

Propagation of water waves through shearing currents in general bathymetry

K.A. Belibassakis

School of Technological Applications, Technological Educational Institute of Athens, Greece

Email: kbel@teiath.gr

ABSTRACT: A coupled-mode model is presented for wave-current-seabed interaction, with application to the problem of wave scattering by steady shearing currents in variable bathymetry regions. We consider obliquely incident waves on a horizontally non-homogeneous current in a variable-depth strip, which is characterized by straight and parallel bottom contours. The flow associated with the current is assumed to be parallel to the bottom contours and it is considered to be known. In a finite subregion containing the bottom irregularity, we assume an arbitrary horizontal current structure. Outside this region, the current is assumed to be uniform (or zero). Based on a variational principle, in conjunction with a rapidly-convergent local-mode series expansion of the wave pressure field in a finite subregion containing the current variation and the bottom irregularity, a coupled-mode system is obtained. The present model can be considered as an extension of the works by Smith (1983, 1987) and McKee (1987,1996), and it can be further elaborated to treat more general current profiles with vertical structure and cross-jet component, and to include the effects of weak nonlinearity.

1 INTRODUCTION

It is well known that, except of depth variations, the presence of currents significantly affects the propagation of waves, especially in the nearshore and coastal environment, Mei (1983), Peregrine (1976). In particular, non-homogeneous, shearing currents following or opposing water wavetrains produce significant changes on their characteristics, which can be further modified by the effects of depth inhomogeneities in variable bathymetry regions. For example, large amplitude waves can be produced when obliquely propagating waves interact with adverse currents, see, e.g., Mei (1983), Smith (1983, 1987). This situation could be further enhanced by inshore effects due to sloping seabeds, and has been reported to be connected with the appearance of “giant waves” in coastal waters; see, e.g., Faulkner (2000), Dysthe (2000). The detailed knowledge of the wave-current system characteristics is important for various applications, as, for example, the design and evaluation of performance of special-type ship and structures operating in nearshore and coastal waters and the study of oil slicks dispersion and pollutant transport.

In this work, a coupled-mode technique is presented for the propagation/diffraction of water waves through shearing currents in general bathymetry, with application to the problem of wave scattering by shearing currents in variable bathymetry regions and current variations on various scales. We consider obliquely incident harmonic waves on a horizontally non-homogeneous current in a variable-depth strip, which is characterized by straight and

parallel bottom contours; see Fig.1. The flow associated with the current is assumed to be parallel to the bottom contours and it is considered to be known. In a finite subregion containing the bottom irregularity we assume an arbitrary horizontal current structure. Outside this region, the current is assumed to be uniform (or zero).

The present coupled-mode system is based on an appropriate variational principle, in conjunction with a rapidly-convergent local-mode series expansion of the wave pressure field in a finite subregion containing the current variation and the bottom irregularity. In contrast with an earlier potential flow approach (Belibassakis & Athanassoulis, 2004), here the wave flow is assumed to be rotational, governed by the Laplace equation on the wave pressure, containing additional terms carrying out the effects of the current gradients (Mei 1983, Eq. 6.17), and subjected to the free-surface boundary condition formulated with respect to the intrinsic frequency, the bottom boundary condition and the conditions at infinity.

Numerical results are presented, including comparisons with other models based on the representation of the shearing current by a series of vertical vortex sheets separating regions of constant velocity, McKee (2003), and simplified mild-slope type models, like the Mild-Shear Equation (McKee, 1987) and the Extended Mild-Shear Equation (McKee, 1996). With the aid of systematic comparisons, the effects of current variations, in conjunction with the bottom slope and curvature variation, on the hydrodynamic characteristics of the wave-current system are presented and discussed.

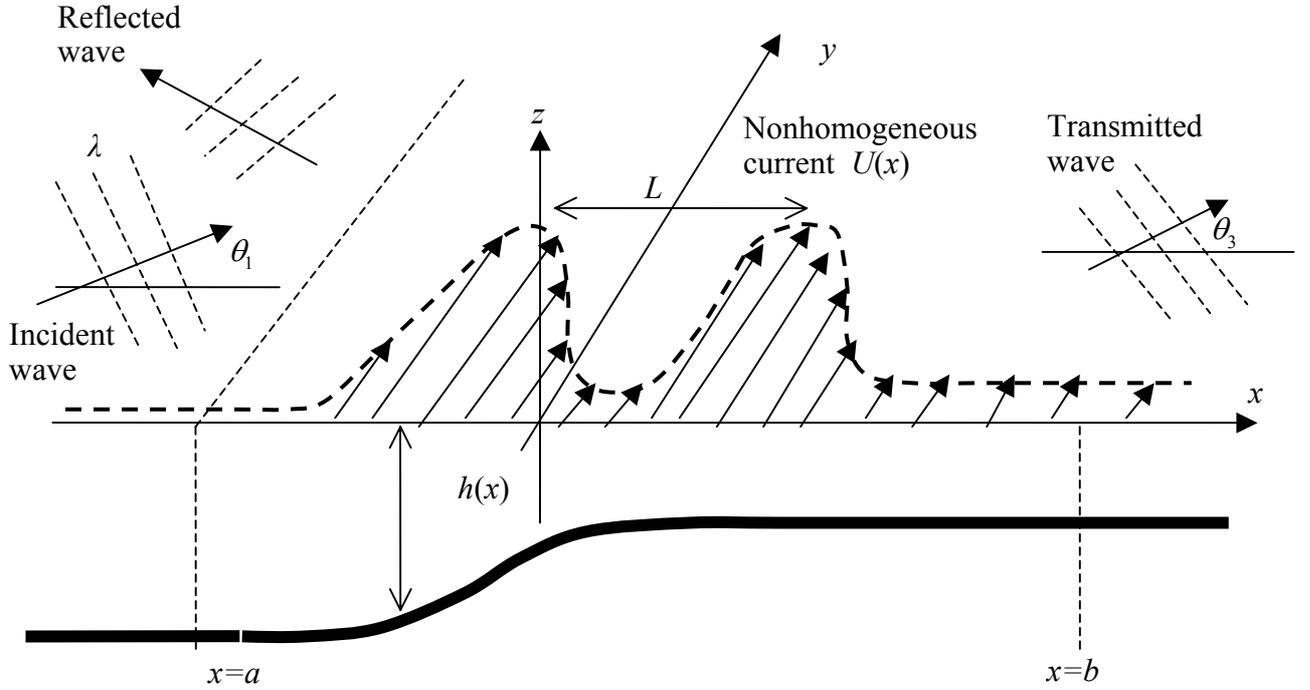


Figure 1. Geometrical configuration and basic notation

2 DIFFERENTIAL FORMULATION

The studied marine environment consists of a water layer bounded above by the free surface and below by a rigid bottom. It is assumed that the bottom surface exhibits an arbitrary 1D variation in a subdomain of finite length, i.e. the bathymetry is characterised by straight and parallel bottom contours lying between two regions of constant but possibly different depth, $h = h_1$ (region of incidence) and $h = h_3$ (region of transmission); see Fig. 1. A Cartesian coordinate system is introduced, with its origin at some point on the mean water level (in the variable bathymetry region), the z -axis pointing upwards and the y -axis being parallel to the bottom contours. The function $h(x)$ represents the local depth, measured from the mean water level. It is considered to be a smooth function, such that $h(x) = h(a) = h_1$, for all $x \leq a$, and $h(x) = h(b) = h_3$, for all $x \geq b$. The fluid domain is $\mathcal{D} = D \times \mathcal{R}$, and the vertical strip D is decomposed in three subdomains $D^{(i)}$, $i=1,2,3$, where $D^{(1)}$ and $D^{(3)}$ are constant-depth subdomains corresponding to $x < a$ and $x > b$, respectively, and $D^{(2)}$ is the variable bathymetry subdomain lying between $D^{(1)}$ and $D^{(3)}$. Without loss of generality, we assume $h_1 > h_3$. The same decomposition is also applied to the free-surface and the bottom boundaries. Finally, we define the vertical interfaces $\partial D_1^{(12)}$ and $\partial D_1^{(23)}$ separating the three subdomains. The latter are vertical segments

(between the bottom and the mean water level) at $x = a$ and $x = b$, respectively.

In this work we consider the scattering problem of monochromatic, obliquely incident plane waves, propagating with direction θ_1 with respect to the bottom contours in the region of incidence, under the combined effects of variable bathymetry and a horizontally non-homogeneous shear current $U(x)$, existing in $x > a$; see Fig.1. The flow associated with the shear current is considered to be steady and directed parallel to the bottom contours (i.e. along the y -axis). Moreover, the steady free-surface displacement associated with this current flow is assumed to be negligible. The current velocity is described by the (given) continuous function $U(x)$, which can be general in the intermediate region, $a \leq x \leq b$, as, e.g., a monotonic one or a periodic one with characteristic width L . Outside this region, the current is assumed to be uniform (or simply zero),

$$U(x) = U_1 = 0, \quad x \leq a, \quad U(x) = U_3, \quad x \geq b. \quad (2.1)$$

Restricting ourselves to linear, monochromatic (harmonic) waves of absolute frequency ω , periodic in the y -direction, the wave pressure can be expressed in the form, Smith (1983, 1987),

$$p(x, y, z, t) = \text{Re} \{ p(x, z) \exp(i(qy - \omega t)) \}, \quad (2.2)$$

where q is the periodicity constant along the y -direction, and $i = \sqrt{-1}$. Under the previous assumptions, the wave flow is governed by the following equations (see Mei 1983, Ch. 3.6, Eqs.

6.17, 6.19 and 6.23) with respect to the (complex) pressure $p(x, z)$,

$$\frac{\partial^2 p}{\partial^2 x} + \frac{\partial^2 p}{\partial^2 z} - q^2 p + \frac{2q}{\sigma} \frac{\partial U}{\partial x} \frac{\partial p}{\partial x} = 0, \quad (2.3a)$$

$$\frac{\partial p}{\partial z} + \mu(x)p = 0, \quad z = 0, \quad (2.3b)$$

$$\frac{\partial p}{\partial z} + \frac{dh}{dx} \frac{\partial p}{\partial x} = 0, \quad z = -h(x), \quad (2.3c)$$

where $\sigma = \sigma(x) = \omega - qU(x)$ is the local intrinsic frequency, $\mu(x) = \sigma^2 / g$ the corresponding frequency parameter, and g is the acceleration due to gravity. From the solution of the above problem, the free-surface elevation is obtained as

$$\eta(x, y; t) = \text{Re} \left\{ \frac{p(x, z)}{\rho g} \exp(-i(qy - \omega t)) \right\},$$

where ρ is the liquid density. Other quantities of interest, as, e.g., the wave velocities, can be obtained in terms of p and its spatial derivatives (Mei 1983).

The problem of water-wave scattering by shearing current, with the effects of variable bathymetry, can be formulated as a transmission problem in the bounded subdomain $D^{(2)}$, with the aid of the following general representations of the complex pressure $p(x, z)$ in the semi-infinite strips $D^{(1)}$ and $D^{(3)}$ (see, e.g., Smith 1983, 1987):

$$p^{(1)}(x, z) = \left(A_0 \exp(ik_0^{(1)}x) + A_R \exp(-ik_0^{(1)}x) \right) Z_0^{(1)}(z) + \sum_{n=1}^{\infty} C_n^{(1)} Z_n^{(1)}(z) \exp(k_n^{(1)}(x-a)) \quad \text{in } D^{(1)}, \quad (2.4a)$$

$$p^{(3)}(x, z) = A_T \exp(ik_0^{(3)}x) Z_0^{(3)}(z) + \sum_{n=1}^{\infty} C_n^{(3)} Z_n^{(3)}(z) \exp(k_n^{(3)}(b-x)), \quad \text{in } D^{(3)}. \quad (2.4b)$$

The terms $\left(A_0 \exp(ik_0^{(1)}x) + A_R \exp(-ik_0^{(1)}x) \right) Z_0^{(1)}(z)$ and $A_T \exp(ik_0^{(3)}x) Z_0^{(3)}(z)$ in the series (2.4) are the *propagating modes*, associated with incident wave (which is considered to be known), and the reflected and the transmitted wave, respectively, while the remaining ones ($n=1,2,\dots$) are the *evanescent modes*. In the above expansions,

$$k_0^{(i)} = \sqrt{\left(\kappa_0^{(i)}\right)^2 - q^2}, \quad k_n^{(i)} = \sqrt{\left(\kappa_n^{(i)}\right)^2 + q^2}, \quad n \geq 1, \quad (2.5a)$$

are the wavenumbers, where $\{i\kappa_0^{(i)}, \kappa_n^{(i)}, n=1,2,\dots\}$, $i=1,3$, are obtained as the roots of the following dispersion relations

$$\mu h_i = -\kappa^{(i)} h_i \tan(\kappa^{(i)} h_i), \quad (2.5b)$$

where $\mu = \sigma^2 / g$, $\sigma = \omega - pV_i$, $i=1,3$. The functions $\{Z_n^{(i)}(z), n=0,1,2,\dots\}$ appearing in Eqs. (2.4) are given by

$$Z_0^{(i)}(z) = \frac{\cosh(\kappa_0^{(i)}(z+h_i))}{\cosh(\kappa_0^{(i)} h_i)}, \quad (2.6a)$$

$$Z_n^{(i)}(z) = \frac{\cos(\kappa_n^{(i)}(z+h_i))}{\cos(\kappa_n^{(i)} h_i)}, \quad n=1,2,\dots, \quad (2.6b)$$

$i=1,3$. Since the current is zero in $D^{(1)}$, $\sigma = \omega$ there, and thus, the periodicity constant is obtained from the wavenumber of the incident wave, $q = \kappa_0^{(1)} \sin \theta_1$. Then, the direction of the transmitted wave in $D^{(3)}$ is

$$\theta_3 = \sin^{-1}(\kappa_0^{(1)} \sin \theta_1 / \kappa_0^{(3)}). \quad (2.7)$$

Given the representations (2.4), the problem can be reformulated as a transmission boundary value problem for the pressure $p^{(2)}(x, z)$ in the bounded subdomain $D^{(2)}$, consisting of the following equations, boundary and matching conditions:

$$\frac{\partial^2 p^{(2)}}{\partial^2 x} + \frac{\partial^2 p^{(2)}}{\partial^2 z} - q^2 p^{(2)} + \frac{2q}{\sigma} \frac{\partial U}{\partial x} \frac{\partial p^{(2)}}{\partial x} = 0, \quad (x, z) \in D^{(2)}, \quad (2.8a)$$

$$\frac{\partial p^{(2)}}{\partial z} - \mu(x)p^{(2)} = 0, \quad z = 0, \quad (2.8b)$$

$$\frac{\partial p^{(2)}}{\partial z} + \frac{dh}{dx} \frac{\partial p^{(2)}}{\partial x} = 0, \quad z = -h(x), \quad (2.8c)$$

$$p^{(2)} = p^{(1)}, \quad \frac{\partial p^{(2)}}{\partial x} = \frac{\partial p^{(1)}}{\partial x}, \quad x = a, \quad -h_1 < z < 0, \quad (2.8d,e)$$

$$p^{(2)} = p^{(3)}, \quad \frac{\partial p^{(2)}}{\partial x} = \frac{\partial p^{(3)}}{\partial x}, \quad x = b, \quad -h_3 < z < 0. \quad (2.8f,g)$$

3 VARIATIONAL FORMULATION

The problem (2.8) admits an equivalent variational formulation, which will serve as the basis for the derivation of an equivalent coupled-mode system of horizontal equations. Consider the functional:

$$\mathcal{F} \left(\varphi^{(2)}, A_R, \{C_n^{(1)}\}_{n \in N}, A_T, \{C_n^{(3)}\}_{n \in N} \right) =$$

$$\begin{aligned}
& \frac{1}{2} \int_{D^{(2)}} \left(\left(\nabla \left(\frac{p^{(2)}}{\sigma} \right) \right)^2 + \left(q^2 + \sigma \frac{\partial^2}{\partial x^2} \left(\frac{1}{\sigma} \right) \right) \left(\frac{p^{(2)}}{\sigma} \right)^2 \right) dx dz \\
& - \frac{1}{2} \int_{\partial D_F^{(2)}} \frac{\mu}{\sigma^2} (p^{(2)})^2 dS - \frac{1}{2} \int_{\partial D_H^{(2)}} \left(\sigma \frac{\partial}{\partial n} \left(\frac{1}{\sigma} \right) \right) (p^{(2)})^2 dS \\
& + \frac{1}{\sigma^2} \int_{\partial D_I^{(12)}} \left(p^{(2)} - \frac{1}{2} p^{(1)} \right) \cdot \frac{\partial p^{(1)}}{\partial x^{(1)}} dS + \\
& - \frac{1}{\sigma^2} \int_{\partial D_I^{(23)}} \left(p^{(2)} - \frac{1}{2} p^{(3)} \right) \cdot \frac{\partial p^{(3)}}{\partial x} dS - A_0 A_R J^{(1)}, \quad (3.1)
\end{aligned}$$

where $J^{(1)} = 2k_0^{(1)} \int_{z=-h_1}^{z=0} (Z_0^{(1)}(z))^2 dz$ and $\partial/\partial n$ denotes the outward normal derivative on the boundary. The functions $p^{(i)}$ and their derivatives $\partial p^{(i)}/\partial x, i=1,3$, appearing in the Eq. (3.1), are considered to be represented by means of their series expansions, Eqs. (2.4), and their horizontal derivatives, respectively.

The function $p^{(2)}(x, z), (x, z) \in D^{(2)}$ and the coefficients $A_R, \{C_n^{(1)}\}_{n \in N}$ and $A_T, \{C_n^{(3)}\}_{n \in N}$ constitute a solution of the problem, if they render the functional F stationary, i.e.

$$\delta \mathcal{F} \left(\varphi^{(2)}, A_R, \{C_n^{(1)}\}, A_T, \{C_n^{(3)}\} \right) = 0. \quad (3.2)$$

To see this we calculate the first variation δF of the above functional (see also, Athanassoulis & Belibassakis, 1999). Making use of the Green's theorem and the properties of the modal representations (2.4) in the two constant-depth strips, the variational equation (3.2) takes the form:

$$\begin{aligned}
& - \int_{D^{(2)}} \frac{1}{\sigma^2} \left(\Delta p^{(2)} - q^2 p^{(2)} + \frac{2q}{\sigma} \frac{\partial U}{\partial x} \frac{\partial p^{(2)}}{\partial x} \right) \delta p^{(2)} + \\
& + \int_{\partial D_F^{(2)}} \frac{1}{\sigma^2} \left(\frac{\partial p^{(2)}}{\partial z} - \mu p^{(2)} \right) \delta p^{(2)} dS + \\
& \int_{\partial D_H^{(2)}} \frac{1}{\sigma^2} \left(\frac{\partial p^{(2)}}{\partial z} - \frac{dh}{dx} p^{(2)} \right) \delta p^{(2)} dS + \\
& - \frac{1}{\sigma^2(a)} \int_{\partial D_I^{(12)}} \left(\frac{\partial p^{(2)}}{\partial x} - \frac{\partial p^{(1)}}{\partial x} \right) \delta p^{(2)} dS + \\
& + \frac{1}{\sigma^2(b)} \int_{\partial D_I^{(23)}} \left(\frac{\partial p^{(2)}}{\partial x} - \frac{\partial p^{(3)}}{\partial x} \right) \delta p^{(2)} dS + \\
& + \frac{1}{\sigma^2(a)} \int_{\partial D_I^{(12)}} \left(p^{(2)} - p^{(1)} \right) \delta \left(\frac{\partial p^{(1)}}{\partial x} \right) dS \\
& - \frac{1}{\sigma^2(b)} \int_{\partial D_I^{(23)}} \left(p^{(2)} - p^{(3)} \right) \delta \left(\frac{\partial p^{(3)}}{\partial x} \right) dS = 0, \quad (3.3)
\end{aligned}$$

where $\Delta p^{(2)} = \frac{\partial^2 p^{(2)}}{\partial^2 x} + \frac{\partial^2 p^{(2)}}{\partial^2 z}$. The proof of the equivalence of the variational equation (3.3) and the transmission problem (2.8) is finally obtained by using standard arguments of the *Calculus of Variations* (see, e.g., Rectorys 1977, ch.22).

4 THE COUPLED-MODE SYSTEM

In this section we shall present a new coupled-mode model for treating the problem (2.3), which is derived from the variational principle (3.3) on the basis of the following enhanced *local-mode* representation of the wave pressure field in the variable bathymetry region $D^{(2)}$ (where also the current velocity $U(x)$ presents variation):

$$\begin{aligned}
p^{(2)}(x, z) = & P_{-1}(x) Z_{-1}(z; x) + P_0(x) Z_0(z; x) + \\
& + \sum_{n=1}^{\infty} P_n(x) Z_n(z; x) \quad (4.1)
\end{aligned}$$

In Eq. (4.1) the term $P_0(x) Z_0(z; x)$ is the *propagating mode* of the wave field and the remaining terms $P_n(x) Z_n(z; x), n=1,2,\dots$ are the *evanescent modes*. The additional term $P_{-1}(x) Z_{-1}(z; x)$ is a correction term called the *sloping-bottom mode*, which properly accounts for the bottom boundary condition on the sloping parts of the bottom, and vanishes identically on the horizontal parts of the bottom. The function $Z_n(z; x)$ represents the vertical structure of the n -th mode. The function $P_n(x)$ describes the horizontal pattern of the n -th mode and is called the *complex amplitude* of the n -th mode. The functions $Z_n(z; x), n=0,1,2,\dots$, appearing in Eq. (4.1) are obtained as the eigenfunctions of local vertical *Sturm-Liouville* problems, and are given by

$$Z_0(z; x) = \frac{\cosh[\kappa_0(x)(z+h(x))]}{\cosh(\kappa_0(x)h(x))}, \quad (4.2a)$$

$$Z_n(z; x) = \frac{\cos[\kappa_n(x)(z+h(x))]}{\cos(\kappa_n(x)h(x))}, \quad n=1,2,\dots, \quad (4.2b)$$

where the eigenvalues $\{i\kappa_0(x), \kappa_n(x)\}$ are obtained as the roots of the local dispersion relation

$$\mu(x)h(x) = -\kappa(x)h(x) \tan[\kappa(x)h(x)], \quad (4.2c)$$

in $a \leq x \leq b$. A specific convenient form of the function $Z_{-1}(z; x)$ is given by

$$Z_{-1}(z; x) = h(x) \left[(z/h(x))^3 + (z/h(x))^2 \right], \quad (4.2d)$$

and all numerical results presented in this work are based on this choice for $Z_{-1}(z;x)$. However, other choices are also possible. Using Eqs. (4.1) and (4.2), we obtain that the additional sloping-bottom mode satisfies:

$$P_{-1}(x) = \frac{\partial p^{(2)}(z;x)}{\partial z} \Big|_{z=-h(x)}, \quad (4.3)$$

and thus, it is needed only in subareas where the bottom surface is not flat, making the series (4.1) compatible with the Neumann bottom boundary condition (2.8c) there, while, at the same time, it significantly accelerates the convergence of the local-mode series. For more details about the role and significance of this term we refer to Athanassoulis & Belibassakis (1999, Sec. 4), Belibassakis *et al* (2001), where this idea is first introduced and discussed for wave propagation/diffraction problems in variable bathymetry regions.

By using the local-mode series representation (4.1) in the variational principle (3.3), and by following exactly the same procedure as in Athanassoulis & Belibassakis (1999), the following coupled-mode system (CMS) with respect to the pressure mode amplitudes is obtained:

$$\sum_{n=-1}^{\infty} a_{mn}(x)P_n''(x) + b_{mn}(x)P_n'(x) + (c_{mn}(x) - a_{mn}q^2)P_n(x) = 0, \quad (4.4)$$

in $a < x < b$, $m = -1, 0, 1, \dots$, where a prime denotes differentiation with respect to x . The coefficients a_{mn} , b_{mn} , c_{mn} of the system (4.3) are given by

$$a_{mn} = \langle Z_n, Z_m \rangle = \int_{z=-h(x)}^{z=0} Z_n(z;x)Z_m(z;x)dz, \quad (4.5a)$$

$$b_{mn} = 2 \left\langle \frac{\partial Z_n}{\partial x}, Z_m \right\rangle + \frac{2q}{\sigma} \frac{dU}{dx} \langle Z_n, Z_m \rangle + \frac{dh}{dx} [Z_n Z_m]_{z=-h}, \quad (4.5b)$$

$$c_{mn} = \langle \Delta Z_n, Z_m \rangle + \left\langle \frac{2q}{\sigma} \frac{dU}{dx} \frac{\partial Z_n}{\partial x}, Z_m \right\rangle + \left[\left(\frac{dh}{dx} \frac{\partial Z_n}{\partial x} + \frac{\partial Z_n}{\partial z} \right) Z_m \right]_{z=-h} \quad (4.4c)$$

4.1 Boundary conditions for the CMS

The CMS (4.3) is supplemented by the following decoupled end-conditions (also obtained from the variational equation 3.3)

$$P_{-1}(a) = P_{-1}'(a) = 0, \quad P_{-1}(b) = P_{-1}'(b) = 0, \quad (4.6a,b)$$

$$P_0'(a) + ik_0^{(1)}P_0(a) = 2ik_0^{(1)}A_0 \exp(ik_0^{(1)}a), \quad (4.6c)$$

$$P_n' \varphi_n(a) - k_n^{(1)}P_n(a) = 0, \quad n = 1, 2, \dots, \quad (4.6d)$$

$$P_0' \varphi_0(b) - ik_0^{(3)}P_0(b) = 0, \quad (4.6e)$$

$$P_n'(b) + k_n^{(3)}P_n(b) = 0, \quad n = 1, 2, 3, \dots, \quad (4.6f)$$

where the coefficients $k_n^{(1)}$, $k_n^{(3)}$, $n = 0, 1, 2, \dots$ are defined by Eqs. (2.5a). Furthermore, the series expansion coefficients defining reflection and transmission coefficients

($K_R = A_R / A_0$, $K_T = A_T / A_0$) are obtained from the solution of the coupled-mode system as follows:

$$A_R = (P_0(a) - A_0 \exp(ik_0^{(1)}a)) \exp(ik_0^{(1)}a), \quad (4.7a)$$

$$A_T = P_0(b) \exp(-ik_0^{(3)}b). \quad (4.7b)$$

An important feature of the solution of the present scattering problem by means of the representation (4.1), is that it exhibits an improved rate of decay of the modal amplitudes $|P_n(x)|$ of the order $O(n^{-4})$. Thus, a small number of modes suffices to obtain a convergent solution to $P(x, z)$, even for large bottom slopes.

4.2 Simplified forms of the CMS

In the case of horizontal bottom ($dh/dx = 0$), the sloping bottom mode is zero, $P_{-1} = 0$. In addition, if we assume a mild shear current, the evanescent modes P_n , $n = 1, 2, 3, \dots$, producing localised second-order effects, can be approximately disregarded. In this case, the CMS (4.3) is simplified to the one-equation model

$$a_{00}(x)P_0''(x) + b_{00}(x)P_0'(x) + (c_{00}(x) - a_{00}q^2)P_0(x) = 0 \quad (4.8)$$

where

$$a_{00} = \langle Z_0, Z_0 \rangle, \quad (4.8a)$$

$$b_{00} = 2 \left\langle \frac{\partial Z_0}{\partial x}, Z_0 \right\rangle + \frac{2q}{\sigma} \frac{dU}{dx} \langle Z_0, Z_0 \rangle, \quad (4.8b)$$

$$c_{00} = \kappa_0^2 \langle Z_0, Z_0 \rangle + \left\langle \frac{\partial^2 Z_0}{\partial x^2} + \frac{2q}{\sigma} \frac{dU}{dx} \frac{\partial Z_0}{\partial x}, Z_0 \right\rangle, \quad (4.8c)$$

which has been first derived and studied by McKee (1996), and is known as the Enhanced Mild-Shear Equation (EMSE). On the basis of very slow current variations, $|dU/dx| \ll 1$, the above coefficient c_{00} can be further simplified by disregarding the last term in the right-hand side of Eq. (4.8c). In that case, Eq. (4.6) is reduced to

$$\left(\Gamma(x) P_0'(x) \right)' + \Gamma(x) (\kappa_0^2(x) - q^2) P_0(x) = 0, \quad (4.9)$$

where $\Gamma(x) = (\sigma/\omega)^2 a_{00}(x)$. The latter, known as

the Mild-Shear Equation (MSE), has been also first derived and studied by McKee (1987).

5 NUMERICAL RESULTS AND DISCUSSION

In this section we shall present numerical results obtained by the present discrete CMS and comparisons with other models. The discrete system is obtained by truncating the local-mode series (4.1) to a finite number of terms (modes), retaining a number of evanescent modes, and by using central, second-order finite differences to approximate the derivatives in Eqs. (4.3). Discrete boundary conditions are obtained by using second-order forward and backward differences to approximate derivatives at the ends. Thus, the discrete scheme is uniformly of 2-order in the

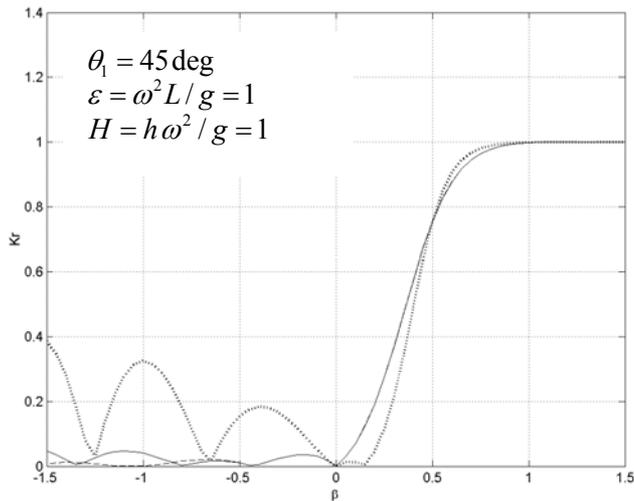


Figure 2. Oblique incident wave in deep water conditions, scattered by opposing and following currents. Comparison of the modulus of the reflection coefficient, as obtained by the present CMS (solid line), EMSE (dashed line) and MSE (dotted line), for various current velocities.

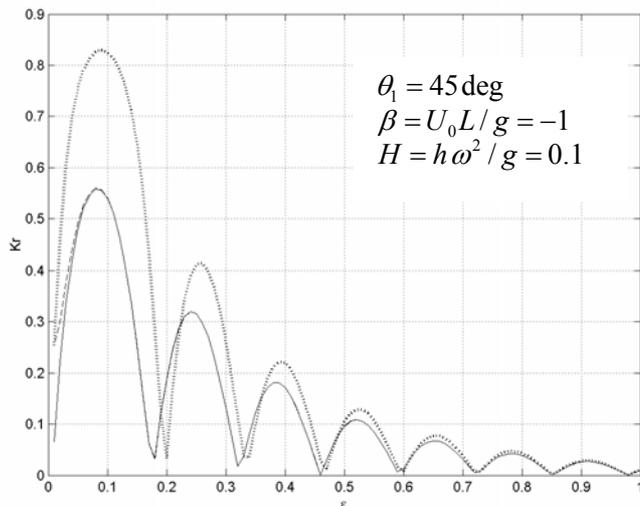


Figure 3. Oblique incident wave in shallow water, scattered by opposing shear current. Comparison of the modulus of the reflection coefficient, as obtained by the present CMS (solid line), EMSE (dashed line) and MSE (dotted line), for various frequencies.

horizontal direction. The forcing appears only in one equation, at the left endpoint $x = a$ (Eq. 4.6c).

5.1 The case of shear current in constant depth

In the first example, taken from McKee (2003), we present in Figs. 2 and 3 results concerning the reflection coefficient, as obtained by the present CMS with 5 modes vs. the EMSE model (4.8) and the MSE model (4.9), in constant depth. In this case, the shear current profile is of the form: $U(x) = U_0 \exp(-(x/L)^2)$, and the important parameters involved (except of the incidence angle) are: $\varepsilon = \omega^2 L / g$, $\beta = U_0 \omega / g$, $H = h \omega^2 / g$, McKee (2003). The present CMS results (shown by solid lines) are in perfect agreement with the ones obtained by the multidomain approximation method by McKee (2003, Figs. 5,7). In addition, in these figures, we are able to observe the enhanced performance of the EMSE vs. the MSE model. This finding is in perfect analogy with the corresponding performance of the modified mild-slope equation (Massel, 1993) vs. the classic Berkhoff's mild-slope equation (Berkhoff, 1972).

5.2 The case of a smooth underwater shoaling

In order to illustrate the combined effects of variable bathymetry and shearing current on the calculated wave field, we examine the case of a smooth but steep underwater shoal, characterised by the following depth function

$$h(x) = \frac{h_1 + h_3}{2} - \frac{h_1 - h_3}{2} \tanh\left(3\pi\left(\frac{x-a}{b-a} - \frac{1}{2}\right)\right), \quad (5.1)$$

in $a = 0 < x < b = 20m$, with $h_1 = 15m$ and $h_3 = 5m$.

This bottom profile has mean slope $s_{\text{mean}} = 0.5$ and maximum slope $s_{\text{max}} = 2.40$. (A sketch of the bottom topography is shown in Fig. 4). The angular frequency of the incident wave is selected to be $\omega = 1.62 \text{ rad/sec}$ ($\kappa_1 h_1 = 4$), implying almost deep water wave conditions in $D^{(1)}$, and its direction is taken $\theta_1 = -45^\circ$. The phase speed of the waves in $D^{(1)}$ is $c_1 = 6.06m/s$. The refracted/diffracted wave field (real part) above the variable bathymetry domain (Eq. 5.1), without any current effects, is shown in Fig. 4 by using equipotential lines. Extension of these lines below the bottom surface is maintained in the figure in order to better visualise the fulfillment of the Neumann boundary condition on $z = -h(x)$, which is equivalent to the fact that these lines intersect the bottom profile normally. At the top of

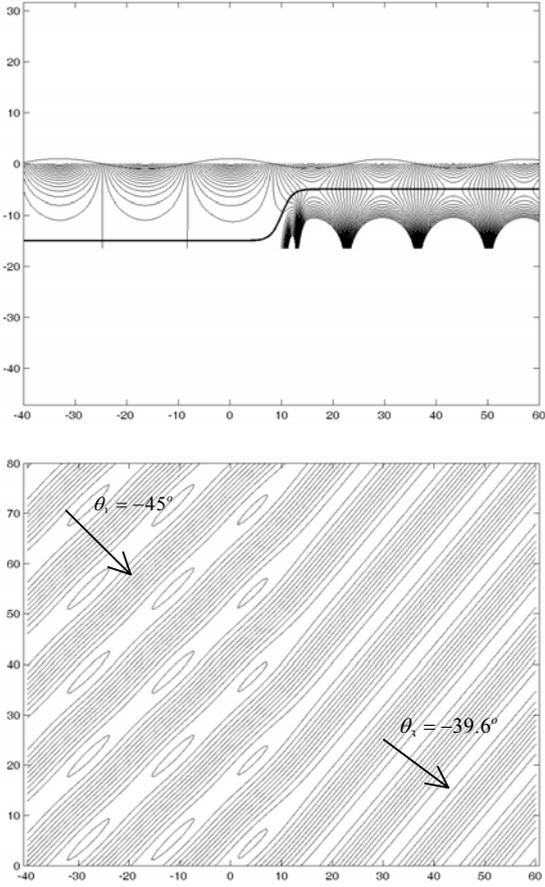


Figure 4. Refraction/diffraction of waves over a smooth and steep shoal, without current. $K_R = 0.025$, $K_T = 0.884$.

the figure, the wave pressure on the free surface, which is proportional with the free surface elevation, is also plotted. In Figs. 5 and 6, we present the corresponding result with the additional effect of following (Fig. 5) and opposing (Fig. 6) shear currents with (transitional) horizontal profile:

$$U(x) = \frac{U_3}{2} + \frac{U_3}{2} \tanh\left(3\pi\left(\frac{x-a}{b-a} - \frac{1}{2}\right)\right), \quad (5.2)$$

in $a = 0 < x < b = 20m$, where $U_3 = \max U = 0.25c_1$. In the region of transmission $D^{(3)}$ the wave directions become $\theta_3 = -57.5^\circ$ and $\theta_3 = -29.3^\circ$, respectively. The effects of the shear current on the wave are comparatively shown in Figs. 5 and 6 on both the horizontal and vertical planes by using equipotential lines. (Again, only the real part of the wave pressure field is plotted). A small number of modes (totally 5 terms) have been retained in the modal series expansions, which has been proved enough for numerical convergence, even for such large gradients of the bathymetry and the shear current. In all cases we observe that the equipotential lines intersect the bottom surface perpendicularly, which is evidence of satisfaction of the bottom boundary condition, both on the horizontal and on the sloping parts of the bottom.

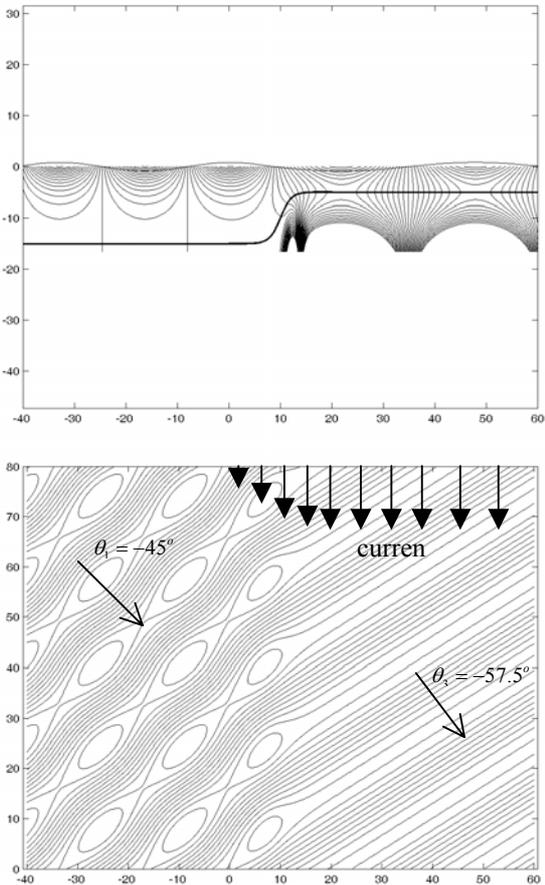


Figure 5. Refraction/diffraction of waves over a smooth and steep shoal, with the effect of a following transitional current. $K_R = 0.176$, $K_T = 0.852$.

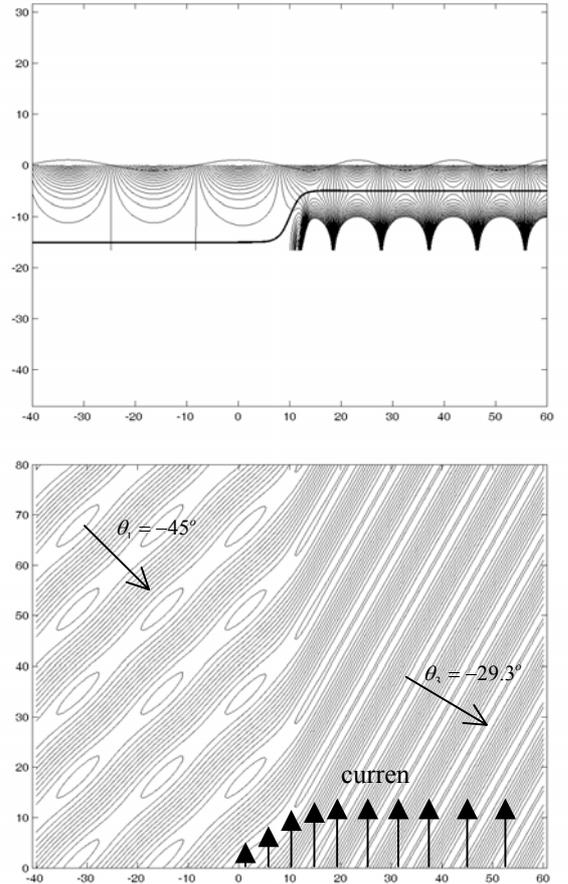


Figure 6. Refraction/diffraction of waves over a smooth and steep shoal, with the effect of opposing transitional current. $K_R = 0.053$, $K_T = 1.0$.

5.3 The case of a sinusoidal current

As a final example, we consider the case of waves of angular frequency $\omega = 1.62$ rad/sec propagating with direction $\theta_1 = -45^\circ$ in a constant depth strip $h = 15$ m, under the effects of an sinusoidal shear current with horizontal profile of the form:

$$V(x) = \frac{c_1}{4} \left(1 - \cos \left(2\pi \frac{x-a}{L} \right) \right), \quad L = \frac{b-a}{2}, \quad (5.3)$$

In this case, the shear current exists only in the region from $x = a = 0$ m to $x = b = 20$ m, it has a periodic structure with characteristic width $L = 10$ m, and its maximum is equal to the half of the phase speed of waves in $D^{(1)}$ and $D^{(3)}$, which is $c_1 = c_3 = 6.06$ m/s. The real part of the calculated wave field, as well as the values of the pressure on the free surface (which is proportional to the free-surface elevation), are shown in Fig. 7, as obtained by the present method using only 5 modes in the series (4.1).

6 CONCLUSIONS

A continuous coupled-mode technique for wave-current-seabed interaction in variable bathymetry regions is presented, with application to the prob-

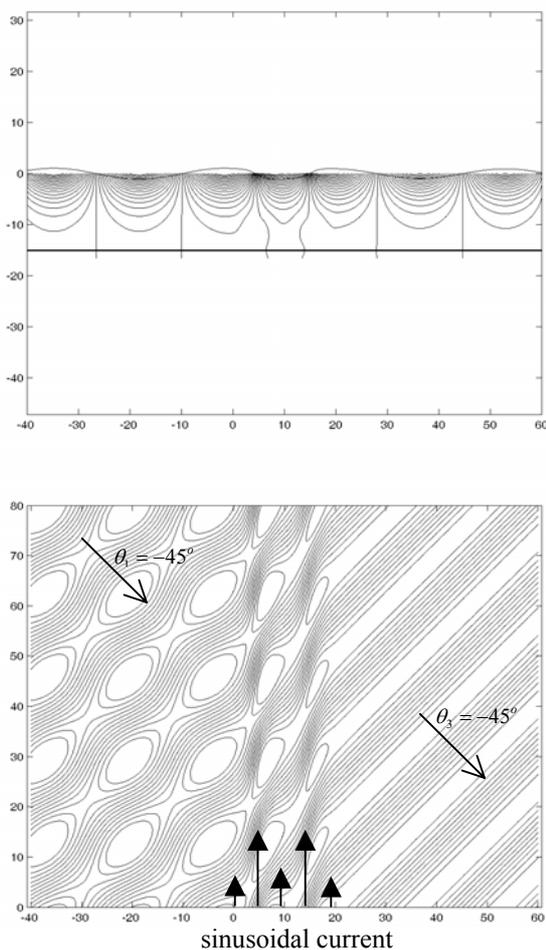


Figure 7. Refraction/diffraction of waves by an opposing sinusoidal current. $K_R = 0.329$, $K_T = 0.944$.

lem of wave scattering by steady shearing currents, characterised by current variations on various scales. The present method does not introduce any simplifying assumptions or other restrictions concerning the bottom slope and curvature, or the structure of the current. The analytical structure of the present model facilitates its extension to various directions as, e.g., to three-dimensional problems and to more complex wave-current systems, including more general vertical current profiles with cross-jet component, and the effects of weak nonlinearity.

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