ANALYSING MULTIDIRECTIONAL WAVE SPECTRA : A TENTATIVE CLASSIFICATION OF AVAILABLE METHODS

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ABSTRACT

Wave gauges and buoys are nowadays available for measuring the directional spreading of wave energy in the ocean or in laboratory basins. These devices allow to analyse the directional characteristics of a wave field in a detailed way, by the full analysis of the directional wave spectrum. This item lies at the core of the present paper, in particular by considering multi-component gauges, recording simultaneously and over a given duration several wave signals (typically 3 to 10), either on the same vertical (the so-called « single-point » systems) or at different locations (wave gauges arrays).

Estimating the directional wave spectrum from so few data is a particularly awkward problem, for which numerous analysis methods have been proposed. A great number of these methods are considered in this paper, in particular the ones used within the IAHR Working Group on Multidirectional Waves. The theoretical backgrounds of these methods are briefly described. Special attention is focused on the underlying assumptions of each type of method, in order to highlight and discuss their capabilities and limitations. To that end, it was attempted to sort the methods and to propose a tentative classification, not with the objective of finding the « best » method, but rather to offer a comparative overview of the existing methods with some comments on their characteristics.

0. INTRODUCTION

This paper is a contribution from the IAHR Working Group on Multidirectional Waves. This Group was composed of 11 participating laboratories from 10 different countries in the World. Although most of the Group members were involved in the discussions related to the various steps of the review and classification of directional wave analysis methods, the paper was prepared for the Working Group by Michel BENOIT, assisted by the co-authors. The contributions and discussions with the other members of the Working Group are anyway highly acknowledged.

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This paper is organized as follows. Chapter I introduces the scope of the work and gives some basic definitions. A brief overview of measuring devices for multidirectional waves is given in Chapter II. Chapter III is devoted to stochastic analysis methods and it represents the bulk of this paper. Chapter IV discusses deterministic analysis methods and Chapter V briefly mentions some time-domain approaches. Chapter VI draws up some conclusions.

I. SCOPE OF THE PAPER - BASIC DEFINITIONS

I.1 Scope of the paper

This paper is mainly devoted to the linear analysis of ocean surface waves in the absence of reflection. This corresponds to the so-called "open ocean" case, for which a lot of methods have been proposed in the scientific literature. It is out of the scope of this paper to review, present in detail and discuss all the methods that have been published in the literature for several decades. Several journal papers or book reviews are available in the bibliography for the reader who is interested in more detailed descriptions or comparative evaluations of a great part of the methods considered in the present work (e.g. Horikawa, 1988 ; Benoit, 1992, 1993 ; Kim *et al.*, 1993 ; Brissette and Tsanis, 1994 ; Benoit and Teisson, 1994 ; Hashimoto, 1997 among other authors).

The main goal of this paper is to provide some general descriptions of the underlying principles, capabilities and limitations of some of the most widely used methods, and in particular (but not only) the ones used within the Working Group.

I.2 The directional wave spectrum

A directional analysis of field or laboratory wave fields basically consists in determining the way the energy (or equivalently the variance) of this wave field spreads over both frequencies (or wave-numbers) and directions of propagation.

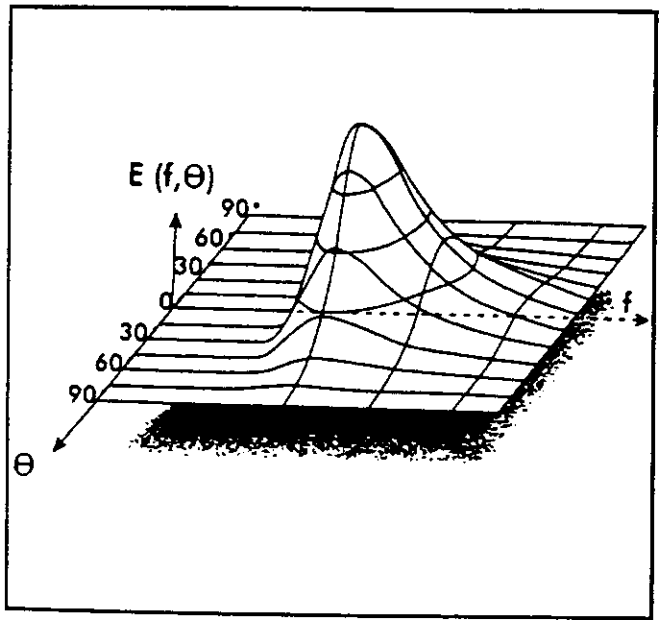

Figure 1 : Example of a directional wave spectrum

This spectral and angular distribution is formally represented by the directional energy spectrum of the wave field $E(f,\theta)$ (unit : $N.m^{-1}.Hz^{-1}.rad^{-1}$), a function of wave frequency f and direction of propagation θ.

Another more widely used variable is the **directional variance spectrum** $S(f,\theta)$ (unit : $m^2.Hz^{-1}.rad^{-1}$), which is directly proportional to the energy spectrum :

$$S(f,\theta) = E(f,\theta) / (\rho g) \qquad (1)$$

where g is the gravitational acceleration ($g \approx 9.81$ m/s^2) and ρ the water density.

$S(f,\theta)$ is often simply referred as the directional spectrum. Its estimation from wave measurements lies at the core of this paper.

A way to describe more practically what the directional spectrum represents may be given on the basis of the decomposition of the wave field into a large number of elementary sine wave components :

$$\eta(x,y,t) = \sum_{m=1}^{M} a_m \cos\left[k_m(x.\cos\theta_m + y.\sin\theta_m) - \omega_m t + \varphi_m\right] \quad (2)$$

In this expression, the surface elevation η is a function of both time t and location (x,y). It is written as the superposition of M elementary sine waves, each having amplitude a_m, angular frequency $\omega_m = 2\pi f_m$, wave-number k_m, direction θ_m and phase φ_m. Each wave component satisfies the linear dispersion relationship (d : water depth) :

$$\omega_m^2 = g.k_m.\tanh(k_m.d) \quad (3)$$

Under the assumption that the phases φ_m are randomly distributed over $[0 ; 2\pi]$ (with an uniform probability density), the following relationship holds between the directional variance spectrum and the amplitudes of the wave components comprised in the bidimensional range $[f ; f+df] \times [\theta ; \theta + d\theta]$.

$$\sum_{f}^{f+df} \sum_{\theta}^{\theta+d\theta} \frac{1}{2} a_m^2 = S(f,\theta)\, df\, d\theta \quad (4)$$

I.3 Classical decomposition of the directional spectrum

The following conventional decomposition of the directional spectrum is often used :

$$S(f,\theta) = E(f).D(f,\theta) \quad (5)$$

E(f) is the classical one-sided variance spectrum, that may be estimated by a single record of free-surface elevation. It is related to the directional spectrum by :

$$E(f) = \int_0^{2\pi} S(f,\theta)\, d\theta \quad (6)$$

$D(f,\theta)$ is the Directional Spreading Function (DSF) satisfying two important properties :

$$D(f,\theta) \geq 0 \text{ over } [0 , 2\pi] \quad (7)$$

$$\int_0^{2\pi} D(f,\theta)\, d\theta = 1 \quad (8)$$

The former condition expresses that the DSF is a non-negative function, whereas the latter is a direct consequence of (6). The DSF thus models the directional spreading of the wave energy at each frequency f. The directional analysis problem thus consists in determining the directional spectrum $S(f,\theta)$, or equivalently the variance spectrum E(f) and the DSF $D(f,\theta)$.

It should be noted that the directional variance spectrum may alternatively be expressed as a function of wave-number and direction (k, θ) or wave-number vector $(k_x, k_y) = (k.\cos\theta ; k.\sin\theta)$. The relationships between these expressions are recalled below :

$$S(f,\theta) = \frac{2\pi}{C_g} S(k,\theta) = \frac{2\pi k}{C_g} S(k_x,k_y) \quad (9)$$

where C_g is the group velocity for frequency f, as given by the linear wave theory.

II BRIEF REVIEW OF EXISTING MEASURING DEVICES :

A great effort has been devoted in the recent years to the determination of the directional wave spectrum. Several measurement techniques have been proposed for in situ or laboratory applications (see e.g. Horikawa (1988) for a more detailed description with some photographs).

They can be divided into several groups depending on the way they proceed :

i. the single-point systems : they measure simultaneously at the same location (i.e. on the same vertical) several properties of waves. In general, these system record 3 wave signals. The most widely used are :

   * the heave-pitch-roll buoy (e.g. Longuet-Higgins *et al.*, 1963 ; Lygre and Krogstad, 1986 ; Mardsen and Juszko, 1987 ; Brissette and Tsanis, 1994), which delivers the sea-surface elevation (heave) and two orthogonal slopes of sea-surface (pitch and roll).
   * the two-component current-meter associated with a pressure sensor or a wave elevation probe (e.g. Briggs, 1984 ; Schäffer and Hyllested, 1994 ; Benoit and Teisson, 1994),

   Other possible combinations include three-dimensional current-meter, a buoy recording three accelerations or three displacements over three axis,...

   Among these systems the heave-pitch-roll buoy is probably the most widely employed for operational use because it is a compact single-point measurement system of moderate cost and easy keeping.

   The clover-leaf buoy developed by Mitsuyasu *et al.* (1975) was designed to measure 6 wave signals, namely the surface curvatures $\left(\frac{\partial^2\eta}{\partial x^2} ; \frac{\partial^2\eta}{\partial y^2} ; \frac{\partial^2\eta}{\partial xy}\right)$ in addition to the three signals delivered by a classical heave-pitch-roll buoy.

ii. the gauge arrays : they are composed of several sensors set up at various locations over a fixed frame. The sensors may be either identical (wave elevation probes) or of various types including for instance current meters and pressure sensors (e.g. Hashimoto *et al.*, 1987, 1997 ; Benoit and Teisson, 1994).

iii. the remote-sensing systems : These techniques proceed in a different way, based on spatial correlations, rather than time correlations. Their common principle is to make a « picture » of the wave field over a given area. The wave field is assumed to be homogeneous over that area, so that the (wave-number) directional spectrum may be computed by Fourier transform of the surface elevation field. These techniques include microwave radars (Jackson *et al.*, 1985), aerial stereo-photography techniques,... They have been significantly developed in the recent years with the use of the Synthetic Aperture Radar (SAR), on the satellite ERS-1 and 2 for example.

Within the frame of this review we mainly consider measuring systems of types i and ii, recording several wave signals at one or several positions and delivering simultaneously sampled time series. These systems will be termed multi-component arrays in the following. We will thus mainly consider analysis methods based on temporal analysis of wave signals at a limited number of locations. Methods based on spatial correlations for systems of type iii will thus not be dealt with in this review.

III.1 Characterization of stochastic methods :

Methods of this class are based on the assumption that the wave field may be expressed in a form close to (2), but written in a continuous way :

$$\eta(x,y,t) = \iint \sqrt{2\, S(f,\theta)\, df\, d\theta}\, \cos\left[k(x.\cos\theta + y.\sin\theta) - 2\pi ft + \varphi\right] \qquad (10)$$

An important assumption is that the phase function φ is randomly distributed over $[0 ; 2\pi]$ (with an uniform probability density), expressing that the wave components are all independent of each other. There are no « phase-locked » waves. This remark clearly emphasizes that these methods are not suitable to deal with situations where such phase-lock relation exist, e.g. close to a reflective structure. Specific additional refinements are necessary to deal with this case (see Chapter VI).

In the so-called « stochastic approach », the information on the phase distribution of the wave field is thus ignored and interest is only focused on the determination of the directional spectrum. These methods proceed in two steps :

1. perform spectral analysis of the recorded time-series, by computing the cross-spectra between each pair of signals (cf. III.2),
2. determine the directional spectrum (or equivalently the DSF at each frequency) by inverting the relationship (established in III.3) between the cross-spectra and the directional spectrum, by making use of one of the methods presented in section III.5 and subsequent ones.

III.2 Cross-spectral analysis :

We consider a general multi-component measuring device composed of N sensors. As seen in chapter II, several of these probes may be located at the same position. Each probe records a wave signal (surface elevation, velocity, surface slope,...), noted here for convenience : $P_n(t)$ (n =1 to N). The locations of the probes are noted : $\vec{x}_n$ (n =1 to N) relative to an arbitrary origin.

The signals are recorded simultaneously at the N probes of the array, over a given duration T and with a sampling time-step Δt.

The analysis of the correlation between each pair of signals is performed in the frequency domain, by estimating of the cross-covariance spectral densities (i.e. the cross-spectra) $G_{mn}(f)$ between each couple $(P_m ; P_n)$, defined by :

$$G_{mn}(f) = \int_{-\infty}^{+\infty} R_{mn}(\tau)\, e^{-i2\pi f\tau}\, d\tau \quad \text{with} \quad R_{mn}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T P_m(t).P_n(t+\tau)\, dt \qquad (11)$$

In practice, the cross-spectra $G_{mn}(f)$ are usually estimated from the discretely sampled and finite duration time-series, by digital procedures based on the Fast Fourier Transform (e.g. Jenkins and Watts, 1968).

These cross-spectra are usually computed for $m \leq n$ only, as one can show that $G_{mn}(f)$ and $G_{nm}(f)$ are complex conjugate quantities. The total number of (complex) cross-spectra to compute for a measuring device composed of N probes is thus $N.(N+1)/2$. Among these, one can note that :

* N are auto-spectra ($m = n$), which are real quantities. If $P_n(t)$ is the surface elevation time-series, then $G_{nn}(f)$ is directly an estimation of the variance spectrum $E(f)$.
* $N.(N-1)/2$ are actual cross-spectra ($m < n$), whose real parts $C_{mn}(f)$ are called "coïncident spectral density functions" or "co-spectra" and imaginary parts $Q_{mn}(f)$ are called "quadrature spectral density functions" or "quad-spectra".

III.3 Relationship between the cross-spectra and the directional spectrum (or the directional spreading function) :

Within the frame of the linear theory and by assuming that the phases of the various components of the wave field are randomly distributed over $[0 ; 2\pi]$, the following equation is obtained between the directional spectrum $S(f,\theta)$ and the cross-spectra $G_{mn}(f)$:

$$G_{mn}(f) = \int_0^{2\pi} H_m(f,\theta)\, H_n^*(f,\theta)\, e^{-i\, \vec{k}.(\vec{x_n}-\vec{x_m})}\, S(f,\theta)\, d\theta \quad (m=1,..,N\ ;\ m<n) \tag{12}$$

which may also be written as an equation over the Directional Spreading Function $D(f,\theta)$, by using the decomposition (5) of the directional spectrum :

$$G_{mn}(f) = E(f) \int_0^{2\pi} H_m(f,\theta)\, H_n^*(f,\theta)\, e^{-i\, \vec{k}.(\vec{x_n}-\vec{x_m})}\, D(f,\theta)\, d\theta \tag{13}$$

$H_m(f,\theta)$ is the transfer function between the surface elevation signal and any other wave signal $P_m(t)$ (pressure, velocity, slope of the surface,...).

The symbol * stands for the conjugate operator, as $H_m(f,\theta)$ is a complex function in the general case.

The transfer function $H_m(f,\theta)$ may be decomposed in the following form :

$$H_m(f,\theta) = h_m(f).\cos^{\alpha_m}\theta\ .\ \sin^{\beta_m}\theta \tag{14}$$

The quantities h_m, α_m and β_m depend on the type of wave signal $P_m(t)$. Their expressions are given in Table 1 for several wave signals and by considering the linear wave theory results (e.g. Isobe *et al.*, 1984). In this table, the variable $\Psi(x,y,z,t)$ represents the velocity potential for a long-crested monochromatic wave :

$$\Psi = i\,\frac{g}{\omega}\,\frac{H}{2}\,\frac{ch(k.(d+z))}{ch(kd)}\,\exp\left(i.(\vec{k}.\vec{x} - \omega.t)\right) \tag{15}$$

For convenience, (13) is also rewritten in the following form :

$$\phi_{mn}(f) = \frac{G_{mn}(f)}{E(f)} = \int_0^{2\pi} q_{mn}(f,\theta)\, D(f,\theta)\, d\theta \tag{16}$$

Wave signal	P_m	$h_m(f)$	α_m	β_m
Surface elevation	$\eta = \frac{1}{g}\frac{\partial \Psi}{\partial t}\Big\|_{z=0} = \frac{H}{2}\exp\left(i.(\vec{k}.\vec{x} - \omega.t)\right)$	1	0	0
Surface slope (x axis)	$\frac{\partial \eta}{\partial x} = [i\,k\cos\theta]\,\eta$	i.k	1	0
Surface slope (y axis)	$\frac{\partial \eta}{\partial y} = [i\,k\sin\theta]\,\eta$	i.k	0	1
Velocity (x axis)	$u_x = -\frac{\partial \psi}{\partial x} = \left[\omega\frac{ch(k.(d+z))}{sh(kd)}\cos\theta\right]\eta$	$\omega\frac{ch(k.(d+z))}{sh(kd)}$	1	0
Velocity (y axis)	$u_y = -\frac{\partial \psi}{\partial y} = \left[\omega\frac{ch(k.(d+z))}{sh(kd)}\sin\theta\right]\eta$	$\omega\frac{ch(k.(d+z))}{sh(kd)}$	0	1
Velocity (z axis)	$u_z = -\frac{\partial \psi}{\partial z} = \left[-i\,\omega\frac{sh(k.(d+z))}{sh(kd)}\right]\eta$	$-i\,\omega\frac{sh(k.(d+z))}{sh(kd)}$	0	0
Vertical surface veloc.	$\frac{\partial \eta}{\partial t} = \frac{\partial \Psi}{\partial z}\Big\|_{z=0} = -i.\omega.\eta$	$-i.\omega$	0	0
Acceleration (x axis)	$a_x = \frac{\partial u_x}{\partial t} = \left[-i\,\omega^2\frac{ch(k.(d+z))}{sh(kd)}\cos\theta\right]\eta$	$-i.\omega^2\frac{ch(k.(d+z))}{sh(kd)}$	1	0
Acceleration (y axis)	$a_y = \frac{\partial u_y}{\partial t} = \left[-i\,\omega^2\frac{ch(k.(d+z))}{sh(kd)}\sin\theta\right]\eta$	$-i\,\omega^2\frac{ch(k.(d+z))}{sh(kd)}$	0	1
Acceleration (z axis)	$a_z = \frac{\partial u_z}{\partial t} = \left[-\omega^2\frac{sh(k.(d+z))}{sh(kd)}\right]\eta$	$-\omega^2\frac{sh(k.(d+z))}{sh(kd)}$	0	0
Vertical surface accel.	$\frac{\partial^2 \eta}{\partial t^2} = \omega^2.\eta$	$-\omega^2$	0	0
Displacement (x axis)	$\xi_x = \int u_x\,dt = \left[i\frac{ch(k.(d+z))}{sh(kd)}\cos\theta\right]\eta$	$i\frac{ch(k.(d+z))}{sh(kd)}$	1	0
Displacement (y axis)	$\xi_y = \int u_y\,dt = \left[i\frac{ch(k.(d+z))}{sh(kd)}\sin\theta\right]\eta$	$i\frac{ch(k.(d+z))}{sh(kd)}$	0	1
Displacement (z axis)	$\xi_z = \int u_z\,dt = \left[\frac{sh(k.(d+z))}{sh(kd)}\right]\eta$	$\frac{sh(k.(d+z))}{sh(kd)}$	0	0
Dynamic pressure	$p = \rho\frac{\partial \psi}{\partial t} = \left[\rho\,g\frac{ch(k.(d+z))}{ch(kd)}\right]\eta$	$\rho\,g\frac{ch(k.(d+z))}{ch(kd)}$	0	0

Ψ = wave potential ; ω = angular frequency ; k = wave-number ; θ = direction ; d = water depth
z = elevation from the still water level ; ρ = water density ; g = gravitational acceleration

Table 1 : Transfer functions for various wave signals (from linear theory).

III.4 Estimation of the directional spectrum :

Solving the system of integral equations (12) or (16) for estimating the directional spectrum of the DSF lies at the core of directional wave analysis. When considering (16), the problem is to find at each frequency $D(f,\theta)$, a continuous function over $[0 ; 2\pi]$ from a finite (and often very limited) number of equations, given by the cross-spectra. If an infinite number of wave signals were simultaneously recorded (i.e. if we had an infinite number of cross-spectra), the DSF could in principle be determined uniquely.

As the operational measuring devices usually have a number of gauges ranging from 3 to 10, the mathematical problem is not fully defined and one has to introduce some additional assumptions or conditions of the DSF in order to get a unique solution.

For instance, for the case of a heave-pitch-roll buoy, recording at the same location the surface elevation and two slopes on two orthogonal axis (N =3), we have the following set of signals : $(P_1(t) ; P_2(t) ; P_3(t)) = \left(\eta(t) ; \frac{\partial \eta}{\partial x}(t) ; \frac{\partial \eta}{\partial y}(t)\right)$

This gives the following set of 6 complex cross-spectra :

$$\left.\begin{array}{ll}
C_{11}(f) = \int_0^{2\pi} S(f,\theta)\, d\theta = E(f) & Q_{11}(f) = 0 \\
C_{22}(f) = E(f).k^2 \int_0^{2\pi} D(f,\theta).\cos^2(\theta) d\theta & Q_{22}(f) = 0 \\
C_{33}(f) = E(f).k^2 \int_0^{2\pi} D(f,\theta).\sin^2(\theta) d\theta & Q_{33}(f) = 0 \\
C_{12}(f) = 0 & Q_{12}(f) = E(f).k \int_0^{2\pi} D(f,\theta).\cos(\theta) d\theta \\
C_{13}(f) = 0 & Q_{13}(f) = E(f).k \int_0^{2\pi} D(f,\theta).\sin(\theta) d\theta \\
C_{23}(f) = E(f).k^2 \int_0^{2\pi} D(f,\theta).\cos(\theta).\sin(\theta) d\theta & Q_{23}(f) = 0
\end{array}\right| \quad (17)$$

From the twelve real cross-spectral coefficients, only six are different from zero. Furthermore $C_{11}(f)$ does not carry any information about the directional distribution, but directly gives the variance spectrum $E(f)$. In addition, $C_{11}(f)$, $C_{22}(f)$, $C_{33}(f)$ are related by the following relationship :

$$C_{22}(f) + C_{33}(f) = k^2.C_{11}(f) \quad (18)$$

This relation is often used for obtaining an empirical value of the wave number, instead of using the linear dispersion relation (3) :

$$k = \sqrt{\frac{C_{22}(f) + C_{33}(f)}{C_{11}(f)}} \quad (19)$$

It is thus possible to compute only five independent coefficients at each frequency, from which one is devoted to the estimation of the frequency spectrum $E(f)$ and the four

may be used for the computation of the DSF $D(f,\theta)$ at this frequency. This mation may be rewritten in terms of Fourier coefficients of the DSF, defined by :

$$\int_0^{2\pi} D(f,\theta).\cos(n.\theta).\,d\theta \qquad b_n = \int_0^{2\pi} D(f,\theta).\sin(n.\theta).\,d\theta \tag{20}$$

For single-point systems (N = 3), only the four Fourier coefficients (a_1, b_1, a_2, b_2) may be computed. They carry all the available information on the DSF. The expressions of these Fourier coefficients from the cross-spectra are gathered in Table 2 for various single-point systems.

In the following section, several directional methods aiming at solving (16) will be briefly reviewed and discussed. These methods are not equivalent. Their resolving capabilities and limitations are related to the modelling assumptions they are based on, as discussed in the next sub-sections.

III.5 Fourier Series Decomposition Methods

III.5.1 Truncated Fourier Series Decomposition (TFS)

As the Directional Spreading Function (DSF) is, at least in the general short-crested case, a continuous function over $[0 ; 2\pi]$ satisfying $D(f,0)=D(f,2\pi)$, it is possible to write its Fourier series decomposition :

$$D(f,\theta) = \frac{a_0}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[a_n.\cos(n.\theta) + b_n.\sin(n.\theta)\right] \tag{21}$$

where a_n and b_n $(n > 0)$ are given by (20),

$$\text{and : } a_0 = \int_0^{2\pi} D(f,\theta)\,d\theta = 1 \tag{22}$$

The most straightforward way to use this approach consists in truncating the above Fourier series decomposition in order to only keep the terms whose coefficients may be determined from the available cross-spectral data. If we note K the maximum value of the rank of the decomposition that can be computed, we then get :

$$\hat{D}_{TFS}(f,\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{K}\left[a_n.\cos(n.\theta) + b_n.\sin(n.\theta)\right] \tag{23}$$

For a single-point measuring devices for instance, only terms of ranks 1 and 2 may be computed from the co- and quad-spectra (cf. III.4). We thus obtain for this estimate :

$$\hat{D}_{TFS}(f,\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\left[a_1.\cos(\theta) + b_1.\sin(\theta) + a_2.\cos(2.\theta) + b_2.\sin(2.\theta)\right] \tag{24}$$

This estimate is easy to implement and computationally efficient. But, as a drawback of its limited number of component terms, it may sometimes produce negative values, which is not acceptable for a DSF. A refinement of this method is presented in the next section (cf. III.5.2), in order to overcome this shortcoming.

n°	Single-point measuring device	E(f)	$a_1(f)$	$b_1(f)$	$a_2(f)$	$b_2(f)$
1	Heave-pitch-roll buoy $[\eta ; \frac{\partial\eta}{\partial x} ; \frac{\partial\eta}{\partial y}]$	C_{11}	$\frac{Q_{12}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{Q_{13}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{C_{22}-C_{33}}{C_{22}+C_{33}}$	$\frac{2.C_{23}}{C_{22}+C_{33}}$
2	Elevation probe + 2D current-meter $[\eta ; u_x(z_2) ; u_y(z_2)]$	C_{11}	$\frac{C_{12}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{C_{13}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{C_{22}-C_{33}}{C_{22}+C_{33}}$	$\frac{2.C_{23}}{C_{22}+C_{33}}$
3	Pressure gage + 2D current-meter $[p(z_1) ; u_x(z_2) ; u_y(z_2)]$	$C_{11}\left(\frac{1}{\rho g}\frac{ch(k.d)}{ch(k(d+z_1))}\right)^2$	$\frac{C_{12}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{C_{13}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{C_{22}-C_{33}}{C_{22}+C_{33}}$	$\frac{2.C_{23}}{C_{22}+C_{33}}$
4	Device recording three displacements $[\eta ; \xi_x(0) ; \xi_y(0)]$	C_{11}	$\frac{Q_{12}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{Q_{13}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{C_{22}-C_{33}}{C_{22}+C_{33}}$	$\frac{2.C_{23}}{C_{22}+C_{33}}$
5	Device recording three velocities $[u_z(z_1) ; u_x(z_2) ; u_y(z_2)]$	$C_{11}\left(\frac{1}{\omega}\frac{sh(k.d)}{sh(k(d+z_1))}\right)^2$	$\frac{Q_{12}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{Q_{13}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{C_{22}-C_{33}}{C_{22}+C_{33}}$	$\frac{2.C_{23}}{C_{22}+C_{33}}$
6	Device recording three accelerations $[a_z(z_1) ; a_x(z_2) ; a_y(z_2)]$	$C_{11}\left(\frac{1}{\omega}\frac{sh(k.d)}{sh(k(d+z_1))}\right)^2$	$\frac{Q_{12}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{Q_{13}}{\sqrt{C_{11}(C_{22}+C_{33})}}$	$\frac{C_{22}-C_{33}}{C_{22}+C_{33}}$	$\frac{2.C_{23}}{C_{22}+C_{33}}$

Table 2: Expressions of the first Fourier coefficients of the DSF for various single-point measuring devices

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$$D_{WFS}$$

an array of N wave elevation gauges, the number of harmonics in the composition may theoretically be increased up to N(N-1)/2, if the gauge pairs are independent (e.g. Borgman, 1969). In fact, it is often preferable to keep harmonics only to rank K = N(N-1)/2 - 2 (or even less) in order to get a stable estimate.

The Fourier coefficients a_n and b_n are thus obtained from the measured co- and quad-spectra by using a lest-squares method. This leads to solve two linear systems of equations : one for the harmonics of even rank and one for harmonics of odd rank (Borgman, 1969).

III.5.2 Weighted Fourier Series Decomposition (WFS) :

In order to get a positive estimate of the DSF from the Fourier series decomposition, it is possible to use a weighting function of the following form (Longuet-Higgins *et al.*, 1963):

$$W_K(\theta) = R_K.\cos^{2K}(\frac{\theta}{2}) \tag{25}$$

where K is the rank of the truncated Fourier series decomposition. The normalisation constant R_K is determined in order to verify :

$$\int_0^{2\pi} W_K(\theta).d\theta = 1 \tag{26}$$

which gives after some algebra :

$$R_K = \frac{2^{2K-1}\,(K!)^2}{\pi\,(2K)!} \tag{27}$$

The weighting function $W_K(\theta)$ has a finite Fourier series decomposition :

$$W_K(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{K} a_k^K.\cos\left(n.\theta\right) \tag{28}$$

It may be shown after little algebra that its Fourier coefficients read :

$$a_k^K = \frac{(K!)^2}{(K-k)!\,(K+k)!} \tag{29}$$

The DSF estimate based on the weighted Fourier series decomposition is thus obtained by the convolution of the TFS estimate and the weighting function $W_K(\theta)$:

$$\hat{D}_{WFS}(f,\theta) = \int_0^{2\pi} D(f,\theta').W_K(\theta'-\theta)\,d\theta' \tag{30}$$

which finally gives :

$$\hat{D}_{WFS}(f,\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{K} a_k^K.\left(a_n.\cos\left(n.\theta\right) + b_n.\sin\left(n.\theta\right)\right) \tag{31}$$

For the case of single-point measuring devices for instance, K = 2 and the weighting function takes the following form :

$$W_2(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\left[\frac{2}{3}\cos(\theta) + \frac{1}{6}\cos(2.\theta)\right] \quad (32)$$

which then gives the following expression of the DSF estimate :

$$\hat{D}_{WFS}(f,\theta) = \frac{1}{2\pi} + \frac{2}{3\pi}\left(a_1.\cos(\theta) + b_1.\sin(\theta)\right) + \frac{1}{6\pi}\left(a_2.\cos(2.\theta) + b_2.\sin(2.\theta)\right) \quad (33)$$

Compared to the TFS estimate, the WFS estimate is always positive, but the use of the weighting function usually results in a diffusion of the energy over the directions surrounding the main peaks. These directional peaks are systematically lowered and broadened by this convolution. Due to that, this estimate is not widely used. It may however be used as a first guess for other iterative methods, due to its computational efficiency.

III.6 Parametrical Methods

III.6.1 Direct fitting to parametrical models :

Unimodal models

The common principle of the fitting methods consists in assuming *a priori* a given parametrical expression for the DSF and determining the low number of parameters of this expression from the measured cross-spectra. This approach allows to significantly reduce the number of unknowns related to the DSF. In the case of a unimodal parametrical model, the problem is indeed reduced to the determination of two parameters : the main direction of propagation and a directional spreading factor, which represents the angular spreading of wave energy around the main direction. These coefficients are best computed for the first rank coefficients a_1 and b_1 of the Fourier series decomposition of the DSF, but higher rank coefficients could also be alternatively used.

Several expressions are used for the parametrical unimodal model. For instance, Mitsuyasu *et al.* (1975) proposed the following model :

$$\hat{D}_{1MFM}(f,\theta) = \Delta(s)\cos^{2s}\left(\frac{\theta - \theta_0}{2}\right) \quad (34)$$

The normalisation coefficient is found from then condition that the integral of the DSF over $[0 ; 2\pi]$ remains equal to 1 (Γ is the gamma function) :

$$\Delta(s) = \frac{2^{2s-1}\left(\Gamma(s+1)\right)^2}{\pi\ \Gamma(2s+1)} \quad (35)$$

The main direction θ_0 and the directional index s may be computed from the Fourier coefficient of rank 1 or 2 :

Rank 1 : $\theta_0 = \mathrm{Arg}(a_1 + i.b_1)$ $\qquad s = \frac{r_1}{1 - r_1}$ (36)

Rank 2 : $\theta_0 = \frac{1}{2}\mathrm{Arg}(a_2 + i.b_2)$ $\qquad s = \frac{1 + 3.r_2 + \sqrt{r_2^2 + 14.r_2 + 1}}{2(1 - r_2)}$ (37)

where : $r_n = \sqrt{a_n^2 + b_n^2}$ (38)

In the latter case, it should be emphasized that the main direction is given modulo π, which makes it quite difficult to use. This and the fact that parameters given by rank 1 are usually more stable than those given by rank 2 lead to recommend the use of the pair $(a_1 ; b_1)$ in determining the parameters of the unimodal model.

Another quite widely used model is the Gaussian model, for a wave direction falling in the range $[\theta_0 - \pi ; \theta_0 + \pi]$:

$$\hat{D}_{1MFG}(f,\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(- \frac{(\theta - \theta_0)^2}{2\,\sigma^2} \right) \qquad (39)$$

Again the main direction θ_0 and the directional width σ are computed from the Fourier coefficient of rank 1 or 2 :

Rank 1 : $\theta_0 = \mathrm{Arg}(a_1 + i.b_1)$ $\qquad \sigma = \sqrt{-2.\ln(r_1)}$ (40)

Rank 2 : $\theta_0 = \frac{1}{2}\mathrm{Arg}(a_2 + i.b_2)$ $\qquad \sigma = \frac{\sqrt{-2.\ln(r_2)}}{2}$ (41)

Other expressions may be found in the literature (e.g. Borgman, 1969), but their principle is always similar to the one presented here. Only the expressions giving the parameters of the model vary from one parametrical model to an other.

Methods of this type were (and are still) widely used as they are very computationally efficient and robust. Furthermore the corresponding estimate satisfies the various properties of the DSF. But these methods suffer from some shortcomings related to the strong assumptions they are based on : by definition, they are only suitable to model unimodal and symmetrical directional peaks. They are thus unable to detect non-symmetrical unimodal peaks, nor bimodal sea-states (two directional peaks at the same frequency). This explains why these methods are not advised here.

The interpretation of results given by these methods thus requires great care. Before concluding that there is no bimodal conditions (at the same frequency) at a given location, it is stressed that one should first check that the method used for directional analysis allows for the representation of two peaks at the same frequency.

Bimodal models

In order to make the fitting methods more suitable, several authors have suggested to use bimodal parametrical models (Van Heteren, 1983 ; Benoit, 1992). Those bimodal models are obtained by a linear combination of two unimodal models such as the one proposed in the previous section. Again various expressions may be used. We only give here, as an example, the bimodal estimate based of the Gaussian model :

$$\hat{D}_{2MFG}(f,\theta) = \frac{\lambda}{\sqrt{2\pi}\,\sigma_1} \exp\left(- \frac{(\theta - \theta_1)^2}{2\,\sigma_1^2} \right) + \frac{1-\lambda}{\sqrt{2\pi}\,\sigma_2} \exp\left(- \frac{(\theta - \theta_2)^2}{2\,\sigma_2^2} \right) \quad \lambda \in [0, 1] \qquad (42)$$

This type of models has 5 unknowns $(\theta_1, \sigma_1, \theta_2, \sigma_2, \lambda)$ and is able to model both unimodal and bimodal sea-states with two peaks at the same frequency.

For single point systems however, the number of available data is limited to 4 independent Fourier coefficients. One has thus to introduce an additional condition in order to solve the problem. It is for instance possible to impose $\sigma_1 = \sigma_2$ (Van Heteren, 1983), but this somewhat restricts the general capabilities introduced by the bimodal

expression introduced above. On the other hand, Benoit (1992) tried to use method of minimisation under constraints by keeping the 5 unknown parameters and introducing additional conditions on the variations of these unknowns. These methods are clearly out of the scope of this presentation and require sophisticated algorithms. They may be quite efficient, but they are not easy to deal with in the general case. They appear to be quite sensitive to the first guess given as input to the algorithm. As they may become unstable in some conditions, they are not advised for operational use with single-point systems.

For the array of wave gauges however, this method is more suitable as the larger number of measured cross-spectra allows for a better determination of the unknown parameters by using a least-squares method. Benoit and Teisson (1994) developed and used such a method with an array of 5 probes in laboratory experiments both under unimodal and bimodal waves conditions. They showed that the method usually depicts the correct shape of the spectrum, but that there are still some numerical problems of stability at frequencies located away from the peak frequency. Further improvements on the numerical algorithms are required in order to make this method more robust in a large variety of situations.

III.6.2 Statistical fitting to unimodal parametrical models :

In the unimodal method presented in section III.6.1, the parameters of the parametrical model of DSF are determined in deterministic way, by making use of only one harmonic of the Fourier series decomposition of the DSF. In order to use more data information and to proceed to a statistical fitting of the model, Isobe (1990) developed a Maximum Likelihood Fitting method (MLF) to be used on data from single-point systems, as a wave elevation probe and a bidimensional horizontal current-meter.

The DSF is still assumed to be given in a standard form. Isobe (1990) used a Mitsuyasu-type expression as in (34):

$$\hat{S}_{MLF}(f,\theta) = E(f).\Delta(s)\cos^{2s}\left(\frac{\theta - \theta_0}{2}\right) \qquad (43)$$

but the values of the parameters of the models (E(f), θ_0, s) are the ones which maximize a likelihood function for the Fourier coefficients of the signals (see Isobe (1990) for more details). In addition, a level of noise, noted ε, is allowed in the data, so that finally we get a set of four unknown parameters at each frequency :

$(\lambda_1 = E(f) ; \lambda_2 = \theta_0 ; \lambda_3 = s ; \lambda_4 = \varepsilon)$

This method requires of course a higher computational effort than the direct fitting technique of the previous section. One has to find the set of parameters which maximize the Likelihood, i.e. that make its partial derivatives vanish. Isobe (1990) used a Newton-Raphson method for that purpose, but this approach also requires the computation of the second derivatives of the Likelihood function, which makes the algorithm not straightforward to implement.

By definition, as presented above, this method is in principle only applicable to unimodal and symmetrical DSF. For single-point systems, it is not able to resolve two directional peaks at the same frequency, but this extension should be possible and interesting for more complex arrays. Isobe (1990) applied his method to field data and found that its results were very similar to the direct estimation of directional parameters such as the main direction and the long-crestedness parameter.

III.7 Maximum Likelihood Methods :

III.7.1 Maximum Likelihood Method (MLM) :

Introduced by Capon *et al.* (1967) in seismic wave detection, and then extended by several authors (e.g. Isobe *et al.*, 1984 ; Krogstad, 1988), the Maximum Likelihood Method (MLM) is based on the assumption that the estimate of the DSF may be expressed as a linear combination of the cross-spectra :

$$\hat{D}_{MLM}(f,\theta) = \frac{1}{\hat{E}(f)} \sum_{m,n} \alpha_{mn}(f,\theta).G_{mn}(f) \qquad (44)$$

One can then show that this estimate is related to the actual DSF $D(f,\theta)$ by the following relationship :

$$\hat{D}_{MLM}(f,\theta) = \int_0^{2\pi} D(f,\theta).w(\theta,\theta')\, d\theta'$$

where : $$w(\theta,\theta') = \sum_{m,\,n} \alpha_{mn}(f,\theta).H_m(f,\theta').H_n^*(f,\theta') \qquad (45)$$

The MLM estimate may thus be seen as the convolution product of the actual DSF by a window function $w(\theta,\theta')$. This estimate will best approach the actual DSF as the window function tends towards a Dirac function $\delta(\theta,\theta')$.

The estimate that best satisfies this condition is found to be (Isobe *et al.*, 1984) :

$$\hat{D}_{MLM}(f,\theta) = \frac{\kappa}{\sum_{m,\,n} H_m(f,\theta).G_{mn}^{-1}(f).H_n^*(f,\theta)} \qquad (46)$$

In the above expression, $G_{mn}^{-1}(f)$ stands for the elements of the inverse of the cross-spectral matrix and κ is determined from the condition that the integral of the MLM estimate over $[0 ; 2\pi]$ is equal to 1. In the case of a single-point measuring system, the cross-spectral matrix is of dimension 3 and can be inverted analytically, which makes the method very computationally efficient. For the case of a multi-component array, a matrix inversion subroutine has to be run. The MLM method is quite widely used and has proven to have a good level of accuracy in estimating the DSF. Tests on numerical and laboratory data (Benoit and Teisson, 1994) however show that this method usually tends to produce broader directional peaks, compared to the target directional spectra.

III.7.2 Iterative Maximum Likelihood Methods (IMLM1 ; IMLM2)

When considering the MLM estimate, one can notice that it is not consistent with the cross-spectral matrix. This means that the cross-spectra computed from the MLM estimate differ from the cross-spectra computed from the wave signals. Pawka (1983) and then Oltman-Shay and Guza (1984) have thus proposed an iterative refinement of the MLM in order to get a consistent estimate. Two iterative schemes were introduced, corresponding respectively to the versions 1 (IMLM1) and 2 (IMLM2) of the Iterative Maximum Likelihood Method, and both based on the following expression :

$$\hat{D}^{i}_{IMLM}(f,\theta) = \hat{D}^{i-1}_{IMLM}(f,\theta) + \varepsilon^{i}(f,\theta) \qquad \text{with} \quad \hat{D}^{0}_{IMLM}(f,\theta) = \hat{D}_{MLM}(f,\theta) \qquad (47)$$

$$\underline{\text{IMLM1 :}} \quad \varepsilon^i(f,\theta) = \frac{|\lambda|^{\beta+1}}{\lambda\,\gamma}\,\hat{D}^{i-1}_{IMLM}(f,\theta) \qquad \text{with} \quad \lambda = 1 - \frac{\Delta^{i-1}_{MLM}(f,\theta)}{\hat{D}_{MLM}(f,\theta)} \tag{48}$$

$$\underline{\text{IMLM2 :}} \quad \varepsilon^i(f,\theta) = \frac{|\lambda|^{\beta+1}}{\lambda\,\gamma} \qquad \text{with} \quad \lambda = \hat{D}_{MLM}(f,\theta) - \Delta^{i-1}_{MLM}(f,\theta) \tag{49}$$

In the above expressions, Δ^{i-1}_{MLM} stands for the MLM estimate computed from the cross-spectra based on $\hat{D}^{i-1}_{IMLM}(f,\theta)$. The parameters β and γ control the convergence of the iterative algorithm. Their standard values are in general of order 1 for β and of order 10 for γ (Oltman-Shay and Guza, 1984). The iterations are stopped after a fixed number of steps or when a convergence criterion is satisfied on $\hat{D}^{i}_{IMLM}(f,\theta)$ (Krogstad *et al.*, 1988).

Tests on numerical and laboratory wave (Benoit, 1992 ; Benoit and Teisson, 1994) show that in most cases the IMLM2 version exhibit more reliable results than the IMLM1 version. In general, 10 to 20 iterations are required to reach a good convergence, which produces a significant increase of the computing time compared to the standard MLM approach. However, the IMLM2 is one of the suitable methods that may be operationally used, as it clearly improves the standard MLM estimate, in particular for single-point measuring devices. It is not very difficult to implement and very stable if an efficient procedure is used for testing the convergence.

III.7.3 Eigenvector methods (EVM ; IEVM1 ; IEVM2) :

This method originally used in acoustic wave detection was applied to directional wave analysis by Mardsen and Juszko (1987) for single-point measuring systems. The theoretical background is similar to the MLM approach, but they introduce the additional assumption that it is possible to split the matrix of cross-spectra **M** into noise and signal components :

$$\mathbf{M} = \hat{\mathbf{S}} + \hat{\mathbf{N}} \tag{50}$$

where $\hat{\mathbf{S}}$ is an estimate of the signal component of the cross-spectral matrix and $\hat{\mathbf{N}}$ is an estimate of the noise component. The partitioning of the noise and signal components is achieved through the diagonalization of the cross-spectral matrix, which allows to find the eigenvalues and eigenvectors of this matrix. The cross-spectral matrix is theoretically Hermitian, definite and positive : all these eigenvalues are thus real and positive. Mardsen and Juszko (1987) proposed that the largest eigenvalue is related to the signal component whereas the two lower ones span the noise component. They furthermore used the fact that the noise component is orthogonal to the signal component. For a multi-component array, the separation of the cross-spectral matrix into signal and noise is more critical. Some indications to apply this method to that type of measuring device may be found in Barrodale *et al.* (1985).

Compared to the MLM, only the noise component of the directional spectrum is minimized, which finally gives an estimate of the form :

$$\hat{D}_{EVM}(f,\theta) = \frac{\kappa}{\sum_{m,\,n} H_m(f,\theta).\hat{N}^{-1}_{mn}(f).H^*_n(f,\theta)} \tag{51}$$

where κ is determined from the condition that the integral of the DSF over $[0 ; 2\pi]$ is equal to 1. As for the MLM, $\widehat{N}^{-1}_{mn}(f)$ stands for the elements of the inverse of the signal component matrix $\widehat{N}$.

Note that if all three eigenvalues were assumed to span the signal, the method would be equivalent to the MLM.

The EVM requires an algorithm to find the eigenvalues and eigenvectors of a complex matrix (dimension 3 for a single-point system) : it is thus less computationally efficient than the MLM (about twice the CPU time on tests performed by Benoit, 1992). This method being more oriented to signal detection than the MLM, the directional peaks of the spectrum are usually more sharp and narrow. Applied to numerical simulated signals, this method usually appears less efficient than the MLM in estimating the target spectrum (Benoit, 1992). In fact, this approach is much more recommended to real measured signals with a significant level of noise in the data. Comparisons performed by Mardsen and Juszko (1987) confirm in comparison to other methods that the reliability of the EVM estimates decreases as the noise level on numerical simulated data decreases.

As for the MLM method, the EVM estimate is not consistent with the cross-spectral matrix. It is thus possible to set up the same iterative procedure as presented for the MLM in the previous section. We thus obtain two Iterative EigenVector Methods, termed IEVM1 and IEVM2 respectively, based on the schemes introduced for the MLM (Mardsen and Juszko, 1987).

III.8 Long-Hasselmann Method (LHM)

This method was proposed by Long and Hasselmann (1979) and Long (1980). It is based on an inverse technique, applicable to the problem of fitting a model to some data, representing integral properties of the function to be estimated. An initial simple estimate (e.g. a uniform DSF, a Fourier series decomposition estimate TFS or WFS, a unimodal fitted parametrical model) is iteratively modified to minimize a "nastiness" function that takes into account the various conditions on the spreading function, namely the equations given by the cross-spectra, the condition of unit integral of the DSF, the fact that the DSF should remain positive).

This method also takes into account the statistical variability of the cross-spectral estimates. The « optimal » estimate is the one which minimizes the nastiness function.

For a single-point system, this estimate finally takes the following general form :

$$\hat{D}_{LHM}(\theta) = \Delta.MAX\left[0 ; \tilde{D}(\theta) + \mu_1 + \mu_2 \cos\theta + \mu_3 \sin\theta + \mu_4 \cos 2\theta + \mu_5 \sin 2\theta\right] \quad (52)$$

where $\tilde{D}(\theta)$ is the initial estimate and μ_j (i=1,...,5) are model coefficients determined by an iterative algorithm, in order to satisfy (in statistical sense) the various constraints mentioned above. The application of this method to buoy data is described in detail by Long (1980).

This method usually produces reliable results for the DSF, although it was observed both on numerical and laboratory tests (Benoit, 1992 ; Benoit and Teisson, 1994) that the peaks of the estimate are often somewhat broader and lower compared to the target peaks. This method is thus not very suitable to analyse sharp and thin directional peaks, unless the initial estimate is already able to do so. Furthermore, at least for single-point

systems, the interpretation of the results of this method may be delicate due to the MAX operator in (52). Due to its truncated Fourier series-like expression, LHM estimate may have some lobes whose negative portions will be truncated by the MAX operator. The remaining positive part of these lobes may be interpreted as secondary directional peaks in the DSF, whereas they are sometimes just spurious peaks produced by the method.

Finally, the method is quite difficult to implement and rather time consuming, so that it is in fact not very widely used for operational analysis.

III.9 Maximum Entropy Methods (MEM1 ; MEM2)

The Maximum Entropy Methods have been adapted from the theory of probability to directional wave analysis due to the similarities between a DSF and a probability density function (PDF). Indeed, both a DSF and a PDF are positive functions, whose integral is equal to 1. In fact, a DSF may be seen as the PDF of the wave energy over the directions of propagation.

The principle of these methods is to define an « entropy » function, which has to be maximized under the various constraints given by the cross-spectra equations. Two such methods have been proposed depending on the definition used for the entropy.

III.9.1 Maximum Entropy Method - Version 1 (MEM1) :

The first definition for the entropy was proposed by Barnard (1969), and then adapted to single-point systems by Lygre and Krogstad (1986). This approach is more specifically used in spectral analysis. Some authors suggest to rather speak of « change in entropy » (Nwogu *et al.*, 1987).

The entropy H_1 is defined by :

$$H_1(\hat{D}) = -\int_0^{2\pi} \ln(\hat{D}(\theta))\, d\theta \qquad (53)$$

The basic idea of MEM1 is to search a estimate that maximizes the entropy H_1 and whose two first harmonics of the Fourier series decomposition are identical to the ones of the TFS estimates (i.e. obtained from the measured co- and quad-spectra). Lygre and Krogstad (1986) established that the estimate satisfying these conditions has the following form :

$$\hat{D}_{MEM1}(\theta) = \frac{1}{2\pi} \frac{1 - F_1 c_1^* - F_2 c_2^*}{\left| 1 - F_1.(\cos\theta - i.\sin\theta) - F_2.(\cos 2\theta - i.\sin 2\theta) \right|^2} \qquad (54)$$

where F_1 and F_2 are complex numbers, obtained from the complex Fourier coefficients c_1 and c_2 of the DSF ($c_1 = a_1 + i\,b_1$ and $c_2 = a_2 + i\,b_2$) :

$$F_1 = \frac{(c_1 - c_2 c_1^*)}{1 - |c_1|^2} \qquad (55)$$

$$F_2 = c_2 - c_1 F_1$$

The main advantage of this method lies in its high computational efficiency. In addition, the MEM1 estimate is always consistent with the cross-spectral data.

But this estimate also presents a quite important drawback in the fact that it consistently overpredicts the height of the directional peaks. Furthermore, it sometimes produces double peaks on unimodal cases (Benoit, 1992 ; Kim *et al.*, 1993 ; Brissette and Tsanis, 1994). This latter shortcoming was also noted by the authors of the method both on numerical test-cases and field observations (Lygre and Krogstad, 1986). This constitutes a quite severe limitation of the approach, because in the presence of an estimate with two peaks, one can not systematically conclude to a bimodal sea-state. Due to this, it is not advised to use only that method for wave analysis, but rather to run it with other methods in a comparative evaluation exercise.

III.9.2 MEM2 or Maximum Entropy Principle (MEP) :

This approach is based on the Shannon definition for the entropy, which is more specifically used in probability theory. It has been applied to directional wave analysis by Kobune and Hashimoto (1986) and Nwogu *et al.* (1987) for single-point systems, and extended to wave probe arrays by Nwogu (1989). In scientific literature, this method is called MEM2 or MEP (for « Maximum Entropy Principle »). The entropy H_2 we try to maximize is defined by :

$$H_2(\hat{D}) = - \int_0^{2\pi} \hat{D}(\theta) \ln(\hat{D}(\theta)) \, d\theta \qquad (56)$$

The constraints to be satisfied are (as for the MEM1 method) the values of the first four Fourier coefficients of the DSF determined from the-cross spectral data, together with the condition of unit integral of the DSF over $[0 ; 2\pi]$.

It may be shown (e.g. Kobune and Hashimoto, 1986) that the estimate satisfying these conditions has the following form :

$$\hat{D}_{MEM2}(\theta) = \exp\left\{ \sum_{I=1}^{L+1} \mu_I . q_I(\theta) \right\} \qquad (57)$$

where the $q_I(\theta)$ functions are defined on (16). The important point is that these functions are some combinations of sin and cosine functions of θ. The μ_I (I=1,..., L+1) coefficients are unknown Lagrange multipliers. L is the number of independent and non-zero real cross-spectral quantities from the system of N sensors. The μ_I coefficients are determined by solving the non-linear system of equations given by the L cross-spectral equations and the condition of unit integral of the estimate.

In the case on single-point measuring systems, we have N=3 and L=4 (see section III.4), thus 5 equations and the estimate (57) takes the following form, suitable to model both unimodal and bimodal DSF :

$$\hat{D}_{MEM2}(\theta) = \exp (-\mu_1 - \mu_2 \cos \theta - \mu_3 \sin \theta - \mu_4 \cos 2\theta - \mu_5 \sin 2\theta) \qquad (58)$$

For wave arrays consisting of more than 3 probes, it is possible to increase the number of harmonics in the Fourier series appearing in (57). Nwogu (1989) introduced a method to find the most suitable order of this decomposition. The numerical procedure then used for solving the system of non-linear equations may be based on classical techniques, e.g. Newton-Raphson or Levenberg-Marquardt.

The MEP or MEM2 estimate appears to be very suitable in most of cases. From its definition (57), it is clear that it is always positive. Tests on numerical or laboratory

tests have proven that this method is equally efficient for dealing with unimodal or bimodal sea-states (Nwogu *et al.*, 1987 ; Nwogu, 1989 ; Benoit, 1992 ; Kim *et al.*, 1993 ; Benoit and Teisson, 1994).

The method is however not simple to implement (especially for wave probe arrays) and requires more computing time than most of methods presented above (e.g. WFS, 1MFG, MLM, IMLM) due to the iterative algorithm needed for solving the set of non-linear equations. Recently, Kim *et al.* (1994) proposed some approximations to the numerical scheme of the MEM2 method, in order to remove occasional convergence problems that may occur with real sea data.

Although the numerical effort associated with MEM2 is quite important, this method is very recommended if one wishes to get a precise and reliable directional analysis. In particular, several authors suggest that it is probably the most powerful and reliable technique for estimating the DSF when using single-point measuring systems (Nwogu *et al.*, 1987 ; Benoit, 1992 ; Kim *et al.* , 1993, 1994 ; Benoit and Teisson, 1994).

III.9.3 Extended Maximum Entropy Principle (EMEP) :

Recently, Hashimoto *et al.* (1994) developed the Extended Maximum Entropy Principle (EMEP), which represents an improvement to the MEP for the general case of a multi-component array, but is equivalent to the MEP for single-point measuring systems (delivering 3 wave signals).

Based on the MEP estimate (57), the following expression is adopted for the EMEP estimate, which again yields only positive values :

$$\hat{D}_{EMEP}(f,\theta) = \frac{1}{\Delta} \exp\left\{\sum_{k=1}^{K} \left[A_k . \cos\left(k.\theta\right) + B_k . \sin\left(k.\theta\right)\right]\right\} \qquad (59)$$

$$\text{with : } \Delta = \int_0^{2\pi} \exp\left\{\sum_{k=1}^{K} \left[A_k . \cos\left(k.\theta\right) + B_k . \sin\left(k.\theta\right)\right]\right\} d\theta$$

In (59) the coefficients A_k and B_k (k = 1,...,K) are unknown parameters. If we note L the number of co- and quad spectra both different from zero and independent from each other, we get L independent equations of the problem. Each equation corresponds to the difference between the measured co- or quad-spectrum and the model co- or quad-spectrum, obtained by substitution of (59) in the general expression of the cross-spectra (12) or (16).

In fact, Hashimoto *et al.* (1994) do not impose this difference to be zero, but consider the possible existence of errors in the cross-spectral data and note ε_l (l=1,...,L) this difference. The ε_l residuals are assumed to be independent of each other and the probability of their occurrence is expressed by a normal distribution having zero mean and a variance σ^2 given by the cross-spectral estimates (see Hashimoto *et al.* (1994) for details). The optimal estimate is then the one which minimizes the sum of the squares of the residuals :

$$\sum_{l=1}^{L} \varepsilon_l^2 \rightarrow 0 \qquad (60)$$

This leads to a non-linear problem which may be solved by applying Newton's technique of local linearization and iterations (Hashimoto *et al.*, 1994). The determination of the optimal finite order K of the model (59) requires the use of Akaike's Information Criterion (AIC) (Akaike, 1973) :

$$AIC = L\,(\ln(2\pi) + 1 + \ln(\hat{\sigma}^2) + 4K + 2 \tag{61}$$

where $\hat{\sigma}^2$ is an estimate of the variance of ε_l (l=1,...,L). In practice, the computation is performed from lower (K=1) to higher orders, in order to stop at the order minimizing the AIC.

This method is superior to the MEP in the sense that the number of harmonics in the estimate (59) is directly adapted to the available cross-spectral information. Another advantage of this method lies in the fact that statistical variability in the measured cross-spectra is accounted for in the analysis. Hashimoto *et al.* (1994) showed on numerical test-cases that this estimate gives the same results as the MEP method for single-point systems. For multi-component measuring devices, the EMEP gives results comparable to the Bayesian Directional Method (see III.10).

III.10 Bayesian Directional Method (BDM)

This method is based on the Bayesian technique used in probability theory. It has been adapted to the problem of directional wave analysis by Hashimoto *et al.* (1987). Among the methods presented in this chapter, it is probably the most awkward to implement for numerical analysis, but also the most powerful in terms on resolving capabilities as no *a priori* assumption is made about the shape of the DSF.

The range [0 ; 2π] is divided in a number of K sub-ranges or segments, each having a width of $\Delta\theta = 2\pi/K$. The BDM estimate of the DSF is simply considered as a piecewise-constant function over each segment. It is defined by a series of K values x_k (each value corresponding to the logarithm of the constant value of the BDM estimate on each segment) :

$$x_k = \ln\left[\hat{D}_{BDM}(\theta_k)\right] \quad \text{where} \quad \theta_k = (k\text{-}1/2)\Delta\theta \tag{62}$$

$$\hat{D}_{BDM}(\theta) = \sum_{k=1}^{K} \exp(x_k).I_k(\theta) \quad \text{with} \quad I_k(\theta) = \begin{cases} 1 & \text{if } (k\text{-}1)\Delta\theta \le \theta \le k\Delta\theta \\ 0 & \text{otherwise} \end{cases} \tag{63}$$

K is usually in the order of 40 to 90, which means a rather high number of unknowns, in particular with respect to the number of available equations, given by the expressions of the co- and quad-spectra. Furthermore possible errors in the estimated cross-spectra are taken into account, so that these equations are assumed to be verified in a statistical sense. This method clearly presents a stochastic feature in dealing with the cross-spectral data.

The system of non-linear equations given by the co- and quad-spectra is completed by a smoothness condition of the estimated DSF. This is mathematically expressed by the following relationship between three consecutive values of the estimate :

$$\sum_{k=1}^{K} (x_{k+1} - 2.x_k + x_{k-1})^2 \rightarrow 0 \tag{64}$$

An hyperparameter u is introduced in order to combine the above smoothness condition and the equations giving the cross-spectra. The value of this hyperparameter is determined as the one which minimizes the ABIC criterion (Akaike Bayesian Information Criterion) (Akaike, 1973).

Due to its large number of degrees of freedom, the BDM is able to represent almost all kinds of DSF shapes : unimodal, bimodal and even trimodal, symmetrical peaks or not,... This method is more specifically devoted to multi-component arrays, for which the number of available cross-spectra is quite large. Benoit (1992) and Benoit and Teisson (1994) have however shown that the method still produce correct results when applied to single-point systems (application to numerical and laboratory simulated data). For single-point systems however, the BDM requires quite a lot of computing time for a result whose accuracy is usually comparable to those of more rapid analysis methods (e.g. MEM2).The BDM is therefore not advised for such single-point systems. For multi-component arrays, it is usually considered as one on the most powerful techniques to analyse any kind of spectra.

IV DETERMINISTIC ANALYSIS METHODS

IV.1 Characterization of deterministic analysis methods :

While stochastic methods presented in chapter III rely on the random phase assumption, the deterministic methods retain the phase information inherent in the data set to be analysed. As opposed to the stochastic approaches, methods for deterministic analysis of directional waves are sparse.

While the stochastic methods rely on the cross-spectra between the different measured quantities, most deterministic methods utilize the complex Fourier coefficients of each signal. The process of splitting a wave field into a number of wave components each given by a direction, an amplitude and a phase is often referred to as deterministic decomposition. The surface elevation may then be seen as a superposition of numerous sine components, in a generalized form of (2) :

$$\eta(x,y,t) = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} \cos\left[k_m(x.\cos\theta_n + y.\sin\theta_n) - \omega_m t + \varphi_{mn}\right] \qquad (65)$$

where subscript m refers to frequency bins and subscript n to directional bins. The deterministic analysis approach assumes a large number M of frequencies, but a low number N (i.e. N = 1 or 2) of directions per frequency, so that both the amplitude a_{mn} and the phase φ_{mn} of each component may be determined from the Fourier coefficients of the recorded time-series. Smoothing the decomposition data can then provide an estimate of the underlying directional spectrum.

Another application of the decomposition data (which will not be pursued here) is the deterministic reproduction of a wave field, also by starting from an expression like (65). It deserves to be mentioned that a wave field constructed using a discrete set of frequencies is not spatially homogeneous if more than one wave component appears at each frequency. Spatial variation is typically present near reflecting structures, while the wave field in open waters is homogeneous.

IV.2 General approaches for deterministic analysis :

Presumably the first deterministic method for directional spectral analysis is the work by Panicker and Borgman (1970, 1974), who derived the so-called "locked phase method". Their work resembles the early stochastic models in using a Fourier expansion of the directional distribution at each frequency. In this respect it differs from the following approaches, which assume a few discrete directional components at each frequency. According to Sand (1979), it seems probable that the "locked phase" method was never used in practice, and we are not aware of any applications of the method since then. The reason for this is unknown, but it appears that the poor resolution inherent in the directional Fourier expansion makes is less attractive than more recent methods.

The locked phase method was derived for wave elevation gauge arrays (Panicker and Borgman, 1970) as well as for measurements of more general wave properties (Panicker and Borgman, 1974). The remainder of deterministic methods all basically assume that the sea-state can be approximated by a sum of a few unidirectional wave components at each "raw" frequency. When smoothing is applied to reduce the variance of the spectral density then a large number of components representing a directional distribution are typically present within each frequency bin. Except for the work by Prislin *et al.* (1996), the remainder of the deterministic approaches were confined to single-point measurements (e.g. the surface elevation and components of particle velocity measured at the same vertical.)

A common feature of the work by Sand (1979, 1984) and Lundgren and Klinting (1987a, 1987b) is that each set of complex Fourier coefficients were used in determining two components.

Let : $C_u = A_u - i\,B_u$ and $C_v = A_v - i\,B_v$ (66)

denote the complex Fourier coefficients resulting from the respective Fourier transforms of two orthogonal velocity time series measurements, then the vector (A_u, A_v) determines the direction of one component and (B_u, B_v) determines the direction of the other. It turns out that this approach has the undesirable property that the two directions depend on the phase of the underlying signals, except in case the two directions coincide. In other words, changing the origin of the time axis yields another set of directions. This makes the approach physically unsound. However, reasonable results have been obtained using this concept and one may conjecture that the problem is reduced in the subsequent smoothing process, although this is not clear.

IV.3 Single and Double Direction Analysis (SDA ; DDA) :

Schäffer and Hyllested (1994) devised two different methods for deterministic decomposition, assuming one and two components per frequency, respectively [N = 1 or 2 in (65)]. The corresponding directional wave analysis were termed Single Direction Analysis (SDA) and Double Direction Analysis (DDA), respectively. The first method was basically to find the principal direction from complex Fourier coefficients of the velocity components and assign this direction to a wave component with amplitude and phase taken from the Fourier coefficient for the surface elevation. The directional spectrum was then obtained by smoothing the raw decomposition data in direction as well as in frequency. The second method resembled the approach by e.g. Sand (1979) in the assumption of two components at each frequency. However, these components were found by determining the directions and complex amplitudes of two crossing wave

trains requiring their superposition to match the complex Fourier coefficients from the measurements of surface elevation and horizontal velocity components. The solution was found analytically and situations where no solution was possible were identified. As in the SDA the raw decomposition data were smoothed to obtain the directional spectrum. The energy at frequencies where no solution could be obtained was discarded (approximately 5-10 % in practical applications). Although the SDA and DDA methods are fundamentally different from stochastic methods, they usually produce results which are very similar to those of e.g. MEM2 method (Hawkes *et al.*, 1997).

IV.4 Other deterministic work

Recently Prislin *et al.* (1996) introduced a hybrid between a deterministic and a stochastic method in a framework allowing for mixed instrumentation. Their idea is to use a stochastic method for determination of the directional distribution at each frequency followed by a deterministic fit to the data to regain the lost phase information. Specifically they used the Maximum Likelihood Method (Isobe *et al.*, 1984) to obtain the directional distribution and then chose a few wave components to distend this distribution at each frequency. In the time domain a subsequent least squares fitting to the measured data then provided the phases for the components chosen. Despite the problem of inhomogeneous wave fields encountered when more than one component is present at each frequency, the use of several components may be attractive for an accurate description of the local wave field. However, the approach of starting with a stochastic method requires spectral smoothing which implies comparatively large frequency bins. Although several wave components are allowed at each frequency in this approach, the total number of wave components for all frequencies may be smaller than e.g. using just one component per frequency in raw decomposition data with a very fine frequency resolution.

The work of Prislin *et al.* (1996) does not really represent a new technique for estimating directional spectra, but rather a way of using the directional information to deterministically describe the local wave field.

V. TIME DOMAIN ANALYSIS METHODS

Time-domain analysis methods are quite sparse in operational applications. We only report below some general ideas about this approach of analysis, as these methods were not used within the IAHR Working Group on multidirectional waves. A first way of presenting this approach is to consider the distribution of kinetic and potential wave energy at each time-step for a single-point measuring system composed of a surface elevation probe (signal η) and a bi-dimensional current-meter (signals U and V) :

Distribution of kinetic energy : The kinetic energy $E = U^2 + V^2$ and the angle of wave energy propagation are computed, allowing for finding the distribution of kinetic energy against direction.

Distribution of potential energy : The potential energy is computed from η^2. By using the wave direction determined above, it is also possible to get a plot of the distribution of potential energy against direction.

These basic remarks represent the starting point of more sophisticated approaches, such as the Instantaneous Direction Spreading Function by Egozcue and Arribas (1991).

VI. CONCLUSIONS - MAIN DEVELOPMENT AXIS

In this paper, we made an attempt to review, present and, to a certain extent, classify a large number of directional analysis methods for linear waves in open water conditions. The exercise is quite difficult because we did not consider all the methods presented in the scientific literature, nor perform any comprehensive comparison of the methods on common test-cases. We rather focused on the main modelling assumptions associated to each method in order to highlight their advantages/shortcomings.

According to the authors, the main points that deserve to be mentioned are :

* one should not think that all the methods have similar capabilities. When looking at a plot of an analysed directional spectrum, one should keep in mind which method was used for the analysis. This may to a certain extent have some influence on the conclusions from the analysis. For instance, one should not conclude that there is no case of crossed waves at the same frequency if the method used is unable to detect such case (e.g. fitting to unimodal model method).
* Some of the methods have a high level of parameterization (e.g. the Fourier Series decomposition methods or the fitting to unimodal parametrical models) while other ones are able to model a large variety of directional shapes (e.g. EMEP, BDM, DDA).
* The methods are not equivalent in terms of implementation and computing time. Benoit (1992) shows that there is a ratio of almost 10 000 between the quickest and the lowest methods for a heave-pitch-roll buoy.
* It seems important to preferably use methods that try to fit to the data in a statistical sense (MLF, EMEP, BDM) than in a deterministic way (TFS, WFS, 1MF) even if the distinction between both approaches is not always easy.
* The method to preferably use also depends on the type of measuring device and in particular of the number of wave signals. For a single-point system, reliable choices for analysis methods are MLM, IMLM2, MEM2 (or MEP), SDA, DDA among others. For wave probe arrays, more sophisticated methods may be used : EMEP, BDM, among others.

Some research axis, not discussed in detail in this paper, concern :

- Analysis of reflection :

In the vicinity of a reflective structure, the wave field results from the superposition of incident and reflected wave components. These components are phase-locked and this phase relationship is in contradiction with the "random phase" assumption the stochastic methods of Chapter III are based on. Special refinements are then introduced : e.g. the Modified Maximum Likelihood Method (MMLM), proposed by Isobe and Kondo (1984) or the Modified Likelihood Fitting Method (MMLFM) from Yokoki *et al.* (1994). It should be noted that the Double Decomposition Analysis (DDA) is applicable to this case as it permits two directions per frequency. However, for alinear structure, a constraint based on Green law of reflection (DDAC) can further improve the results of DDA.

- Analysis of non-linear waves, with in particular :
  - the analysis of second-order (superharmonic) spectrum (e.g. Sekimoto, 1995)
  - the analysis of the second-order long wave spectrum (e.g. Sand, 1981).

All these items are enough to show that the analysis of multidirectional waves still need to receive a lot of work and attention.

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