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SIMULATION OF WAVE INTERACTIONS AND TURBULENCE IN ONE-DIMENSIONAL WATER WAVES*

KURT M. BERGER† AND PAUL A. MILEWSKI‡

Abstract. The weak- or wave-turbulence problem consists of finding statistical states of a large number of interacting waves. These states are obtained by forcing and dissipating a conservative dispersive wave equation at disparate scales to model physical forcing and dissipation, and by predicting the spectrum, often as a Kolmogorov-like power law, at intermediate scales. The mechanism for energy transfer in such systems is usually triads or quartets of waves. Here, we first derive a small-amplitude nonlinear dispersive equation (a finite-depth Benney–Luke-type equation), which we validate, analytically and numerically, by showing that it correctly captures the main deterministic aspects of gravity wave interactions: resonant quartets, Benjamin–Feir-type wave-packet stability, and wave-mean flow interactions. Numerically, this equation is easier to integrate than either the full problem or the Zakharov integral equation. Some additional features of wave interaction are discussed such as harmonic generation in shallow water. We then perform long time computations on the forced-dissipated model equation and compute statistical quantities of interest, which we compare to existing predictions. The forward cascade yields a spectrum close to the prediction of Zakharov, and the inverse cascade does not.

Key words. water waves, wave turbulence, finite depth, quartets

AMS subject classifications. 74J15, 74J30, 76B15, 76F99

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1. Introduction.

The weak- or wave-turbulence problem consists of finding statistical states of a large number of interacting waves. These states are obtained by forcing and dissipating a conservative dispersive wave problem at disparate scales and predicting the spectrum, often as a Kolmogorov-like power law, at intermediate scales. In dispersive waves, the energy transfer between waves occurs mostly amongst *resonant* sets of waves, usually triads or quartets of waves. Here we consider only quartets since triads do not exist in surface gravity waves. Quartet resonances occur when the product of a pair of waves has a component with the same frequency and wavenumber as the product of two other waves. For simple waves $e^{i(\mathbf{k}_j\cdot\mathbf{x}-\omega(\mathbf{k}_j)t)}$, this means

$$\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4, \tag{1.1}$$

$$\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)=\omega(\mathbf{k}_3)+\omega(\mathbf{k}_4). \tag{1.2}$$

In dispersive problems, these resonant sets are sparse, in contrast to nondispersive problems, where interactions are dense in Fourier space ((1.2) is always satisfied). The deterministic dynamics of *isolated* resonant quartets are modeled by sets of coupled nonlinear differential equations for the wave amplitudes and are well understood (see [10]). The dynamics of quartets which are not isolated (allowed to interact with other quartets) are poorly understood. In the limit in which all possible quartets are active, statistical theories of wave turbulence apply.

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†Department of Mathematics, The Ohio State University, 231 W. 18th Ave., Columbus, OH 43210 (berger@oblon.com).
‡Department of Mathematics, University of Wisconsin, 480 Lincoln Dr., Madison, WI 53706 (milewski@math.wisc.edu). The research of this author was partially supported by NSF-DMS and a Sloan Research Fellowship.

The initial work on wave turbulence was done by Hasselmann [14], Benney and Saffmann [7], and Benney and Newell [5], who introduced the statistical closures based on the resonant wave interactions. Zakharov [29], through conformal transformations, solved the resulting kinetic equation and obtained the power law for the Kolmogorov spectrum. The particular physical context for these initial results was the ocean surface gravity wave spectrum.

Majda, McLaughlin, and Tabak [19] started the numerical investigation of the predictions of weak turbulence theory using a nonlinear Schrödinger-like (NLS) model equation. Adding large-scale forcing and dissipation to their one-dimensional model, they investigated the turbulent cascades of energy and initially showed that Zakharov's [29] prediction for the energy spectrum did not hold, proposing a simpler, yet unrigorous quartet-based scaling to explain their results. More recent work by Cai et al. [9] and Zakharov et al. [30] shows that several meta-stable spectra can coexist in the system, with the Zakharov spectra being among those observed.

Classical, small-amplitude periodic gravity waves, discovered by Stokes, are unstable to small modulations through the Benjamin–Feir instability. This result was derived independently by Lighthill [15], Benjamin [1], and Whitham [26], and confirmed experimentally by Benjamin and Feir [2]. One can obtain the result by analyzing the slow modulation of gravity waves and deriving an NLS equation for the evolution of the wave envelope (see Hasimoto and Ono [13] and Zakharov [28], among others). A plane wave solution of the NLS equation corresponds to the Stokes wave, and it can be shown for waves of wavelength $\frac{2\pi}{k}$ in water of depth H to be unstable when $kH > 1.363$. (Waves in deeper water are unstable, and the NLS switches from "defocusing" to "focusing.") Davey and Stewartson [11] generalized this result to two spatial dimensions, deriving a more complicated NLS-type equation. This result for two-dimensional waves was derived independently a few years earlier, however, by Benney and Roskes [6], albeit in a slightly different form.

Here we investigate wave interaction and turbulence numerically for an equation describing small-amplitude gravity water waves. We perform wave interaction experiments and long time wave turbulence computations using a finite-depth Benney–Luke (fBL) equation [21]. To validate this model, we first show, analytically and numerically, that the fBL equation correctly captures the main deterministic aspects of resonant gravity wave interactions: resonant quartets and the Benjamin–Feir-type wave-packet stability. Some additional features of our numerical results are discussed: the generation of harmonics in shallow water and the long time frequency downshift of unstable wavepackets. For the wave-turbulence experiments, we compare the computed wave spectrum to predicted spectra. We note that the use of a single partial differential equation, rather than the full water wave equations, makes computing complex surface wave dynamics possible. All of our work is for a one-dimensional free surface. Although the computation of the two-dimensional free surface problem is not fundamentally different, we restrict our attention to the one-dimensional problem because of computational time constraints.

We note that there is a fundamental difference in the wave interaction problem between the one-dimensional and two-dimensional free surface. In two dimensions and infinite depth, the fundamental interaction mechanism is resonant quartets. The quartet interaction coefficients, however, vanish as the waves become parallel to each other, and therefore, for one-dimensional infinite depth, quartets are not important. Thus in a one-dimensional deep water system, the strongest mechanism for energy exchange between Fourier modes is the Benjamin–Feir instability, which is local in Fourier space, and the slower quintet interaction, which requires quartic terms in the equations to be modeled correctly. The Benjamin–Feir instability is also relevant for two-dimensional free surface problems.

For a one-dimensional free surface over water of finite depth, there exist quartet interactions (which vanish as $|\mathbf{k}| \to \infty$ to agree with the deep water limit). Therefore the one-dimensional finite depth problem is a computationally accessible useful test for the more relevant two-dimensional problem. That is why we restrict our numerical calculations to waves that are long enough to be influenced by the bottom.

The remainder of this paper is organized as follows. In section 2, we derive the fBL equation. Next, in section 3, we derive the nonlinear Schrödinger equation from the one-dimensional fBL equation using a multiple-scales approach, in a manner similar to Hasimoto and Ono, who started from the full water wave equations. This NLS equation correctly predicts the Benjamin–Feir instability limit, which we verify numerically using the fBL equation. In section 4 we derive a set of new partial differential equations that describe the coupled evolution of quartets and the induced mean flow for one-dimensional finite-depth gravity waves. We then show that solutions to these quartet equations closely match numerical solutions of the fBL equation, when initialized with four waves that satisfy the resonance conditions. We also study a model of the interaction of a primary wave and its quasi-resonant second harmonic in shallow water to explain the quartet simulation results. Finally, in the last section, we investigate wave turbulence numerically using the fBL model.

2. The Benney–Luke equation for gravity waves in finite depth. The Benney–Luke equation [4] describes the evolution of three-dimensional, weakly nonlinear waves in shallow water. Recently Milewski and Keller [22] derived a more general Benney–Luke model for waves in water of finite depth, shown here in a slightly different (and corrected) form:

$$u_{tt} + \mathcal{L}u + \epsilon\mathcal{N}_1(u,u) + \epsilon^2\mathcal{N}_2(u,u,u) = 0 \tag{2.1}$$

with quadratic terms

$$\mathcal{N}_1 = (\nabla u)_t^2 + (\mathcal{L}u)_t^2 + u_t\Delta u - u_t\mathcal{L}u_{tt} \tag{2.2}$$

and cubic terms

$$\begin{aligned}\mathcal{N}_2 = \frac{1}{6}\nabla\cdot(\nabla u\,(\nabla u)^2) + (\Delta u - \mathcal{L}^2 u)\left(u_t\mathcal{L}u_t - \frac{1}{2}(\mathcal{L}u)^2\right) + 2u_t\Delta u_t\mathcal{L}u\\ - 2u_t(\nabla u\cdot\nabla\mathcal{L}u)_t + 2\mathcal{L}u(\nabla u\cdot\nabla\mathcal{L}u) + \frac{1}{2}\mathcal{L}^2u(\nabla u)^2.\end{aligned} \tag{2.3}$$

In this equation, $\epsilon = a/H \ll 1$ is the ratio of wave amplitude a to depth H, $u(x,y,t)$ is the velocity potential at the undisturbed free surface $z = H$, and $\mathcal{L}$ is the operator $\mathcal{L} = (-\Delta)^{\frac{1}{2}}\tanh[(-\Delta)^{\frac{1}{2}}]$, resulting in the dispersion relation $\omega^2 = |\mathbf{k}|\tanh(|\mathbf{k}|)$. The water surface is given by $H + \eta(x,y,t)$, where, to leading order, $\eta = -u_t$. The fBL is derived as follows. Using the depth H as both the horizontal and vertical length scale, a as the scale for typical free surface displacements, $a\sqrt{gH}$ as the velocity potential scale, and $\sqrt{H/g}$ as the time scale, the dimensionless water wave equations can be written in terms of the velocity potential $\phi(x,y,z,t)$ and free surface displacement $\eta(x, y, t)$ as

$$\Delta\phi + \phi_{zz} = 0, \quad 0 < z < 1 + \epsilon\eta, \tag{2.4}$$

$$\phi_z = 0, \quad z = 0, \tag{2.5}$$

$$\eta_t + \epsilon(\nabla\eta \cdot \nabla\phi) - \phi_z = 0, \quad z = 1 + \epsilon\eta, \tag{2.6}$$

$$\phi_t + \frac{\epsilon}{2}(\nabla\phi)^2 + \frac{\epsilon}{2}\phi_z^2 + \eta = 0, \quad z = 1 + \epsilon\eta. \tag{2.7}$$

Expanding the two surface boundary conditions about $z = 1$ and eliminating η leads to a single boundary condition in ϕ at $z = 1$, correct to $\mathrm{O}(\epsilon^2)$:

$$\phi_{tt} + \phi_z + \epsilon\mathcal{Q}_1(\phi, \phi) + \epsilon^2\mathcal{Q}_2(\phi, \phi, \phi) = 0, \tag{2.8}$$

where the quadratic terms are

$$\mathcal{Q}_1(\phi, \phi) = \left[\frac{1}{2}((\nabla\phi)^2 + \phi_z^2) - \phi_t\phi_{tz}\right]_t + \nabla \cdot (\phi_t\nabla\phi) \tag{2.9}$$

and the cubic terms are

$$\begin{aligned} \mathcal{Q}_2(\phi, \phi, \phi) = &\left[-\frac{1}{2}\phi_t((\nabla\phi)^2 + \phi_z^2)_z + \phi_t\phi_{tz}^2 + \frac{1}{2}\phi_{tzz}\phi_t^2\right]_t \\ &+ \nabla \cdot \left[\frac{1}{2}(\nabla\phi)((\nabla\phi)^2 + \phi_z^2) - (\nabla\phi)\phi_t\phi_{tz} - \frac{1}{2}(\nabla\phi_z)\phi_t^2\right]. \end{aligned} \tag{2.10}$$

Next, we solve Laplace's equation with the bottom boundary condition, obtaining

$$\phi(x, y, z, t) = \cosh[z(-\Delta)^{\frac{1}{2}}]\Phi(x, y, t), \tag{2.11}$$

with

$$u(x, y, t) = \phi(x, y, 1, t) = \cosh[(-\Delta)^{\frac{1}{2}}]\Phi(x, y, t) \tag{2.12}$$

being the velocity potential at $z = 1$. With this notation, it follows that $\phi_z(x, y, 1, t) = \mathcal{L}u$ and $\phi_{zz}(x, y, 1, t) = -\Delta u$, where $\mathcal{L}$ is defined as $\mathcal{L} = (-\Delta)^{\frac{1}{2}} \tanh[(-\Delta)^{\frac{1}{2}}]$ and has the symbol $\hat{\mathcal{L}}(\mathbf{k}) = |\mathbf{k}| \tanh(|\mathbf{k}|)$. Thus if $\hat{u}(\mathbf{k}, t)$ is the Fourier transform of $u(\mathbf{x}, t)$, then

$$\mathcal{L}u = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{k}| \tanh(|\mathbf{k}|)e^{i\mathbf{k}\cdot\mathbf{x}}\hat{u}(\mathbf{k}, t)d\mathbf{k}. \tag{2.13}$$

Substitution into the boundary condition (2.8) yields, after some simplification, the fBL equation (2.1).

We note that since ϵ is the ratio of the amplitude of the free surface displacement to depth, the wave slope appears to be arbitrary. However, note that for $|\mathbf{k}|$ large the wave slope $\hat{\eta}_x = O(|\mathbf{k}|^{3/2}\hat{u})$ and that in (2.1), $\mathcal{N}_j = O(|\mathbf{k}|^{(1+3/2j)}\hat{u}^j)$, thus implying that solutions of the fBL are relevant only if the wave slope is also small as waves get short compared to depth. For the shallow limit, $|\mathbf{k}|$ small, $\hat{\eta} = O(|\mathbf{k}|\hat{u})$, $\hat{\eta}_x = O(|\mathbf{k}|^2\hat{u})$, and $\mathcal{N}_j = O(|\mathbf{k}|^{(2+j)}\hat{u}^j)$, requiring only that η be small.

A similar equation applies for gravity waves in water of infinite depth, now with ϵ being the wave slope (ratio of the amplitude of the surface displacement to a characteristic length scale), and $\hat{\mathcal{L}} = |k|$. Therefore, $\mathcal{L} = (-\Delta)^{\frac{1}{2}}$, which, in the case of one horizontal dimension, is $\mathcal{L} = -\partial_x \mathcal{H}$, where $\mathcal{H}$ is the Hilbert transform. For the deep-water limit, the derivation must be modified slightly. The origin of the vertical axis is shifted to the undisturbed fluid level, and the bottom boundary condition becomes $|\nabla\phi| \to 0, z \to -\infty$. Expanding the two surface boundary conditions about $z = 0$ and eliminating η again leads to (2.8). Solving Laplace's equation with the new bottom boundary condition modifies the depth dependence of the velocity potential:

$$\phi(x, y, z, t) = e^{z(-\Delta)^{\frac{1}{2}}} \Phi(x, y, t). \tag{2.14}$$

Correspondingly, the velocity potential at $z = 0$ is just

$$u(x, y, t) = \phi(x, y, 0, t) = \Phi(x, y, t), \tag{2.15}$$

and $\mathcal{L}$ is now defined as $\mathcal{L} = (-\Delta)^{\frac{1}{2}}$ and has the symbol $\hat{\mathcal{L}}(\mathbf{k}) = |\mathbf{k}|$.

In the remainder of this paper we assume that the free surface is one-dimensional.

3. Nonlinear modulation of gravity waves.

3. Nonlinear modulation of gravity waves. We consider the slow modulation of one-dimensional gravity waves in water of finite depth using the fBL equation, obtaining an NLS equation, in agreement with earlier results. This equation predicts instability for $kH > 1.363$. Of critical importance in the derivation of this NLS equation is a wave-induced mean flow, which vanishes in the deep water limit.

3.1. Derivation of an NLS equation. In what follows, we employ the method of multiple scales, introducing the slow space and time scales $X = \epsilon x$, $T = \epsilon t$, and $\tau = \epsilon^2 t$. The NLS equation governs the evolution of wave packets or, alternatively, of a narrowly peaked Fourier spectrum centered at k_c. Thus one expands the governing equations with $k = k_c + \epsilon\Delta k$. The equation in physical space is then recovered with the duality $\partial_X \leftrightarrow i\epsilon\Delta k$. Thus, in (2.1) we make the substitutions $\partial_t \to \partial_t + \epsilon\partial_T + \epsilon^2\partial_\tau$ and $\partial_x \to \partial_x + \epsilon\partial_X$ and, for $\mathcal{L}$,

$$\mathcal{L} \to \mathcal{L} - \epsilon i \frac{\partial\hat{\mathcal{L}}}{\partial k}\partial_X - \frac{1}{2}\epsilon^2 \frac{\partial^2\hat{\mathcal{L}}}{\partial k^2}\partial_{XX}, \tag{3.1}$$

where k_c is denoted k. The dispersion relation is

$$\omega^2(k) = \hat{\mathcal{L}} = |k| \tanh(|k|), \tag{3.2}$$

and

$$\frac{\partial\hat{\mathcal{L}}}{\partial k} = 2\omega c_g(k), \tag{3.3}$$

$$\frac{\partial^2\hat{\mathcal{L}}}{\partial k^2} = 2c_g^2 + 2\omega\frac{\partial c_g}{\partial k}, \tag{3.4}$$

where $c_g(k)$ is the group velocity.

After substitution, we have the following equation for $u(x, t, X, T, \tau)$:

$$\begin{aligned} &u_{tt} + \mathcal{L}u + \epsilon\left(2u_{tT} - i\frac{\partial\hat{\mathcal{L}}}{\partial k}u_X + \mathcal{N}_1(u, u)\right) \\ &\quad + \epsilon^2\left(2u_{t\tau} + u_{TT} - \frac{1}{2}\epsilon^2\frac{\partial^2\hat{\mathcal{L}}}{\partial k^2}u_{XX} + \mathcal{N}_2(u, u, u) + \mathcal{M}(u, u)\right) = 0, \end{aligned} \tag{3.5}$$

where

$$\mathcal{M}(u,u) = u_T(u_{xx} - \mathcal{L}u_{tt}) + 2u_t(u_{xX} - \mathcal{L}u_{tT}) + iu_t\frac{\partial\hat{\mathcal{L}}}{\partial k}u_{ttX} + 2u_x(u_{xT} + u_{Xt})$$
$$+ 2u_X u_{xt} - 2i\mathcal{L}u\frac{\partial\hat{\mathcal{L}}}{\partial k}u_{tX} + 2\mathcal{L}u\mathcal{L}u_T - 2i\frac{\partial\hat{\mathcal{L}}}{\partial k}u_X\mathcal{L}u_t. \tag{3.6}$$

Next, we expand u in the small parameter ϵ as $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$ and look for a single plane wave of slowly varying amplitude and wavelength $\frac{2\pi}{k}$:

$$u_0(x,t,X,T,\tau) = A(X,T,\tau)e^{i\theta} + * + B(X,T,\tau), \tag{3.7}$$

where $\theta = kx - \omega t$, B is the "mean-flow" component, and the $*$ denotes the complex conjugate of the preceding terms. We note that although the waves are $O(\epsilon)$, the mean flow B_x is $O(\epsilon^2)$. Substitution into (3.5) leads to a series of equations at various orders of ϵ. The O(ϵ) equation is

$$u_{1tt} + \mathcal{L}u_1 = -\left(2u_{0tT} - i\frac{\partial\hat{\mathcal{L}}}{\partial k}u_{0X}\right) - \mathcal{N}_1(u_0,u_0). \tag{3.8}$$

The first terms on the right of the above equation are secular and impose that A is moving at the group velocity. Thus, with $\xi = X - c_gT$, the right-hand side becomes $3i\omega|k|^2(\sigma^2-1)A^2e^{2i\theta} + *$, where $\sigma = \tanh(|k|)$, $A = A(\xi,\tau)$, and

$$u_1 = \frac{3i|k|^2(1-\sigma^4)}{4\sigma^2\omega}A(\xi,\tau)^2e^{2i\theta} + *. \tag{3.9}$$

Proceeding to O(ϵ^2) terms, the equation is

$$u_{2tt} + \mathcal{L}u_2 = \left(2u_{1tT} - i\frac{\partial\hat{\mathcal{L}}}{\partial k}u_{1X}\right) - \left(2u_{0t\tau} + u_{0TT} - \frac{1}{2}\frac{\partial^2\hat{\mathcal{L}}}{\partial k^2}u_{0XX}\right)$$
$$- \left(\mathcal{N}_1(u_0,u_1) + \mathcal{N}_1(u_1,u_0) + \mathcal{M}(u_0,u_0) + \mathcal{N}_2(u_0,u_0,u_0)\right). \tag{3.10}$$

In this equation, eliminating the secular terms in $e^{i\theta}$ and mean flows (e^{i0}) leads to the two equations

$$B_{\xi\xi} = \frac{\gamma}{c_g^2 - 1}(AA^*)_\xi \tag{3.11}$$

and

$$iA_\tau + \alpha A_{\xi\xi} = \bar{\beta}|A|^2A + \frac{\gamma}{2\omega}B_\xi A, \tag{3.12}$$

where

$$\gamma(k) = 2k\omega + c_g|k|^2(1-\sigma^2),$$
$$\alpha(k) = \frac{1}{2}\frac{\partial c_g}{\partial k},$$
$$\bar{\beta}(k) = \frac{9 - 12\sigma^2 + 13\sigma^4 - 2\sigma^6}{4\omega\sigma^2}|k|^4.$$

Integrating (3.11) to obtain the induced horizontal mean flow $B_\xi = \frac{\gamma}{c_g^2-1}|A|^2$ (we ignore the constant of integration, which would correspond to an imposed weak flow)

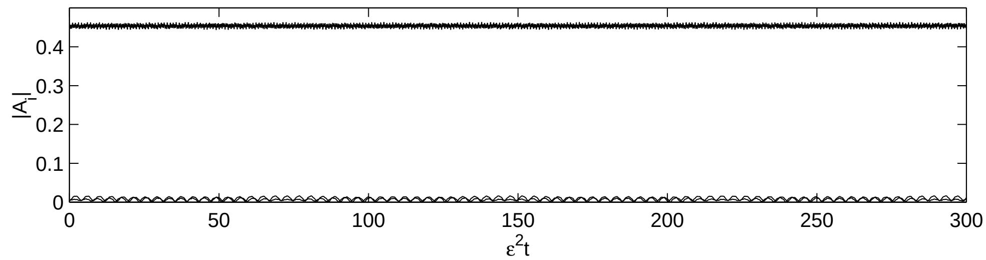

FIG. 3.1. *Evolution of a single plane wave and two small side-bands using the one-dimensional fBL equation. Here* $kH = 1.344 < 1.363$, *and stability is expected.*

and substituting into (3.12) yields a nonlinear Schrödinger equation for the complex amplitude $A(\xi, \eta)$:

$$iA_\tau + \alpha A_{\xi\xi} = \beta|A|^2 A, \tag{3.13}$$

with

$$\beta(k) = \bar{\beta} + \frac{\gamma^2}{2\omega(c_g^2 - 1)}. \tag{3.14}$$

The well-known fact that the mean flow vanishes in the deep water limit can be obtained by writing the mean flow in dimensional variables and taking $H \to \infty$.

3.2. Benjamin–Feir instability. The plane wave solution of the NLS equation $A = A_0 e^{-i\beta|A_0|^2\tau}$ for constant A_0 corresponds to the Stokes wave train to $O(\epsilon^2)$ (see [13]). Moreover, linear stability analysis (see [10] and [13]) shows that a plane wave solution to (3.13) will be unstable if the product $\alpha\beta < 0$. Given the finite-depth dispersion relation, we find $\alpha(k) < 0$ for all k, and $\beta(k)$ changes sign at $k \approx 1.363$, becoming positive for k larger than this value. This is the well-known Benjamin–Feir instability criterion. Note that the induced mean-flow plays an important result in this derivation, and in the deep-water limit this flow is not present.

To verify the stability predictions of this NLS equation, we numerically solve the fBL equation with initial condition $u(x, 0) = Ae^{ikx} + a(e^{i(k+\Delta k)x} + e^{i(k-\Delta k)x}) + *$ corresponding to a primary plane wave of wavenumber k and two side-bands of the next adjacent wavenumbers. We use a relative amplitude of $a = 0.01A$ with $\Delta k = \frac{1}{32}$. Dimensionally, our wavenumber k corresponds to kH, and we take two values on either side of the $kH = 1.363$ limit. Figures 3.1 and 3.2 show the results for $kH = 1.344$ and $kH = 1.438$, respectively, on the long time scale $\tau = \epsilon^2 t$. Note the instability of the primary mode and the side-bands in the second figure. The calculations do not show cyclic modulation and demodulation (or recurrence, present for some limits of the Benjamin–Feir instability) due to the relatively large amplitude of the carrier wave.

The extension to two dimensions (two-dimensional instabilities of plane waves) is straightforward, and the fBL equation is an appropriate starting point for an asymptotic study (such as that of Davey and Stewartson and of Benney and Roskes) or numerical experiments.

We note that in our calculations of the unstable Benjamin–Feir regime it is the lower Fourier side-bands that dominate the spectrum. This "frequency downshift"

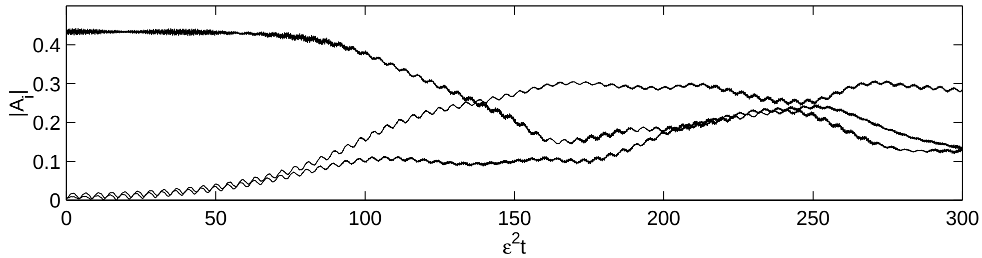

FIG. 3.2. *Evolution of a single plane wave and two small side-bands using the one-dimensional fBL equation. Here* $kH = 1.438 > 1.363$, *and growth of the side-bands is observed.*

has been observed experimentally [18] and is thought to be a three-dimensional phenomenon requiring a combination of nonlinear wave modulation and dissipation [25]. We do not perform here detailed calculations of this phenomenon; however, we believe that the equations used here (at least for two-dimensional free surface waves) could be used for this purpose.

4. Resonant interaction of gravity waves. Nonlinear resonance, an important mechanism for the transfer of energy among periodic wave trains, was pioneered by Phillips, Benney, Longuet-Higgins, and others in the 1960s (see below). The basic idea is that two or more distinct wave trains can combine to produce a perturbation with a frequency that corresponds to the natural frequency of a free wave with the same wavenumber. When this occurs, we have resonance, and the amplitude of the response grows linearly. Resonance with three waves, known collectively as a *triad*, is only possible when the dispersion curve has an inflection point (such as in capillary-gravity waves). For pure gravity waves, resonance is possible only among sets of four waves, known as *quartets*.

The idea of resonance for dispersive waves was first suggested by Phillips [24], who showed that three gravity surface waves could resonantly force a fourth wave, forming a quartet. Using the method of multiple scales, Benney [3] derived a coupled set of ordinary differential equations describing the amplitude evolution of a quartet of deep water gravity waves. Bretherton [8] showed that these types of coupled ordinary differential equations could be solved exactly using Jacobi elliptic functions. Experimental confirmation of the existence and importance of resonant water wave interactions was provided by Longuet-Higgins and Smith [17] and McGoldrick et al. [20]. Hammack and Henderson [12] provide a review of experimental results concerning resonant interaction theory for water waves, while the book by Craik [10] gives a comprehensive treatment of wave interactions in general, including triads and quartets in surface waves.

Here we derive a set of equations describing the resonant interaction of four gravity waves in water of finite depth using the fBL equation. The derivation of these "quartet equations" will closely parallel that of the NLS equation in the previous section, except that we will consider the amplitudes of *four* surface waves as well as the induced mean flow. A similar derivation in infinite depth leads to a set of equations whose primary interaction coefficients are zero, indicating that there is *no* quartet interaction in the one-dimensional "deep water" case. This is predicted by the analysis of Longuet-Higgins [16] and shown analytically by Zahkarov [27]. This is not true of two-dimensional infinite-depth gravity waves, nor of the one-dimensional *finite-depth* waves, which we consider here (although the quartet coefficients vanish for $|k| \to \infty$).

4.1. Derivation of quartet/mean-flow equations. Beginning with the one-dimensional fBL equation (2.1), we proceed with the method of multiple scales as before, obtaining (3.5). Again, we expand u in the small parameter ϵ as

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \tag{4.1}$$

but now consider a set of plane waves of slowly varying amplitude:

$$u_0(x, t, X, T, \tau) = B(X, T, \tau) + \sum_{j=1}^{4} A_j(X, T, \tau) e^{i(k_j x - \omega_j t)} + *. \tag{4.2}$$

Furthermore, we assume that the four plane waves form a resonant quartet satisfying (1.1)–(1.2). Using the notation $\theta = kx - \omega(k)t$, the resonance condition is $\theta_1 + \theta_2 = \theta_3 + \theta_4$. Substitution into (3.5) again leads to a series of equations at various orders of ϵ. The $\mathrm{O}(\epsilon)$ equation is (3.8), and we introduce four frames $\xi_j = X - c_g(k_j)T$ moving at the four group velocities $c_g(k_j)$ and assume that $A_j = A_j(\xi_j, \tau)$.

The quadratic term $\mathcal{N}_1$ is the product of two sums of eight terms each (four plane waves and their conjugates). We need to keep track of only a subset of the sixty-four possible quadratic terms, since we are interested in only those terms that can combine to form quartets at the next order. We will ignore the creation of the conjugate modes, since they are derivable from the main result for the primary modes. For example, at this order we need to account for the $\theta_1 + \theta_2$ term, since $\theta_4 = \theta_1 + \theta_2 - \theta_3$, but can ignore the $-\theta_1 - \theta_2$ term, since this is used only in forming the conjugate of the $e^{i\theta_4}$ wave. With this in mind, the $\mathrm{O}(\epsilon)$ problem can be written

$$u_{1tt} + \mathcal{L}u_1 = -i \sum_{a,b} G(a, b) A_a A_b e^{i(\theta_a + \theta_b)}, \tag{4.3}$$

in which

$$G(a, b) = \omega_a(k_b^2 - \omega_b^2 \hat{\mathcal{L}}(k_b)) + 2\omega_b(k_a k_b - \hat{\mathcal{L}}(k_a)\hat{\mathcal{L}}(k_b)) \tag{4.4}$$

and $a, b \in \{1, -1, 2, -2, 3, -3, 4, -4\}$, with the notation $k_{-1} = -k_1$, $\omega_{-1} = -\omega_1$, $A_{-1} = A_1^*$, etc. Here the sum is over those combinations of (a, b) that are relevant to forming quartets, and it varies for which of the four waves is being created.

A particular solution to the $\mathrm{O}(\epsilon)$ equation (4.3) is

$$u_1 = -i \sum_{a+b \neq 0} \frac{G(a, b)}{\hat{\mathcal{L}}(k_a + k_b) - (\omega_a + \omega_b)^2} A_a A_b e^{i(\theta_a + \theta_b)}, \tag{4.5}$$

where, again, the details of the sum (discussed below) depend on the primary wave being formed. The restriction on the sum ($a+b \neq 0$) is added because both numerator and denominator vanish (there is no mean flow generated at this order). Proceeding to $\mathrm{O}(\epsilon^2)$ terms, the equation is

$$\begin{aligned} u_{2tt} + \mathcal{L}u_2 = {} & \left(2u_{1tT} - i\frac{\partial \hat{\mathcal{L}}}{\partial k} u_{1X} \right) - \left(2u_{0t\tau} + u_{0TT} - \frac{1}{2}\frac{\partial^2 \hat{\mathcal{L}}}{\partial k^2} u_{0XX} \right) \\ & - \left(\mathcal{N}_1(u_0, u_1) + \mathcal{N}_1(u_1, u_0) + \mathcal{M}(u_0, u_0) + \mathcal{N}_2(u_0, u_0, u_0) \right). \end{aligned} \tag{4.6}$$

In this equation, we seek to eliminate the secular terms in $e^{i\theta_j}, j = 1, \ldots, 4$, and the zero-mode terms (e^{i0}). Upon transforming to the moving frame $\xi_j = X - c_g(k_j)T$, the linear terms on the right-hand side reduce to

$$\left(-2i\omega_j A_{j\tau} - \omega_j \frac{\partial c_g}{\partial k}(k_j) A_{j\xi_j\xi_j}\right) e^{i\theta_j} + B_{TT} - B_{XX}, \tag{4.7}$$

where $j = 1, \ldots, 4$.

On the right-hand side, the terms $\mathcal{N}_1(u_0, u_1) + \mathcal{N}_1(u_1, u_0)$ will yield cubic terms, since u_1 contains quadratic terms (4.5) and u_0 has the original plane waves. We are interested in only combinations that yield a member of the quartet, i.e., terms in $e^{i\theta_j}$. No e^{i0} terms are created here. The relevant contributions of the terms $\mathcal{N}_1(u_0, u_1) + \mathcal{N}_1(u_1, u_0)$ can be written

$$\sum_{a,b,c} \frac{G(a,b)\,[G(a+b,c) + G(c,a+b)]}{\hat{\mathcal{L}}(k_a + k_b) - (\omega_a + \omega_b)^2} A_a A_b A_c e^{i(\theta_a+\theta_b+\theta_c)} + *, \tag{4.8}$$

in which we keep the notation $a, b, c \in \{1, -1, 2, -2, 3, -3, 4, -4\}$. Note that c always comes from the u_0 term, and a and b come from the u_1 term that was solved at $\mathrm{O}(\epsilon)$. Also, we keep the restriction that $a + b \neq 0$. The notation $G(a + b, c)$ means to use $k_a + k_b$ and $\omega_a + \omega_b$ for k_a and ω_a in the expression (4.4).

The term $e^{i(\theta_a+\theta_b+\theta_c)}$ will be equal to $e^{i\theta_j}$ for one of the original θ_j when a, b, and c are chosen appropriately. For example, to form terms in $e^{i\theta_1}$, we are interested in the six permutations of the set $\{-2, 3, 4\}$ since $\theta_1 = -\theta_2 + \theta_3 + \theta_4$. Evaluating the sum in (4.8) with these six sets of a, b, c yields the term $q_1 A_2^* A_3 A_4 e^{i\theta_1}$, where q_1 is a coefficient. However, we must also consider permutations of the sets $\{1, 1, -1\}$, $\{1, 2, -2\}$, $\{1, 3, -3\}$, and $\{1, 4, -4\}$, since they also lead to terms in $e^{i\theta_1}$. (In using these values of a, b, c, however, care must be taken to avoid duplicates and the cases when $a + b = 0$.) Thus, the contribution of $\mathcal{N}_1(u_0, u_1) + \mathcal{N}_1(u_1, u_0)$ to the quartet equations will be

$$\left(q_j A_l^* A_m A_n + \sum_{k=1}^{4} p_{jk} |A_k|^2 A_j\right) e^{i\theta_j} \tag{4.9}$$

for $j = 1, \ldots, 4$, where l, m, and n depend on j and satisfy $\theta_j + \theta_l = \theta_m + \theta_n$.

The contributions of the cubic terms $\mathcal{N}_2(u_0, u_0, u_0)$ are very similar to those of the quadratic terms $\mathcal{N}_1(u_0, u_1) + \mathcal{N}_1(u_1, u_0)$, except that they come from u_0 directly. We again sum over five distinct groups of six permutations and obtain the same twenty terms as in (4.9), but with different coefficients. We add these coefficients to those obtained from the quadratic terms. The contribution of $\mathcal{N}_2(u_0, u_0, u_0)$ is

$$\sum_{a,b,c} H(a,b,c) A_a A_b A_c e^{i(\theta_a+\theta_b+\theta_c)} + *, \tag{4.10}$$

in which

$$\begin{aligned} H(a,b,c) = {} & \omega_a\omega_b \hat{\mathcal{L}}(k_b)(k_c^2 + \hat{\mathcal{L}}^2(k_c)) + 2\omega_a\omega_b \hat{\mathcal{L}}(k_c)(k_b\omega_b - k_c\omega_b - k_c\omega_c) \\ & + \frac{1}{2}k_a^2(3k_bk_c + \hat{\mathcal{L}}(k_b)\hat{\mathcal{L}}(k_c)) + \frac{1}{2}\hat{\mathcal{L}}^2(k_a)(\hat{\mathcal{L}}(k_b)\hat{\mathcal{L}}(k_c - k_bk_c) - 2k_ak_c\hat{\mathcal{L}}(k_b)\hat{\mathcal{L}}(k_c)). \end{aligned}$$

Note that, like the quadratics, the cubics do not contribute any terms in e^{i0}.

Finally, $\mathcal{M}(u_0, u_0)$ gives both terms in $e^{i\theta_j}$ and "mean-flow" terms in e^{i0}:

$$\sum_{j=1}^{4}(-2k_j\omega_j - c_g(k_j)(k_j^2 - \hat{\mathcal{L}}^2(k_j)))(A_jA_j^*)_{\xi_j} \\ + \sum_{j=1}^{4}(2k_j\omega_j B_X + (\hat{\mathcal{L}}^2(k_j) - k_j^2)B_T)A_j e^{i\theta_j}. \tag{4.11}$$

Equating terms from (4.6) in like powers of $e^{i\theta}$ leads to a coupled set of *five* partial differential equations governing the evolution of the resonant quartet and the "mean-flow" term B:

$$B_{TT} - B_{XX} = \sum_{j=1}^{4}(2k_j\omega_j + c_g(k_j)(k_j^2 - \hat{\mathcal{L}}^2(k_j)))(A_jA_j^*)_{\xi_j}, \tag{4.12}$$

$$iA_{j\tau} + \frac{1}{2}\frac{\partial c_g}{\partial k}(k_j)A_{j\xi_j\xi_j} = \frac{\alpha_j}{2\omega_j}A_l^*A_mA_n + \frac{1}{2\omega_j}\left(\sum_{k=1}^{4}\beta_{jk}|A_k|^2A_j\right) \\ + \frac{1}{2\omega_j}(2k_j\omega_j B_X + (\hat{\mathcal{L}}^2(k_j) - k_j^2)B_T)A_j \tag{4.13}$$

for $j = 1, \ldots, 4$, where $\theta_j + \theta_l = \theta_m + \theta_n$. Note that we have intentionally glossed over some notational inconsistency by using X, T, and ξ_j as independent variables in (4.12). Furthermore, the induced mean flows on any particular member of the quartet from the other three members are rapidly varying on the time scale of (4.13), unless group velocities are close. However, we will not be concerned with initial conditions for (4.12) and (4.13) that involve spatial modulation of the four primary waves and will thus treat (4.13) as a set of ordinary differential equations (see below).

Historically, the derivation of quartet equations was done for deep water, for which *spatial* modulation effects are ignored since the mean-flow is known to be zero. Thus (4.12) would not be present, $B = 0$ in (4.13), and these equations become ordinary differential equations. Of course the dispersion relation $\omega^2 = \hat{\mathcal{L}}$ also changes for deep water. As noted by Bretherton [8] for the two-dimensional deep-water case, the primary interaction coefficients α_i turn out to be *equal*. This is also true for finite-depth quartet equations, which we have confirmed using the fBL model. For deep water we find $\alpha_i = 0$, as expected. Since the α_i are primarily responsible for the exchange of energy among the four waves (the β_{jk} modify the period and amplitude), there is no interaction in the one-dimensional deep-water case.

4.2. Quartet simulations. To verify that (4.12) and (4.13) correctly capture the finite-depth quartet interactions, we compare the solutions to these equations to a simulation using the fBL equation. Since computing with these five coupled partial differential equations is computationally intensive, we seek a simpler special case. By choosing an initial condition in which the amplitude of the four primary waves is not spatially modulated, we ignore the second term in (4.13) and (4.12) altogether, leaving a set of four coupled ordinary differential equations. Although an exact solution involving Jacobi elliptic functions is known for a set of ordinary differential equations of this form (see [8]), we solve them numerically using a fourth-order Runge–Kutta method, while we evolve the fBL equation in time using a pseudospectral method. The initial condition for both computations is the same. We pick four plane waves among a discrete set of wavenumbers that satisfy the resonance conditions $k_1 + k_2 = k_3 + k_4$ and $\omega_1 + \omega_2 = \omega_3 + \omega_4$, and give them each an initial amplitude. The quartet equations govern the four amplitudes as a function of $\tau = \epsilon^2 t$, while the simulation of the fBL equation computes the evolution of *all* the Fourier modes.

The pseudospectral method used here was developed by Milewski and Tabak [23] and involves the factoring of the fBL equation. Equation (2.1) can be also written as

$$(\partial_{tt} + L^2)u = \mathcal{G}(u), \tag{4.14}$$

in which $L^2 = \mathcal{L} = (-\partial_{xx})^{\frac{1}{2}} \tanh[(-\partial_{xx})^{\frac{1}{2}}]$ and $\mathcal{G}(u) = -\epsilon\mathcal{N}_1(u,u) - \epsilon^2\mathcal{N}_2(u,u,u)$. We factor the left-hand side by introducing $U(x,t) = (\partial_t - iL)u(x,t)$ and recast the equation in terms of U as

$$U_t + iLU = \mathcal{G}(U). \tag{4.15}$$

Thus the free surface is, to leading order, $\eta = -u_t = -Re(U)$. To solve (4.15), we transform it to Fourier space, introduce an integrating factor, and numerically integrate using a Runge–Kutta scheme. Since we compute with U directly and not u, we choose to initially set the quartet amplitudes to have equal values *in terms of* U. The conversion to the amplitudes of u is straightforward. If $u(x,t) = \sum_{j=1}^4 A_j e^{i\theta_j} + *$, then $U(x,t) = \sum_{j=1}^4 -2i\omega_j A_j e^{i\theta_j}$ and $\eta(x,t) = \sum_{j=1}^4 i\omega_j A_j e^{i\theta_j} + *$. In the figures below, we graph the absolute value of the relevant Fourier modes of $U(x,t)$, i.e., $|U_j| = 2\omega_j|A_j|$.

A slight modification to the fBL equation (2.1) must be made before using our pseudospectral method. The problem lies with the O(ϵ) quadratic terms which have the term $-u_t\mathcal{L}u_{tt}$. Because of this term, we cannot integrate (2.1) in the form given. With the substitution $u_{tt} = -\mathcal{L}u - \epsilon\mathcal{N}_1(u,u) + O(\epsilon^2)$, the quadratic term becomes

$$\bar{\mathcal{N}}_1(u,u) = 2u_x u_{xt} + 2\mathcal{L}u\mathcal{L}u_t + u_t u_{xx} + u_t\mathcal{L}^2 u, \tag{4.16}$$

and there is an additional cubic term in the equation which becomes

$$u_{tt} + \mathcal{L}u + \epsilon\bar{\mathcal{N}}_1(u,u) + \epsilon^2\left(\mathcal{N}_2(u,u,u) + u_t\mathcal{L}[\bar{\mathcal{N}}_1(u,u)]\right) = 0. \tag{4.17}$$

These modifications are similar to the formal manipulations that one uses to "regularize" the Korteweg–de Vries (KdV) equation and obtain the Benjamin–Bona–Mahoney (BBM) equation. There are also corresponding changes to the details of the quartet equations, in particular to the definitions of the functions $G(a,b)$ and $H(a,b,c)$ in (4.4) and (4.11), respectively. The new cubic term adds

$$\omega_a\omega_b(k_c^2 - \hat{\mathcal{L}}(k_c))\hat{\mathcal{L}}(k_b + k_c) + 2\omega_a\omega_c(k_b k_c - \hat{\mathcal{L}}(k_b)\hat{\mathcal{L}}(k_c))\hat{\mathcal{L}}(k_b + k_c) \tag{4.18}$$

to the function H.

Figure 4.1 shows the numerical solution of the four coupled quartet equations using a fourth-order Runge–Kutta scheme. We use the quartet wavenumbers $(k_1, k_2, k_3, k_4) = (81, 46, 142, -15)\Delta k$, where $\Delta k = \frac{1}{64}$, and the corresponding frequencies ω_j to precompute the twenty quartet coefficients. The initial amplitude of each mode U_j is 0.2. For this quartet, energy is periodically exchanged between the four waves which is inherently on the $\tau = \epsilon^2 t$ time scale. The total energy, given by $\sum_{j=1}^4 \frac{1}{\alpha_j}|A_j|^2$, remains constant.

Figure 4.2 shows the pseudospectral simulation of the fBL equation initialized with energy only in the same four wavenumbers considered above. We use a total of

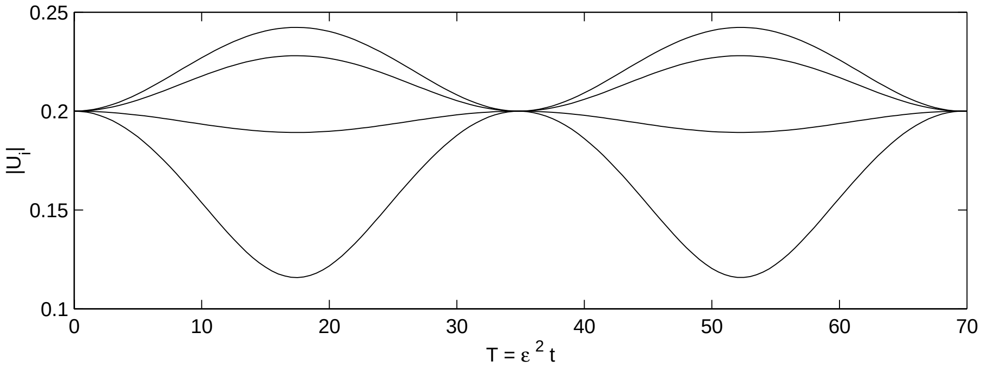

FIG. 4.1. *Numerical solution of the quartet equations using a fourth-order Runge–Kutta scheme. The wavenumbers are* $(k_1, k_2, k_3, k_4) = (81, 46, 142, -15)\Delta k$*, where* $\Delta k = \frac{1}{64}$*. The initial amplitude is* $U_j = 0.2$*.*

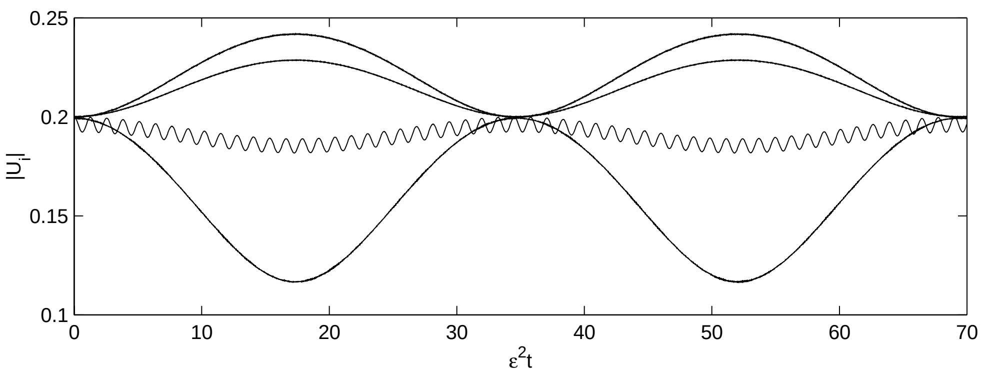

FIG. 4.2. *Simulation of the fBL equation initialized with four waves of the same initial amplitude. The wavenumbers are* $(k_1, k_2, k_3, k_4) = (81, 46, 142, -15)\Delta k$*, where* $\Delta k = \frac{1}{64}$*. The initial amplitude is* $U_j = 0.2$*,* $\epsilon = .05$*, and* $\Delta t = 0.1$*. We use* 1024 *wavenumbers in this computation.*

1024 wavenumbers with the initial amplitude of each member of the quartet being $U_j = 0.2$. Note that energy is periodically exchanged between the four waves on the long time scale $\tau = \epsilon^2 t$, as predicted (here $\epsilon = .05$). On a shorter time scale, the longest wave in the quartet $k_4 = -15/64$ periodically exchanges energy with its near-resonant second harmonic, the $k = -30/64$ mode. This accounts for the smaller oscillations in the amplitude of this mode. In the next section, we derive the equations governing this interaction and show that they can be combined with the quartet equations to correctly predict the simulation results.

Quartets containing wavenumbers closer to the shallow water regime ($kH < 1$) will exhibit prominent second harmonic interaction, as the dispersion curve is nearly linear in this range. This draws energy from the primary quartet and may account for the slight variation in period between the two quartet graphs. Quartets without this second-harmonic interaction do not exist for this one-dimensional model because quartets containing many larger wavenumbers ($kH > 1$) are very weakly coupled since the $\alpha_j \to 0$ as $H \to \infty$, and other mechanisms such as nonresonant interactions and Benjamin–Feir instability are relatively more significant.

4.3. Derivation of second-harmonic interaction. Beginning with the one-dimensional fBL equation (2.1), we proceed with the method of multiple scales. We restrict our attention to the slow time scale $T = \epsilon t$ since we expect the interaction to occur on this scale. We also ignore slow spatial variation, consistent with our integration of the quartet equations above. With the substitution $\partial_t \to \partial_t + \epsilon \partial_T$ we have the following equation for $u(x, t, T)$:

$$u_{tt} + \mathcal{L}u + \epsilon \left(2u_{tT} + \mathcal{N}_1(u, u)\right) = 0, \tag{4.19}$$

where higher-order terms are unnecessary. Next, we expand u as $u = u_0 + \epsilon u_1 + \cdots$ with

$$u_0(x, t, T) = A_1(T)e^{i(kx-\omega t)} + A_2(T)e^{i(2kx-\omega(2k)t)} + *. \tag{4.20}$$

With the notation $\omega_1 = \omega(k)$, $\omega_2 = \omega(2k)$, $\theta_1 = kx - \omega_1 t$, and $\theta_2 = 2kx - \omega_2 t$, the balance of terms at $\mathrm{O}(\epsilon)$ in (4.19)

$$2\omega_1 A_{1T}e^{i\theta_1} + 2\omega_2 A_{2T}e^{i\theta_2} + * = \sum_{a,b} G(a, b)A_a A_b e^{i(\theta_a+\theta_b)}, \tag{4.21}$$

in which the function $G(a, b)$ is the same as that derived before for the quadratic terms (4.4). We can only get terms in e^{2ik} with $a = b = 1$, for which the right-hand side becomes $G(1, 1)A_1^2 e^{i(2kx-2\omega_1 t)} = G(1, 1)A_1^2 e^{i\theta_2} e^{-i\Delta t}$, where the frequency mismatch is $\Delta = 2\omega_1 - \omega_2$. In a similar way, we can create terms in e^{ik} with $(a, b) = (-1, 2)$ or $(2, -1)$, giving a right-hand side of $(G(-1, 2) + G(2, -1))A_1^* A_2 e^{i\theta_1} e^{i\Delta t}$. Thus the wave-second-harmonic interaction equations are

$$\frac{dA_1}{dT} = \delta_1 A_1^* A_2 e^{i\Delta t}, \qquad \delta_1 = \frac{G(-1, 2) + G(2, -1)}{2\omega_1} < 0, \tag{4.22}$$

$$\frac{dA_2}{dT} = \delta_2 A_1^2 e^{-i\Delta t}, \qquad \delta_2 = \frac{G(1, 1)}{2\omega_2} > 0. \tag{4.23}$$

For $k \ll 1$, $\Delta = k^3$, $\delta_1 = -3k^2$, and $\delta_2 = (3/2)k^2$. The transformations $A_1 \to A_1 e^{i\Delta t}$, $A_2 \to A_2 e^{i\Delta t}$ remove the periodic coefficient (detuning term), yielding

$$\frac{dA_1}{dT} = -i\Delta A_1 + \delta_1 A_1^* A_2 e^{i\Delta t}, \qquad \frac{dA_2}{dT} = -i\Delta A_2 + \delta_2 A_1^2 e^{-i\Delta t}. \tag{4.24}$$

These equations for $\epsilon = O(\Delta)$ can be solved analytically. Writing $A_1 = \rho_1 e^{i\phi_1}$, $A_2 = \rho_2 e^{i\phi_2}$, the equations (4.22), (4.23) conserve

$$E = -\frac{1}{\delta_1}\rho_1^2 + \frac{1}{\delta_2}\rho_2^2, \tag{4.25}$$

$$H = \rho_1^2 \rho_2 \sin(\phi_2 - 2\phi_1) - \frac{\Delta}{4}\left(\frac{1}{\delta_1}\rho_1^2 + \frac{1}{\delta_2}\rho_2^2\right), \tag{4.26}$$

where H is the Hamiltonian in appropriate coordinates. From these, one can conclude $|A_1|^2 = \delta_1 I(t) + c_1$, $|A_2|^2 = \delta_2 I(t) + c_2$, where

$$\left(\frac{dI}{dT}\right)^2 = 4(\delta_1 I + c_1)^2(\delta_2 I + c_2) - 4\left(H + \frac{\Delta}{2\delta_2}(\delta_2 I + c_2)\right)^2. \tag{4.27}$$

The solution can be written in terms of elliptic functions. For the results described below, $I(0) = 0$, $c_1 = -\delta_1 E$, $c_2 = 0$, and $H + (\Delta/4)E = 0$.

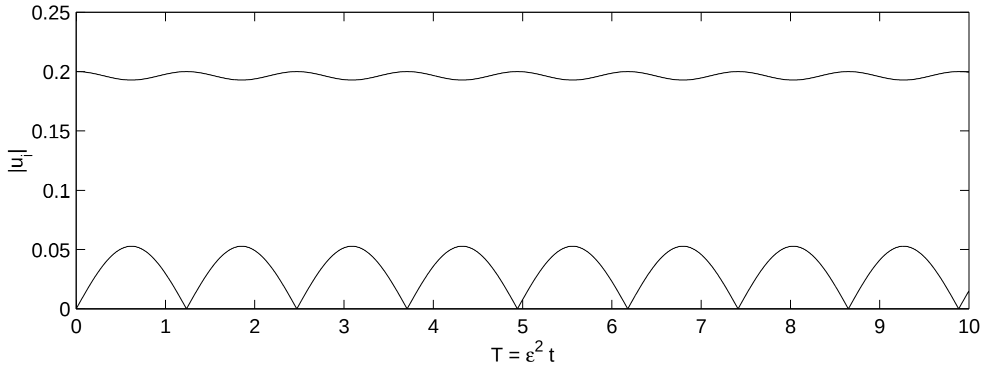

FIG. 4.3. *Solution of the second-harmonic interaction equations with $k = -15/64$. The initial amplitude is $U_1 = 0.2$ for the primary mode and $U_2 = 0$ for the second harmonic. ($\epsilon = .05$.)*

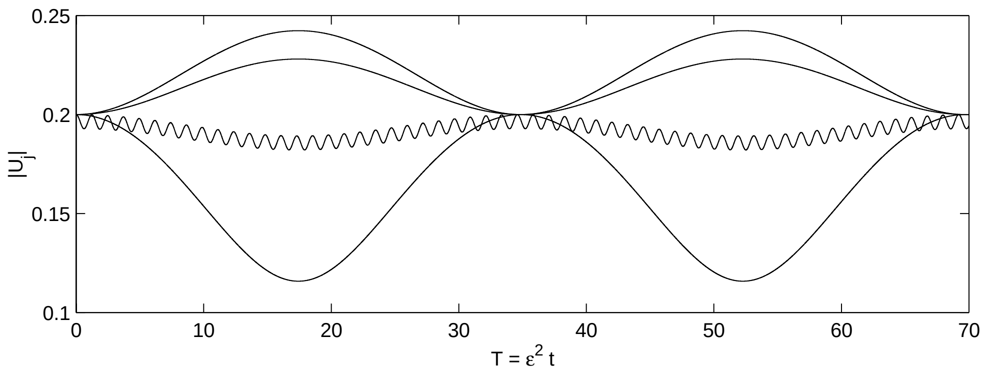

FIG. 4.4. *Numerical solution of the quartet equations with second-harmonic interaction using a fourth-order Runge–Kutta scheme. The wavenumbers are $(k_1, k_2, k_3, k_4) = (81, 46, 142, -15)\Delta k$, where $\Delta k = \frac{1}{64}$. The initial amplitude is $U_j = 0.2$.*

For the quartet of waves that we consider here, only the mode $k_4 = -\frac{15}{64}$ will generate significant second harmonic energy. Figure 4.3 shows the solution of (4.22), (4.23). The primary mode is $k = -\frac{15}{64}$ with initial amplitude $U_1 = 0.2$, as in the quartet simulation. The $2k$ mode has zero initial amplitude. (Note that, for consistency, we show the results in terms of our computational variable $U = (\partial_t - iL)u$, as discussed above.)

A pseudospectral simulation of the fBL equation initialized with energy in only the single mode $k = -\frac{15}{64}$ yields a virtually identical result.

Finally we augment the quartet equations (4.13) with the second-harmonic interaction term (for the $k_4 = -\frac{15}{64}$ mode only) and the second harmonic equation (4.23). Although these equations combine two time scales and thus are not formally correct, they give results virtually identical (see Figure 4.4) to those of the fBL simulation of Figure 4.2.

5. Gravity wave turbulence simulations. Since we have shown that the one-dimensional fBL equation captures the deterministic dynamics of the water wave problem, we turn our attention to the simulation of dispersive wave turbulence using this equation. Statistical dispersive wave-turbulence theory relies on a closure (see [14], [7], [5], [29]) that, in essence, restricts the dynamics to the resonant set of waves satisfying (1.1), (1.2). Briefly, the closure is based on writing (4.15) in Fourier space,

$$\hat{U}_t + i\omega\hat{U} = \int Q(k_1, k_2, k_3, k)\hat{U}_1\hat{U}_2\hat{U}_3\delta(k_1 + k_2 + k_3 - k)dk_1dk_2dk_3, \tag{5.1}$$

where $\hat{U}_1 = \hat{U}(k_1, t)$. The expression for the "collision" kernel Q is essentially the quartet coefficients computed previously. From (5.1) one obtains the equation for the second-order moment $n_k = \langle\hat{U}(k, t)\hat{U}^*(k, t)\rangle$:

$$\frac{dn_k}{dt} = \int 2 \text{ Re } Q\langle\hat{U}_1\hat{U}_2\hat{U}_3\hat{U}^*\rangle\delta(k_1 + k_2 + k_3 - k)dk_1dk_2dk_3, \tag{5.2}$$

where $\langle\cdot\rangle$ denotes the ensemble average. Next, one writes the equation for the evolution of the fourth-order moments appearing in the integrand of (5.2) in terms of sixth-order moments. The closure consists in reducing these sixth-order moments to products of second-order moments (a quasi-Gaussianity assumption). This leads to a relation of the form

$$\langle\hat{U}_1\hat{U}_2\hat{U}_3\hat{U}^*\rangle \sim Q\frac{n_2n_3n_k + n_1n_3n_k + n_1n_2n_k - n_1n_2n_3}{i(\omega_1 + \omega_2 + \omega_3 - \omega_k)}. \tag{5.3}$$

Now, substituting (5.3) into (5.2) and replacing the reciprocal of the sum of frequencies by $\delta(\omega_1+\omega_2+\omega_3-\omega_k)$, one obtains a closed equation that concentrates the dynamics on the resonant set. (The difference in the sign in front of k_3 and ω_3 in these delta functions and in (1.1), (1.2) is just a matter of convention.)

The steady state ($\frac{dn_k}{dt} = 0$) of this resulting equation has two types of solutions: solutions in statistical equilibrium and solutions with finite fluxes (cascades) of energy. The latter have been of particular interest in attempts to describe the ocean's wave spectrum.

Since these cascades require that the governing equation (4.15) be forced and dissipated, we augment the equation by adding forcing and dissipation terms (which are meant to model physical processes such as wind forcing, viscous damping, etc.) at various ranges of wavenumbers. Then, from long time computations, we observe the evolution of the energy spectrum until a statistical steady state is reached. Since both energy and wave action are conserved in this system, we must dissipate at both ends of the Fourier spectrum and force at some intermediate scale. Thus, the factored form of the fBL equation (4.15) in Fourier space, with forcing and dissipation, becomes

$$\hat{U}_t + i\hat{L}\hat{U} = \hat{G}(\hat{U}) + \hat{F}, \tag{5.4}$$

in which we define the forcing-dissipation function $\hat{F}$ as

$$\hat{F}(k) = \begin{cases} f_r\hat{U} & \text{for } k_{fl}\Delta k \le |k| \le k_{fh}\Delta k, \\ d_{r1}|k|^{-2}\hat{U} & \text{for } k_{dl}\Delta k \le |k| \le k_{dh}\Delta k, \\ d_{r2}|k|^2\hat{U} & \text{for } |k| \ge K_d\Delta k, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Here k_{fl}, k_{fh}, k_{dl}, k_{dh}, and K_d are integers which define the range of forcing and the two dissipation ranges, the latter being $|k| > K_d\Delta k$. The forcing rate f_r is positive, while the dissipation rates d_{r1} and d_{r2} should be negative. The closure theory described above appears insensitive on the particular form of the forcing, and this is confirmed by the numerical simulations. Various forms of forcing (both deterministic and random) and dissipation ("standard" viscosity and "hyper" viscosity) were experimented with, and the results did not change appreciably. Our approach is similar to that of Majda and coworkers (see [19] and [9]), who perform computations with a simpler NLS-like model equation.

6. Direct cascades. To generate a direct (or forward) cascade over a significant range, we force at low wavenumbers and dissipate at both the lowest and highest wavenumbers. We construct the experiment such that the *finite-depth* regime lies in the inertial range (the range of wavenumbers that are neither forced nor dissipated) for reasons mentioned in the introduction. Figure 6.1 compares the dispersion relations of the shallow water ($\omega = k$), infinite-depth ($\omega = |k|^{\frac{1}{2}}$), and arbitrary-depth ($\omega = (|k| \tanh |k|)^{\frac{1}{2}}$) problems. We will arbitrarily denote the range $0.5 < k < 2.5$ as the finite-depth regime and indicate this range in our numerical result.

We compute the correlation function

$$p(k) = \overline{\hat{u}(k,t)\hat{u}^*(k,t)}, \tag{6.1}$$

where the overbar denotes time average (after a statistical steady state is reached). It can be shown that

$$p(k) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{u(x,t)u(x+r,t)} e^{-ikr} dr, \tag{6.2}$$

when correlations are spatially independent.

Figure 6.2 shows a typical weak turbulence spectrum that we obtain using the one-dimensional fBL model. The computation uses 2048 dealiased modes with $\Delta k = \frac{1}{100}$. The parameters chosen are $\epsilon = 0.05$, $f_r = 0.00001$, $d_{r1} = -0.0009$, and $d_{r2} = -0.01$. Initially, the system has significant energy only in the lower wavenumbers $\hat{U}(4\Delta k \le k \le 12\Delta k) = 0.5$. All other modes are initialized to $\hat{U}(k) = 0.00001$. We show the average spectrum from $t = 150,000$ to $t = 200,000$, computed with the methods described earlier with $\Delta t = 0.1$.

There are six regions of the spectrum divided by five vertical dotted lines. From left to right, these are (1) the low wavenumber dissipation range, (2) the forcing range, (3) the "shallow" inertial range, (4) the "finite depth" inertial range, (5) the "infinite depth" inertial range, (6) the high wavenumber dissipation range.

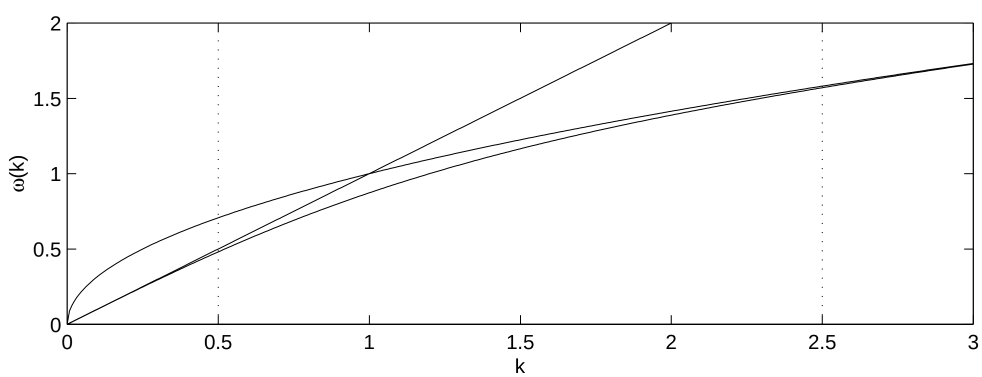

FIG. 6.1. *Finite-depth dispersion relation* $\omega(k) = (|k| \tanh(|k|))^{\frac{1}{2}}$. *As indicated in the figure,* $\omega \sim |k|$ *as* $k \to 0$, *and* $\omega \sim (|k|)^{1/2}$ *for* k *large. The vertical lines in the figure reflect an arbitrary choice for the transition region* $(0.5 < k < 2.5)$ *between these two power laws.*

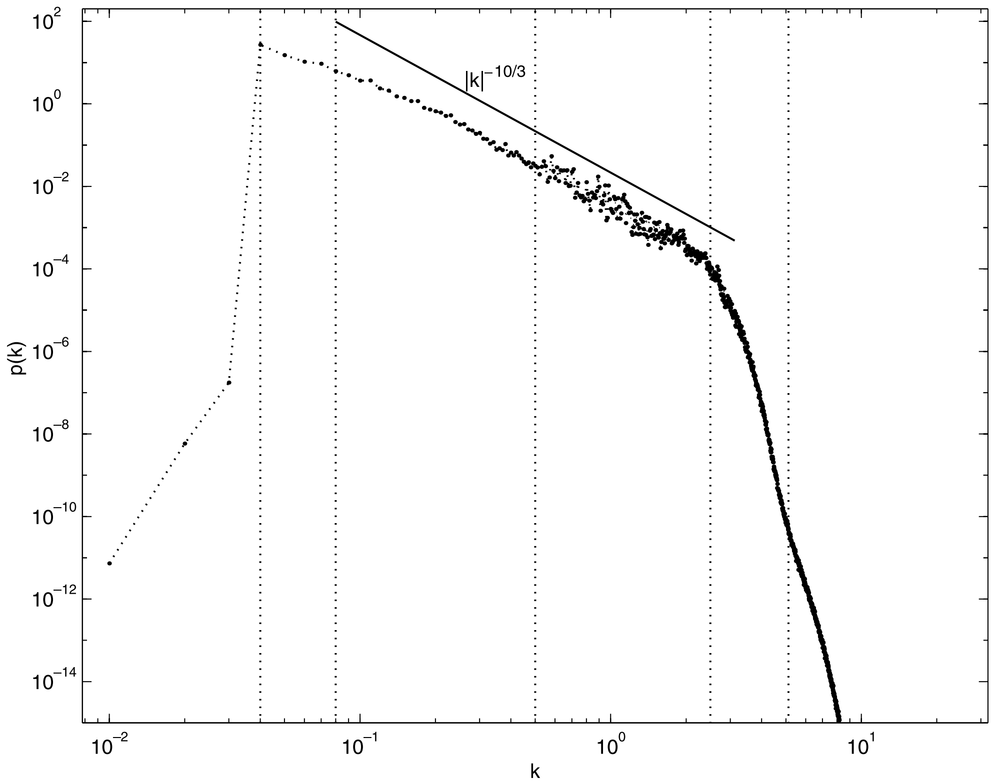

FIG. 6.2. Experiment 1. *Direct cascade using the fBL model with forcing between* $k = 4\Delta k$ *and* $k = 8\Delta k$ *and dissipation from* $k = \Delta k$ *to* $k = 3\Delta k$ *and for* $k > 512\Delta k$. *Here* $\Delta k = \frac{1}{100}$ *and* $\epsilon = 0.05$. *We estimate* $p(k)$ *by the time average of* $\hat{u}(k,t)\hat{u}^*(k,t)$ *for* $t = 150{,}000$ *to* $t = 200{,}000$ *with data every* $t = 500$.

We note that some features of the spectrum seem to change in correlation with the shape of the dispersion curve. In the shallow and finite-depth regimes, there is a good agreement with the weak turbulence theory of Zakharov [27] described above. He predicts a direct cascade of $p(k) \sim |k|^{-10/3}$ (in the present variables), subject to some strict conditions on the wave amplitudes (which are not strictly satisfied in the computations). Using least-square interpolation of the data yields $p(k) \sim |k|^{\alpha}$ with $-3 > \alpha > -3.4$, depending on where the endpoints of the inertial range are chosen. The Zakharov slope is shown in Figure 6.2 for comparison. We also note that in the finite-depth region (region 4) the data is more spread. This is probably because in this regime the discrete quartets are sparser, whereas in shallower water, wave interaction is denser, and nondispersive wave steepening plays a more important role.

6.1. Inverse cascade. To obtain an inverse cascade (from high to low wavenumbers) we modified the forcing and dissipation parameters from the previous experiment. We force near the deep water regime, between wavenumbers 2.25 and 2.50. Again, we use 2048 dealiased modes, now with $\Delta k = \frac{1}{200}$, $\epsilon = 0.05$, $f_r = 0.0006$, $d_{r1} = -0.75$, and $d_{r2} = -0.50$. Initially, all modes are initialized to $\hat{U}(k) = 0$. We show the average spectrum from $t = 150{,}000$ to $t = 200{,}000$, computed with the methods described earlier with $\Delta t = 0.1$.

The results are shown in Figure 6.3. The results here are less clear. In the shallow water regime there appears to be a region with $p(k) \sim |k|^{\alpha}$, with $-2.2 > \alpha > -2.4$.

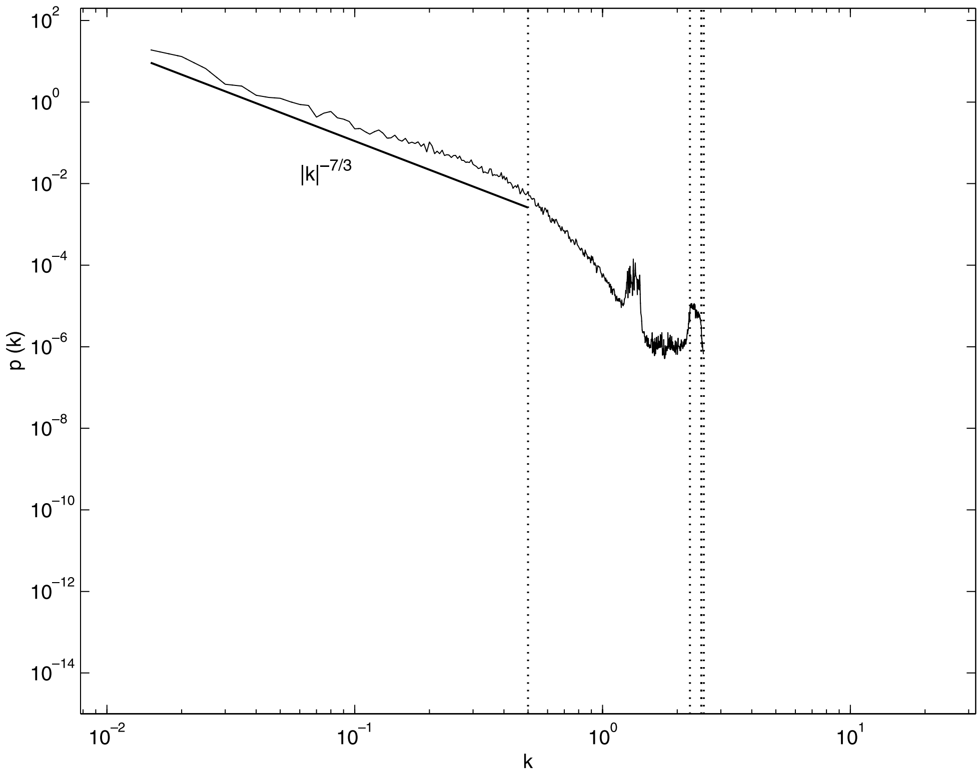

FIG. 6.3. *Inverse cascade using the fBL model with forcing between $k = 450\Delta k$ and $k = 500\Delta k$ and dissipation from $k = \Delta k$ to $k = 2\Delta k$ and for $k > 512\Delta k$. Here $\Delta k = \frac{1}{200}$ and $\epsilon = 0.05$. We estimate $p(k)$ by the time average of $\hat{u}(k,t)\hat{u}^*(k,t)$ for $t = 250,000$ to $t = 300,000$ with data every $t = 1000$.*

(We show a slope of $\alpha = -7/3$ for reference.) Zakharov's [27] prediction for the shallow water inverse cascade is $|k|^{-3.0}$. The reasons for this difference may be related to the generation of coherent structures (solitons) which are excluded from the theory (by assuming sufficiently small amplitudes compared to dispersive effects). In fact, the more recent work on NLS [9], [30] explores the role of coherent structures in the various spectra observed.

At finite depth, our computed spectrum drops much more steeply. The two visible peaks in the spectrum are due to the forcing: the peak at higher wavenumbers is over the forcing region, and the second peak is a direct subharmonic generation from the forced modes.

7. Conclusion. We have derived a Benney–Luke model for waves in arbitrary depth and verified its utility by demonstrating its accuracy in important deterministic water wave phenomena: Benjamin–Feir wave packet instability, resonant quartet interactions, and harmonic generation in shallow water. We have then used the model, together with forcing and dissipation, to simulate wave turbulence. The numerical spectra that we obtain agree with Zakharov's prediction for the direct cascades but not for inverse cascades. Possible reasons for the departure from Zakharov's prediction include the narrow range of applicability of his theory in this regime to avoid solitons. The present work validates the use of the fBL equation for the more interesting problem of two-dimensional turbulent simulations.

REFERENCES

[1] T. B. Benjamin, *Instability of periodic wave trains in nonlinear dispersive systems*, Proc. Roy. Soc. London A, 299 (1967), pp. 59–75.

[2] T. B. Benjamin and J. E. Feir, *The disintegration of wave trains on deep water, Part* 1, J. Fluid Mech., 27 (1967), pp. 417–430.

[3] D. J. Benney, *Nonlinear gravity wave interactions*, J. Fluid Mech., 14 (1962), pp. 577–584.

[4] D. J. Benney and J. C. Luke, *Interactions of permanent waves of finite amplitude*, J. Math. Phys., 43 (1964), pp. 309–313.

[5] D. J. Benney and A. C. Newell, *Random wave closures*, Stud. Appl. Math., 1 (1969), pp. 39–53.

[6] D. J. Benney and G. J. Roskes, *Wave instabilities*, Studies Appl. Math., 48 (1969), pp. 377–385.

[7] D. J. Benney and P. G. Saffman, *Nonlinear interaction of random waves in a dispersive medium*, Proc. Roy. Soc. A, 289 (1965), pp. 301.

[8] F. P. Bretherton, *Resonant interactions between waves. The case of discrete oscillations*, J. Fluid Mech., 20 (1964), pp. 457–479.

[9] D. Cai, A. J. Madja, D. W. McLaughlin, and E. G. Tabak, *Spectral bifurcations in dispersive wave turbulence*, Proc. Natl. Acad. Sci. USA, 96 (1999), pp. 14216–14221.

[10] A. D. D. Craik, *Wave Interactions in Fluid Flows.* Cambridge University Press, Cambridge, UK, 1985.

[11] A. Davey and K. Stewartson, *On three-dimensional packets of surface waves*, Proc. R. Soc. London A, 338 (1974), pp. 101–110.

[12] J. L. Hammack and D. M. Henderson, *Resonant interactions among surface water waves*, Ann. Rev. Fluid Mech., 25 (1993), pp. 55–97.

[13] H. Hasimoto and H. Ono, *Nonlinear modulation of gravity waves*, J. Phys. Soc. Japan, 33 (1972), pp. 805–811.

[14] K. Hasselmann, *On the nonlinear energy transfer in a gravity wave spectrum*, J. Fluid Mech., 12 (1962), pp. 481–500.

[15] M. J. Lighthill, *Contributions to the theory of waves in nonlinear dispersive systems*, J. Inst. Math. Appl., 1 (1965), pp. 269–306.

[16] M. S. Longuet-Higgins, *Resonant interactions between two trains of gravity waves*, J. Fluid Mech., 12 (1962), pp. 321–332.

[17] M. S. Longuet-Higgins and N. D. Smith, *An experiment on third-order wave interactions*, J. Fluid Mech., 25 (1966), pp. 417–435.

[18] B. M. Lake, H. C. Yuen, H. Rungaldier, and W. E. Ferguson, *Nonlinear deep-water waves: Theory and experiment. Part* 2: *Evolution of a continuous wave train*, J. Fluid Mech., 83 (1977), pp. 49–74.

[19] A. J. Majda, D. W. McLaughlin, and E. G. Tabak, *A one-dimensional model for dispersive wave turbulence*, J. Nonlinear Sci., 7 (1997), pp. 9–44.

[20] L. F. McGoldrick, O. M. Philips, N. E. Huang, and T. H. Hodgson, *Measurements of third-order resonant wave interactions*, J. Fluid Mech., 25 (1966), pp. 437–456.

[21] P. A. Milewski, *A formulation for water waves over topography*, Stud. Appl. Math., 100 (1998), pp. 95–106.

[22] P. A. Milewski and J. B. Keller, *Three-dimensional water waves*, Stud. Appl. Math., 97 (1996), pp. 149–166.

[23] P. A. Milewski and E. G. Tabak, *A pseudospectral procedure for the solution of nonlinear wave equations with examples from free-surface flows*, SIAM J. Sci. Comput., 21 (1999), pp. 1102–1114.

[24] O. M. Philips, *On the dynamics of unsteady gravity waves of finite amplitude. Part* 1. *The elementary interactions*, J. Fluid Mech., 9 (1960), pp. 193–217.

[25] K. Trulsen and K. B. Dysthe, *Frequency downshift in three-dimensional wave trains in a deep basin*, J. Fluid Mech., 352 (1997), pp. 359-373.

[26] G. B. Whitham, *Nonlinear dispersion of water waves*, J. Fluid Mech., 27 (1967), pp. 399–412.

[27] V. E. Zakharov, *Statistical theory of gravity and capillary waves on the surface of a finite-depth fluid*, Eur. J. Mech. B Fluids, 18 (1999), pp. 327–344.

[28] V. E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Appl. Mech. Tech. Phys., 2 (1968), pp. 190–194.

[29] V. E. Zakharov, *Kolmogorov spectra in weak turbulence problems*, in Handbook of Plasma Physics, Vol. 2, Elsevier, New York, 1984, pp. 1–36.

[30] V. E. Zakharov, P. Guyenne, A. N. Pushkarev, and F. Dias, *Wave turbulence in one-dimensional models*, Phys. D, 152–153 (2001), pp. 572–619.