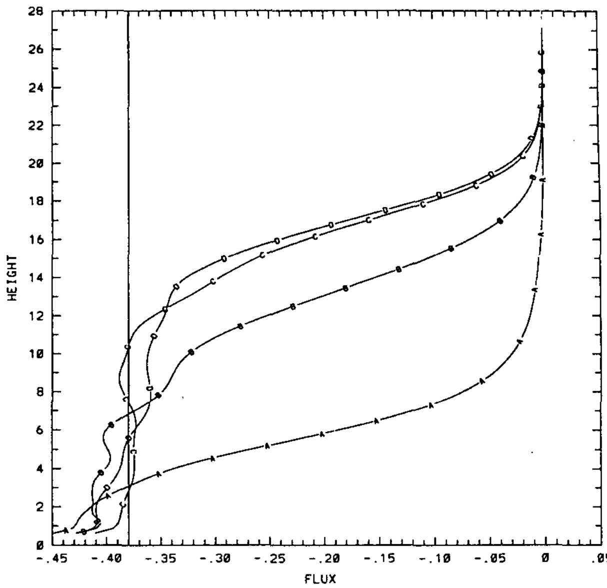

FIG. 6. Temporal evolution of the vertical flux of horizontal momentum compared to its theoretical value. Same parameters as in Fig. 5.

7. The pressure solver

The pressure solver is the heart of an anelastic model dynamical part, even without orography. It uses most of the computer resources, so it is important for it to be efficient and simple, without tuning of parameters, when orography, domain size, or grid spacing are changed.

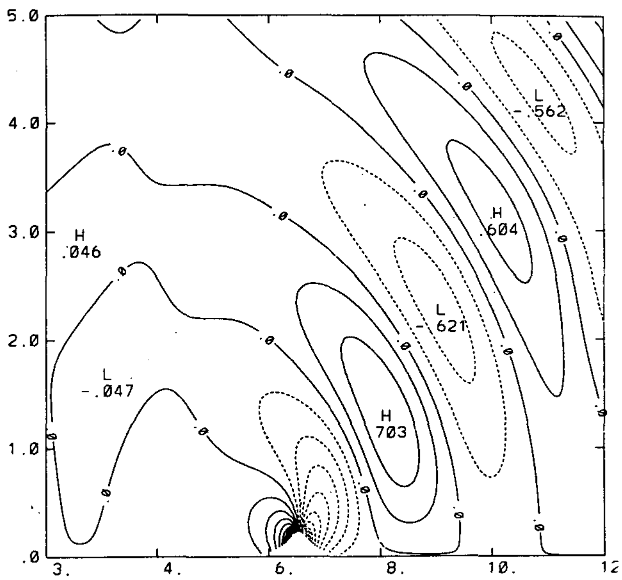

FIG. 5. Stationary 2D flow over a bell-shaped mountain with $NH/U = NL/U = 0.6$. Displayed: vertical velocity.

The elliptic equation for pressure (2.7) is symmetric and will be solved by a preconditioned conjugate gradient. It is well known (Golub and Meurant 1983) that conjugate gradient methods solve symmetric linear systems through a minimization problem. It is here

$$J(p) = \frac{1}{2}\langle \mathbf{D}^*p;\, g^{1/2}\bar{\rho}\mathbf{D}^*p\rangle_{u_{1/2}} - \langle\langle p;\, \mathbf{D}g^{1/2}\bar{\rho}\mathbf{s}\rangle\rangle.$$

The gradient of J is

$$\nabla J = \mathbf{D}g^{1/2}\bar{\rho}(\mathbf{D}^*p + \mathbf{s})$$

and is null when (2.7) is satisfied.

A conjugate-gradient method has an efficiency depending upon the condition number of the matrix to invert [see (B.1)]; the condition number κ of the Laplacian depends roughly upon the ratio of the extreme wavenumbers allowed for by the discretization

$$\kappa \approx I^2 + K^2\frac{L^2}{H^2} \tag{7.1}$$

when $L > H$. We want to precondition the conjugate gradient method so that the efficiency does not depend on the number of points in each direction. For this, we define the operator $\bar{\bar{\mathbf{D}}}$ by

$$\bar{\bar{\mathbf{D}}} = \bar{\mathbf{D}}\bar{\mathbf{C}},$$

with $\bar{g}^{1/2}$ and $\bar{\bar{\rho}}$ are $\bar{x}$-averaged quantities and $\bar{\mathbf{C}}$ the horizontal average of

$$\begin{pmatrix} \delta_z z g^{-1/2} & 0 \\ 0 & \delta_x x g^{1/2} \end{pmatrix},$$

where only diagonal terms are retained. Vertical and horizontal directions are decoupled. After the horizontal Fourier transform we are left, for each mode, with a tridiagonal matrix on the vertical to invert. The operator

$$\bar{\Delta} = \bar{\bar{\mathbb{D}}} g^{1/2} \bar{\bar{\bar{\rho}}} \bar{\bar{\mathbb{D}}}^* \qquad (7.2)$$

is negative definite and fast to invert, and so qualifies as a preconditioning for our problem.

The preconditioned gradient is then

$$\nabla J = \bar{\Delta}^{-1} \mathbb{D} g^{1/2} \bar{\rho} (\mathbb{D}^* p - \mathbb{s}).$$

The condition number of the operator $A = \bar{\Delta}^{-1} \mathbb{D} \times g^{1/2} \bar{\rho} \mathbb{D}^*$ determines the speed of convergence; it is nearly independent of truncation and aspect ratio $\Delta x / \Delta z$ (see appendix B). In Fig. 7 we show a projected velocity field for an extreme geometry.

8. Conclusions and summary

The anelastic system in a closed domain with rigid-wall boundary conditions or inflow–outflow-type of boundary conditions can be handled by a straightforward extension of the Hamiltonian methods of fluid dynamics. Pressure appears as the Lagrange multiplier for these constraints, and the pressure gradient appears in the momentum equation as the reaction force. Conservation of energy is the principle for the derivation of the equations of motion, including the pressure term. The pressure term has no action upon kinetic energy.

However, it is not necessary to go into the details of the Hamiltonian formulation to find the appropriate form of the pressure force, provided we know it should conserve energy. A simple mathematical argument based upon properties of adjoint operators leads us to the correct form both for the anelastic and the pseudocompressible systems. Moreover, it shows boundary constraints easier to incorporate.

In the discretized model, the divergence operator is augmented with the normal component of velocity at the boundary, so it expresses all the constraints at the same time. Pressure is defined at divergence nodes and on the boundary. Conservation of kinetic energy requires that the elliptic operator of the pressure problem is made by composition of the augmented divergence operator and its adjoint and that the pressure force is given by the adjoint of this augmented divergence.

However, extrapolations are necessary to define derivatives at the boundary, and the adjoint of a discretized divergence operator is not necessarily a consistent discretization of a gradient. Numerical examples show this leads to Gibbs-like error in the pressure term near the boundaries [what is referred to in Bernard (1992) as "creation of vorticity"] except with some specific definitions of the metric coefficients, extrapolations, and kinetic energy. This is consistent with known mathematical results.

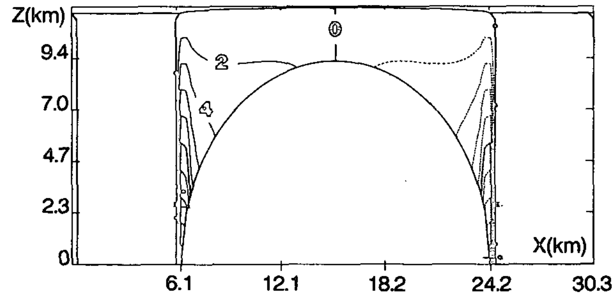

Fig. 7. Projection of a constant $u = 18$ m s^{-1} velocity field. Shown is the resulting field of w. The domain is periodic with $\Delta x = \Delta z$. The orography is half a circle occupying three-fourths of the height.

With these appropriate definitions, we have shown analytically that the nonconservative gradient can be written as in (5.16) and is a consistent approximation.

The current practice (Clark 1977; Bernard and Kapitza 1992) is to sacrifice energy conservation. Separate, consistent discretizations are designed for the divergence, gradient, and elliptic operator. Their definition of the metric coefficients and extrapolations is the same as here. Clark's and Bernard and Kapitza's elliptic operator appears to be very similar to ours, with the redundant averaging operator removed and with a different placement of some of the other operators. Redundant averaging operators are necessary for a strict energy conservation. They appear because our elliptic operator is the composition of the divergence and the gradient.

It has to be recognized that, even properly designed, the elliptic operator is not symmetric. It is self-adjoint for the scalar product, giving kinetic energy with a half weight to velocities defined at the boundary.

We have also shown that the nonconservative form of the gradient is equivalent to the conservative form outside the boundaries. However, it should be borne in mind that in order to get only boundary contributions to the total horizontal angular momentum both forms are equivalent and require the nontrivial property that the gradient of $\bar{\rho}$ be vertical.

The model using this pressure solver and extrapolations proves accurate for the study of orographic processes and compares favorably with Long's solution even with semi-elliptic mountains. It is not too sensitive to the aforementioned details of discretization for shallow orography.

The preconditioned conjugate gradient is the choice method to solve the elliptic problem for pressure, as shown in Kadogliu (1992) or Kapitza (1987). However, when the problem is not symmetric, less stable and more expensive variants have to be employed [such as GMRES (Kadogliu 1992) or ORTHOMIN (Kapitza 1987)]. The discretizations advocated here permit the use of plain conjugate-gradient iterations.

Clark (1977) uses the same Laplacian operator, discretized on a rectangular grid (no orography), in his iterations. We use this operator for preconditioning. It is solved with an FFT direct method.

Accuracy for a given number of iterations depends primarily upon the condition number of the preconditioned Laplacian. It is studied in appendix B. A bound upon condition number has been given. It was found to be nearly independent of the discretization mesh but to depend on domain geometry through the slope and the relative height of orography. The pressure solver handles even pathological orography easily.

Energy conservation is necessary in the design of sound systems of equations. Solution of the discretized hydrostatic equations is usually sought to conserve energy (Arakawa 1983); for the pressure equation of the anelastic model, we saw it was difficult to achieve in the presence of orography; semi-implicit time stepping or treatment of advection terms by semi-Lagrangian methods (Ritchie 1987) emphasize economy or accuracy at the expense of strict energy conservation.

However, we feel it is desirable, at least as a guideline, to examine discretization methods that preserve the symmetry properties of the Hamiltonian formalism. Sensitivity studies (Errico and Vukicevic 1992) provide the need to develop linearized and adjoint equations. Following the Hamiltonian structure presents the advantage that most adjoint operators are already part of the direct model. The anelastic system is a choice testbed to study such an approach. The pressure term is used to project tendencies onto velocities satisfying the anelastic constraint, and acts as a separate layer in the evolution equations.

Convergence of the model assumes that discretized adjoint operators are sound discretizations of the continuous operator they are to represent. Purnell and Revell (1993) present the equations under a clever systematic form that, although not symplectic, guarantees energy conservation. The convergence of their discretization should be ascertained. Use of finite elements normally precludes the type of problem we found with gridpoint methods.

This study is the first step necessary to derive convergent finite-difference discretizations that would follow the Hamiltonian structure of the anelastic model.

Acknowledgments. J. Stein (Météo-France) introduced me to the former version of the 2D anelastic model in generalized coordinates and participated in code restructuring prior to adjoint developments. He was the main user and main thrust for the present developments. Frequent discussions with L. Amodei (Météo-France) about convergence of discretized variational problems were particularly fruitful. Remarks from P. Smolarkiewicz greatly helped to improve the paper.

APPENDIX A

Derivation of the Anelastic Equations from the Hamiltonian Perspective

In this appendix we derive the anelastic equations from the Hamilton principle. We first derive the form of the pressure term and the Lagrangian form of the equations, without taking care of the boundary conditions. We then discuss the general form of the canonical equations with a constraint in order to give some properties of the reaction force to be used in section 2a with boundary conditions taken into account.

Following Salmon (1983), we define a marker $\mathbb{a} = (a, b)$ attached to each fluid parcel. It should be thought of as related to its original position. We chose the markers so that *dadb* represents the mass of the parcel. For each fluid parcel we should determine the evolution of its position $\mathbb{x}(\mathbb{a}, t)$ and velocity $\mathbb{u}(\mathbb{a}, t)$ in time. The total energy of the anelastic system, according to Eq. (2.2), will be

$$\mathcal{H} = \int \left[\frac{1}{2}\mathbb{u}\cdot\mathbb{u} + c_p\overline{\pi}(\mathbb{x})\theta(\mathbb{a})\right]dadb.$$

The density is a dependent variable determined by the positions of the fluid parcels $\mathbb{x}$:

$$\rho = \left[\frac{\partial(\mathbb{x})}{\partial(\mathbb{a})}\right]^{-1},$$

where

$$\frac{\partial(\mathbb{x})}{\partial(\mathbb{a})} = \frac{\partial x}{\partial a}\frac{\partial z}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial z}{\partial a}$$

is the Jacobian operator, so that the anelastic constraint is

$$\mathcal{D}(\mathbb{x}) \equiv \frac{\partial(\mathbb{a})}{\partial(\mathbb{x})} - \overline{\rho}(\mathbb{x}) = 0.$$

We assume no diabatic processes are present. Potential temperature of a parcel is unchanged during motion; thus, it is a function of the particle label only.

The modified Hamilton principle is augmented to express that the system obeys the constraint (Goldstein 1980, p. 377); it states that the following integral

$$\mathcal{I}(\mathbb{x}, \mathbb{u}, p) = \int (\mathbb{u}\cdot\dot{\mathbb{x}} - \mathcal{H})dadb + \int p\mathcal{D}(\mathbb{x})dadb, \quad \text{(A.1)}$$

or, replacing,

$$\mathcal{I}(\mathbb{x}, \mathbb{u}, p) = \int \left[\mathbb{u}\cdot\dot{\mathbb{x}} - \frac{1}{2}\mathbb{u}\cdot\mathbb{u} - c_p\overline{\pi}(\mathbb{x})\theta(\mathbb{a})\right]dadb + \int \left[\frac{\partial(\mathbb{a})}{\partial(\mathbb{x})} - \overline{\rho}(\mathbb{x})\right]pdadb,$$

is stationary under a variation of the position $\mathbf{x}$, the momentum $\mathbf{u}$, or the Lagrange multiplier for the constraint p. The Euler equations of this variational problem are

$$\dot{\mathbf{x}} = \mathbf{u}$$
$$\dot{\mathbf{u}} = \theta \nabla c_p \overline{\pi} + \mathbf{P}$$
$$\rho = \overline{\rho},$$

where $\mathbf{P}$ is the reaction force coming from the variation of the last integral containing the constraint; $\mathbf{P}$ appears in the momentum equation, as the constraint depends on position only.

To determine $\mathbf{P}$, as in Salmon (1983), we will use the following:

$$\delta\left[\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}\right] = \left[\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}\right]^{-2} \delta\left[\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}\right]$$

$$\delta\left[\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}\right] = \frac{\partial(\delta\mathbf{x})}{\partial(\mathbf{a})} = \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}\frac{\partial(\delta\mathbf{x})}{\partial(\mathbf{x})}$$

$$\frac{\partial(\delta\mathbf{x})}{\partial(\mathbf{x})} = \nabla\cdot\delta\mathbf{x},$$

which, when substituting δ for d/dt, are used to derive the continuity equation

$$\frac{d\rho}{dt} = \frac{d}{dt}\left[\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}\right] = \rho\nabla\cdot\mathbf{u}.$$

Along with the constraint $\rho = \overline{\rho}$, this gives the usual anelastic constraint

$$\nabla\cdot\overline{\rho}\mathbf{u} = 0.$$

Integration by parts,

$$\int \rho p \nabla\cdot\delta\mathbf{x}\,dadb = \int \rho^2 p\nabla\cdot\delta\mathbf{x}\,dxdz$$

$$= -\int \delta\mathbf{x}\cdot\nabla\rho^2 p\,dxdz = -\int \delta\mathbf{x}\cdot(\nabla\rho p + p\nabla\rho)\,dadb,$$

is also used in the following transformations:

$$\delta\int\left[\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})} - \overline{\rho}(\mathbf{x})\right]p\,dadb$$

$$= \int(-\rho\nabla\cdot\delta\mathbf{x} - \delta\mathbf{x}\nabla\overline{\rho})p\,dadb$$

$$= \int \delta\mathbf{x}\cdot[\nabla\overline{\rho}p + p\nabla(\rho - \overline{\rho})]\,dadb,$$

and, finally, using the constraint

$$\mathbf{P} = \nabla p\overline{\rho}(\mathbf{x})$$

so the pressure term is a gradient.

A general derivation of the equations of motion in canonical form from (A.1) will highlight the role of the constraint $\mathcal{D}$ and its adjoint. We need to define scalar products

$$\langle \mathbb{u}; \mathbb{v}\rangle = \int \mathbb{u}\cdot\mathbb{v}\,dadb = \int \rho\mathbf{u}\cdot\mathbf{v}\,dxdz,$$

or scalars

$$\langle\langle p; q\rangle\rangle = \int pq\,dadb$$

and the functional derivative of $\mathcal{H}$ with respect to $\mathbb{x}$ is defined by

$$\delta\mathcal{H} = \left\langle \frac{\delta\mathcal{H}}{\delta\mathbf{x}}; \delta\mathbf{x}\right\rangle;$$

$\mathcal{D}$ is linear, so the variation of the integral expressing the constraint is

$$\delta\langle\langle p; \mathcal{D}\mathbb{x}\rangle\rangle = \langle\langle p; \mathcal{D}\delta x\rangle\rangle = \langle \mathcal{D}^* p; \delta x\rangle.$$

The first two equations may be written as

$$\begin{pmatrix}\dot{x}\\ \dot{u}\end{pmatrix} = \mathbb{J}\begin{pmatrix}\dfrac{\delta\mathcal{H}}{\delta x} + \mathcal{D}^* p\\ \\ \dfrac{\delta\mathcal{H}}{\delta u}\end{pmatrix}, \qquad (A.2)$$

where $\mathbb{J}$ is the antisymmetric tensor

$$\mathbb{J} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

This form of the equations of motion is said to be canonical.

Here $\mathcal{D}$ does not depend on time. It is linear, so, differencing in time,

$$\mathcal{D}(\dot{\mathbb{x}}) = \mathcal{D}\left(\frac{\delta\mathcal{H}}{\delta\mathbb{u}}\right) = 0.$$

The time derivative of energy is

$$\frac{d\mathcal{H}}{dt} = \left\langle\frac{\delta\mathcal{H}}{\delta\mathbf{x}}; \dot{\mathbf{x}}\right\rangle + \left\langle\frac{\delta\mathcal{H}}{\delta\mathbf{u}}; \dot{\mathbb{u}}\right\rangle$$
$$= \left\langle\frac{\delta\mathcal{H}}{\delta\mathbf{u}}; \mathcal{D}^* p\right\rangle = 0, \qquad (A.3)$$

so the reaction force does not produce energy. The pressure force $\mathcal{D}^* p$ is orthogonal to any $\mathbb{x}$ or $\dot{\mathbb{x}}$ satisfying the constraint. This is the property we rely on in the main part of the paper, with scalar products defined in the Cartesian space and not as here in the phase space. The form (A.2) and property (A.3) hold only

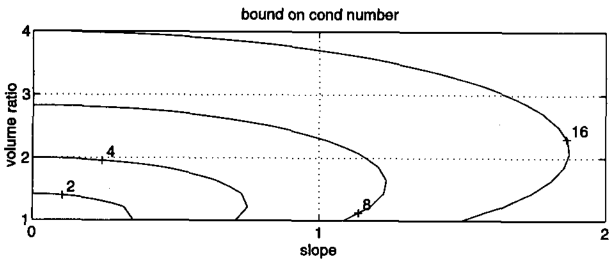

FIG. B1. Bound on condition number in terms of volume ratio and slope of orography.

for a holonomous, linear constraint depending on positions only.

APPENDIX B

Efficiency of the Solver

For moderate steepness and heights, the "flat Laplacian" $\bar{\Delta}$ defined in Eq. (7.2) is a good enough approximation to the true one $\tilde{\Delta}$ so that the Richardson method (Golub and Meurant 1983, p. 193) is a possible method of solution, even with a nonsymmetric operator.

Let us define $A = \bar{\Delta}^{-1}\tilde{\Delta}$. The Richardson iteration is

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \alpha\bar{\Delta}^{-1}(b - \tilde{\Delta})\mathbf{u}_n,$$

and the error is

$$\mathbf{e}_n = (I - \alpha\mathbf{A})^n\mathbf{e}_0,$$

so the method is convergent when we choose α so that $\rho(I - \alpha\mathbf{A}) < 1$. For nonsymmetric matrices, the error bound is expressed as

$$\langle \mathbf{e}_n; \mathbf{e}_n \rangle < \rho[(I - \alpha\mathbf{A})^n(I - \alpha\mathbf{A}^*)^n]\langle \mathbf{e}_0; \mathbf{e}_0 \rangle, \quad \text{(B.1)}$$

where $\rho(\mathbf{A})$ is the spectral radius of $\mathbf{A}$. As for a nonsymmetric matrix, $\rho(\mathbf{AA}^*) > [\rho(\mathbf{A})]^2$, no majoration of the error norm is given in terms of the eigenvalues of $\mathbf{A}$. Only in the case of symmetric matrices will it be

$$\langle \mathbf{e}_n; \mathbf{e}_n \rangle < [\rho(I - \alpha\mathbf{A})]^{2n}\langle \mathbf{e}_0; \mathbf{e}_0 \rangle.$$

The conjugate gradient method works for any symmetric matrix $\mathbf{A}$; a majoration for the error is expressed as a function of the condition number κ, ratio of the extreme eigenvalues $\kappa = \lambda_{max}/\lambda_{min}$ (Golub and Meurant 1983, p. 222):

$$\langle e_n; \mathbf{A}e_n \rangle < 4\left(\frac{\kappa - 1}{\kappa + 1}\right)^{2n+2}\langle e_0; \mathbf{A}e_0 \rangle.$$

For $n > 1$, with low orography, we have observed $\lambda_{max} - 1 \approx 1 - \lambda_{min} \approx 0.5(\kappa - 1)$. Residual error is better for the conjugate gradient than for the Richardson method with $\alpha = 1$ in a ratio 2^{n-1}.

Let $\tilde{\Delta} = \bar{\mathbf{D}}\mathbf{CC}^*\bar{\mathbf{D}}^*$ and $\bar{\Delta} = \bar{\mathbf{D}}\bar{\mathbf{C}}\bar{\mathbf{C}}^*\bar{\mathbf{D}}^*$, and let us obtain bounds upon the eigenvalues of $\bar{\Delta}^{-1}\tilde{\Delta}$. We assume $\mathbf{C}$ and $\bar{\mathbf{C}}$ are invertible.

Let $\mathcal{P}$ be the orthogonal projection upon $\mathrm{Im}(\bar{\mathbf{C}}^*\bar{\mathbf{D}}^*)$ and $\mathbf{v}$ be an eigenvector of $\mathcal{P}(\bar{\mathbf{C}}^{-1}\mathbf{CC}^*\bar{\mathbf{C}}^{-*})$; then,

$$\bar{\mathbf{C}}^{-1}\mathbf{CC}^*\bar{\mathbf{C}}^{-*}\mathbf{v} = \lambda\mathbf{v} + \mathbf{n},$$

with $\mathbf{v} = \bar{\mathbf{C}}^*\bar{\mathbf{D}}^*p$, $\mathbf{n}$ in $\mathrm{Ker}\bar{\mathbf{D}}\bar{\mathbf{C}}$, and then

$$\bar{\Delta}^{-1}\bar{\mathbf{D}}\bar{\mathbf{C}}(\bar{\mathbf{C}}^{-1}\mathbf{CC}^*\bar{\mathbf{C}}^{-*})\mathbf{v} = (\bar{\mathbf{D}}\bar{\mathbf{C}}\bar{\mathbf{C}}^*\bar{\mathbf{D}}^*)^{-1}(\bar{\mathbf{D}}\mathbf{CC}^*\bar{\mathbf{D}}^*)p = \lambda p,$$

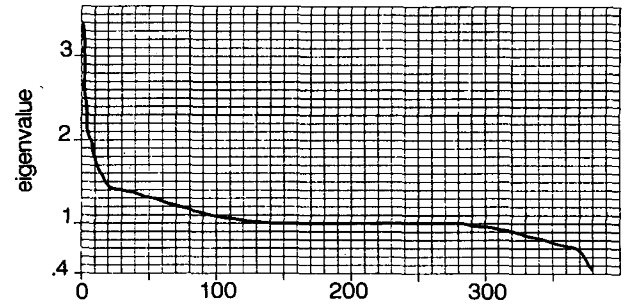

FIG. B3. Same as Fig. B2 but for a semicircle $a = 5$.

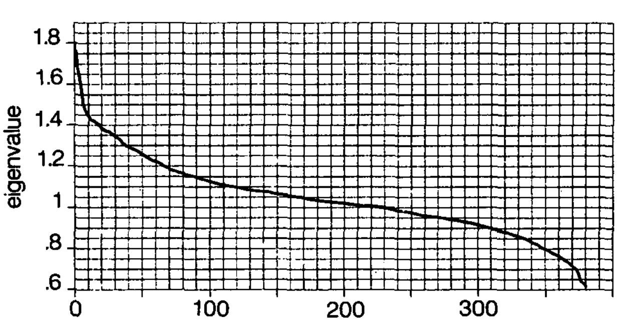

FIG. B2. Eigenvalues of the preconditioned elliptic operator for pressure. Scalar product $u_{1/2}$. Domain width equals 20. Domain height equals 17. Orography is a Witch of Agnesi $h = a = 5$.

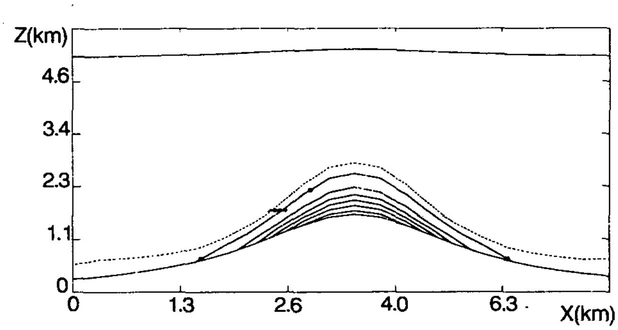

FIG. B4. Leading eigenvector of the preconditioned elliptic operator for pressure. Geometry is identical to Fig. B2.

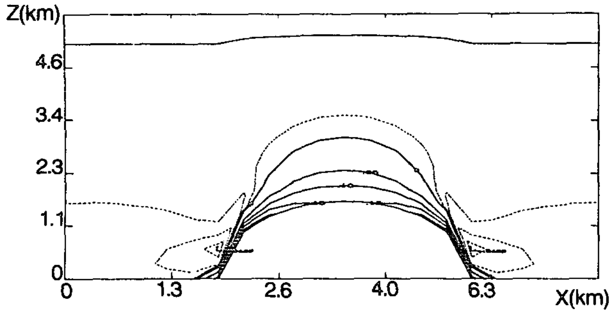

FIG. B5. Same as Fig. B4 but for the semicircle of Fig. B3.

so λ, the eigenvalue of $\mathcal{P}(\bar{\mathbf{C}}^{-1}\mathbf{C}\mathbf{C}^{*}\bar{\mathbf{C}}^{-*})$, is also an eigenvalue of $\bar{\Delta}^{-1}\tilde{\Delta}$ and is bounded by the spectral radius ρ of the symmetric matrix $\bar{\mathbf{C}}^{-1}\mathbf{C}\mathbf{C}^{*}\bar{\mathbf{C}}^{-*}$:

$$\lambda < \rho(\bar{\mathbf{C}}^{-1}\mathbf{C}\mathbf{C}^{*}\bar{\mathbf{C}}^{-*}).$$

In the same way, considering $\tilde{\Delta}^{-1}\bar{\Delta}$, we find

$$\rho^{-1}(\mathbf{C}^{-1}\bar{\mathbf{C}}\bar{\mathbf{C}}^{*}\mathbf{C}^{-*}) < \lambda,$$

so considering both,

$$\kappa(\bar{\Delta}^{-1}\tilde{\Delta}) < \kappa(\bar{\mathbf{C}}^{-1}\mathbf{C}\mathbf{C}^{*}\bar{\mathbf{C}}^{-*}),$$

where $\bar{\mathbf{C}}^{-1}\mathbf{C}$ pertain to the transformation $x' \rightarrow x$. For a Gal-Chen type of coordinate,

$$\bar{\mathbf{C}}^{-1} = \begin{pmatrix} \Delta x' & 0 \\ & \Delta z' \end{pmatrix},$$

$$\mathbf{C} = \frac{1}{\Delta x \Delta z}\begin{pmatrix} \Delta z & 0 \\ \Delta h m_{xz} e_u & \Delta x \end{pmatrix},$$

$$\bar{\mathbf{C}}^{-1}\mathbf{C} = \begin{pmatrix} \dfrac{\Delta x'}{\Delta x} & 0 \\ \dfrac{\Delta z'}{\Delta x \Delta z}\delta_x z m_{xz} & \dfrac{\Delta z'}{\Delta z} \end{pmatrix}.$$

With an x-uniform grid, the bound on the condition number depends upon the two quantities: volume ratio, $\Delta z'/\Delta z \approx H/H - z$, where H is the top height of the domain, and slope $s = (\Delta z'/\Delta x \Delta z)\delta_x z \approx (H/H - z)(\partial_{\bar{z}} z/\partial x)$. Figure B1 shows the condition number of the above local matrix (with no averaging operator $m_{xz}e_u$) in terms of maximum s and maximum volume ratio. Averaging tends to decrease the effect of the slope for high-wavenumber fields. Except for the averaging operator, the $\mathbf{C}$ matrices are local. Vector fields that are gradients have no special orientation, so we expect these bounds to be meaningful. For a given orography and total height H, these bounds are independent of the number of points used in the discretization.

Numerical experiments confirm these bounds on the condition number κ. The domain has $\Delta x = \Delta z$, $H = 18\Delta z$, $L = 20\Delta x$; a is the half-width of the bell-shaped {Witch of Agnesi, $z(x) = h/[1 + (x/a)^2]^{-1}$} orography. Maximum height of orography is h:

a, h	5, 1	5, 5	5, 9	9, 9
κ observed	1.2	2.8	7.7	4.5
bound on κ	1.3	4.5	17	6.6 .

At the difference of the preconditioning used by Kapitza (1988), the spectrum of $\bar{\Delta}^{-1}\tilde{\Delta}$ (Figs. B2 and B3) presents no eigenvalues near zero. For the Witch of Agnesi, extreme eigenvalues are inverse of each other. Leading eigenvectors (Figs. B4 and B5) point at maximum height for the Witch of Agnesi and at maximum slope for the ellipse.

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