A phase-resolving, coupled-mode model for wave-current-seabed interaction over steep 3D bottom topography. Parallel architecture implementation

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ABSTRACT

A phase-resolving, coupled-mode model is developed for the wave-current-seabed interaction problem, with application to wave scattering by steady currents over steep three-dimensional bottom topography. The vertical distribution of the wave potential is represented by a series of local vertical modes. This series consists of the vertical eigenfunctions associated with the propagating and all evanescent modes, plus an additional mode, accounting for the bottom boundary condition when the bottom slope is not negligible, as thoroughly discussed in Athanassoulis & Belibassakis (1999). Using the above representation in conjunction with a variational principle, Luke (1967), the problem is reduced to a coupled system of differential equations on the horizontal plane. If only the propagating mode is retained in the vertical expansion of the wave potential, and after additional simplifications, the above coupled-mode system is reduced to the one-equation model derived by Kirby (1984) with application to the problem of wave-current interaction over slowly varying topography. To treat the problem of wave-current-seabed interaction in unbounded domain, the present model is applied in conjunction with open sea/lateral boundary conditions, see, e.g., Chen *et al* (2005). The present system is discretized by using a second-order finite difference scheme and numerically solved by means of a parallel implementation, developed using the message passing programming paradigm on a commodity computer cluster; see, e.g., Gerostathis *et al* (2005). Thus, direct numerical solution of the present system is made feasible for realistic domains corresponding to areas with size of the order of several kilometers. The analytical structure of the present model facilitates its extension to treat non-linear waves, and it can be further elaborated to study wave propagation over random bottom topography and currents.

KEY WORDS: water waves, non-homogeneous currents, variable bathymetry, modified mild slope, couple-mode theory

INTRODUCTION

The prediction of wave propagation in nearshore and coastal areas is critical to engineering applications associated with coastal management and harbour maintenance. In regions where ambient tidal and other currents are strong, their effect on wave transformation can be substantial. They create a Doppler shift and cause wave refraction, reflection, and breaking, which can completely change the wave energy pattern. In particular, the characteristics of surface gravity waves present significant variations as they propagate through non-homogeneous currents, in the presence of depth inhomogeneities in variable bathymetry regions. For example, large amplitude waves can be produced when obliquely propagating waves interact with opposing currents, see, e.g., Mei (1983, Ch.3.7). This situation could be further enhanced by inshore effects due to sloping seabeds, and has been reported to be connected with the appearance of "giant waves"; see, e.g., Lavrenov & Porubov (2006). Extensive reviews on the subject of wave-current interaction in the nearshore region have been presented by Peregrine (1976), Jonsson (1990) and Thomas & Klopman (1997).

The study of spatial evolution of water waves and the investigation of scattering of realistic wave spectra over irregular currents, with characteristic length of variation comparable to the dominant wavelength, including the effects of bottom irregularities, can be supported by theoretical models treating the simpler problem of monochromatic waves interacting with steady inhomogeneous currents. Wave-current interaction models over slowly varying bottom topography have been developed and studied by various authors. Under the assumption of irrotational wave motion, Kirby (1984) derived a phase-resolving one-equation model, generalizing the mild-slope equation by Berkhoff (1972) in regions with slowly varying depth and ambient currents; see also Liu (1990). The latter model in its elliptic time-harmonic form has been exploited, in conjunction with numerical (finite-element, finite difference etc) solvers, to numerous wave-current-seabed interaction applications; see, e.g., Chen *et al* (2005) and the references cited there.

On the other hand, if the wave flow is assumed to be weakly rotational, as happens to be the case when waves are scattered by shearing currents characterised by stronger horizontal gradients, McKee (1987) derived another one-equation model, called the mild-shear equation. Still however, the validity of the mild-shear equation is based on the assumption of slow current and depth variations compared to the typical wavelength. In the case of flat bottom, the mild-shear model has been further enhanced by McKee (1996) by including an extra term and obtaining the so called enhanced mild-shear equation. The latter model is applicable to cases where the shearing current is varying on the scale of the wavelength. In the above works by McKee (1987, 1996) the current is considered to be flowing along one horizontal direction while the bottom topography varies in the other horizontal direction. Thus, the mild-shear model is more appropriate for problems of wave scattering by slowly varying depth and longshore-type ambient shearing currents.

In both the above approaches (mild-slope model, mild-shear model) the effects of evanescent modes, describing higher-order localised effects due to bottom and current variations, have been ignored. Based on complete normal-mode type expansions, another class of wave-current-seabed interaction models has been developed, applicable to cases where the lateral length scale on which the medium is changing is much smaller than the typical wavelength. In this case, the problem is treated by means of step discontinuities and vertical vortex sheets, separating subregions of essentially potential flow, in conjunction with appropriate matching conditions ensuring continuity of pressure and normal flow following the vortex sheet(s). In this context, generalising the work by Evans (1975) for the transmission of deep-water waves across a vortex sheet, Smith (1983, 1987) presented models for waves crossing uniform current jets in constant finite depth and crossing a step with horizontal shear, respectively. Also, Kirby *et al* (1987) studied the propagation of obliquely incident waves over a trench with uniform current flowing along it. In the latter works complete representations of the wave potentials in the various subregions have been used, containing both the propagating and the evanescent modes, which are necessary in order to satisfy the matching/boundary conditions at the vertical interfaces (vortex sheets and depth discontinuities). Finally, the approach by Smith (1987) and Kirby *et al* (1987) has been further exploited by McKee (2003) to study scattering of waves by shearing currents of general horizontal structure in water of constant depth. In the latter work the current is modelled by a series of vertical vortex sheets separating subregions of constant current velocity, and the solution is again obtained by using complete representations of the wave potential in each subregion and matching conditions at the vertical interfaces. Also, in McKee (2003) systematic comparisons have been presented between the predictions by the mild-shear equation(s) and the piecewise-constant current velocity approximation, showing that the accuracy of the enhanced mild-shear equation is generally better than the original mild-shear equation. Finally, in a recent work by Belibassakis (2007), a continuous coupled-mode model has been developed for the scattering of water waves by horizontally shearing currents in variable bathymetry regions, without any asymptotic assumption or restriction concerning the smallness of the bottom and current variation lengths with respect to the local wavelength. In the case of mild bottom topography and slow current variations, the evanescent modes, producing localised second-order effects, can be approximately disregarded. Also, due to bottom mildness the sloping-bottom mode can be neglected. In that case, the bove coupled-mode system simplifies to a new one-equation model, called the *mild-slope and shear* equation. The latter has the property to exactly reduce to the Modified Mild-Slope equation, derived by Massel (1993), Chamberlain & Porter (1995), in the case of no-current, and to the Enhanced Mild-Shear Equation, derived McKee (1996), in the case of a flat domain.

In this work we consider wave-current flow of a homogeneous liquid in a general bathymetry region, in the case where momentum and gravity forces are dominant, and thus we can approximately ignore the effects of capillary and viscous forces. The flow associated with the current is assumed to be self-existent (background current). Moreover, we will restrict ourselves to large-scale currents, where the horizontal and time variations of the background current are small within a characteristic wavelength and wave period, respectively. Under the above assumptions, a new phase-resolving, coupled-mode model is developed using Luke's (1967) variational principle, with application to wave scattering by slowly varying steady currents over steep three-dimensional bottom topography. The vertical distribution of the wave potential is represented by a series of local vertical modes. This series consists of the vertical eigenfunctions associated with the propagating and all evanescent modes, plus an additional mode, accounting for the bottom boundary condition when the bottom slope is not negligible, which is thoroughly discussed in Athanassoulis & Belibassakis (1999). Using the above representation in conjunction with Luke's (1967) variational principle (see also Massel 1989, Ch.1.3), the problem is reduced to a coupled system of differential equations on the horizontal space. If only the propagating mode is retained in the vertical expansion of the wave potential, the above coupled-mode system is reduced to an one-equation model, which after additional simplifications reduces to the mild-slope model derived by Kirby (1984) with application to the problem of wave-current interaction over slowly varying topography. To treat the problem of wave-current-seabed interaction in unbounded domain, the present model is applied in conjunction with approximate open sea/lateral boundary conditions, see, e.g., Chen *et al* (2005). Our system is discretized by using a second-order finite difference scheme and numerically solved by means of a parallel implementation of the model, which is developed using the message passing programming paradigm on a commodity computer cluster; see, e.g., Gerostathis *et al* (2005). Thus, direct numerical solution of the present system is made feasible for realistic domains corresponding to areas with size of the order of several kilometers.

The present work is structured as follows: first the complete non-linear problem is considered. The flow is decomposed to a steady part associated with the presence of the ambient current and a disturbance field associated with the wave flow, for which an approximate Bernoulli equation is derived. Then the equivalent variational formulation of the problem for the unknown wave disturbance field is formulated, through the application of the Luke's (1967) functional, which is based on the integration of pressure in the domain. Restricting ourselves to time harmonic problems and introducing an enhanced local-mode series expansion of the wave potential, first developed by Athanassoulis & Belibassakis (1999) for the propagation of water waves in variable bathymetry region, the variational formulation leads to the new coupled-mode system of equations. The local-mode series contains except of the propagating and evanescent modes, all associated with the local-intrinsic frequency of the wave, an additional term called the sloping-bottom model, also first introduced in the above work. This term permits the consistent satisfaction of the Neumann boundary condition on the sloping parts of the bottom, and makes the local-mode series to converge fast. Numerical results are presented and discussed, in the case of vortex ring current and a rip current. The analytical structure of the present model facilitates its extension to treat non-linear waves, and it can be further elaborated for studying wave propagation over random bottom topography and currents.

THE BACKGROUND CURRENT FLOW

We consider wave propagation, in the presence of ambient, non-homogeneous current, in a variable bathymetry region; see Fig. 1. The liquid is assumed inviscid and homogeneous, and the flow associated with the background current is assumed to be self-existent, steady and possibly weakly rotational. On the other hand, the wave flow perturbing the background current flow, generated by an incident wave system coming from the far up-wave region (see Fig.1), is assumed to be irrotational. The background current flow is assumed to be nearly horizontal. Moreover, its velocity is assumed to be small and slowly varying, and thus, the associated mean free-surface elevation (set-down) is also small. We introduce however, no assumption as regards the mildness of the bottom slope.

A Cartesian coordinate system is used, having its origin at some point on the unperturbed free-surface ($z=0$). The z-axis is pointing upwards and one of the horizontal axes is taken to be (approximately)

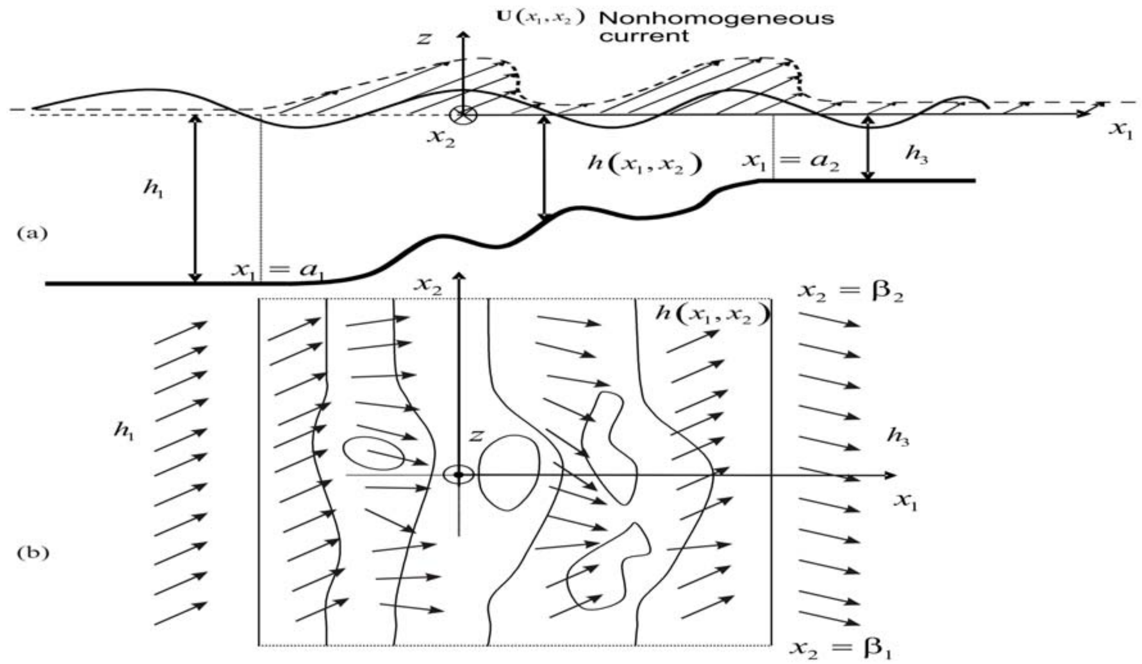

Fig. 1. Geometrical configuration and basic notation: (a) Vertical section, (b) Horizontal plane.

aligned with the mean direction of the transmitted wave field; see Fig. 1. The current $\mathbf{q} = (U_1, U_2, W)$ has been assumed to be steady and self existent, and the kinematics of this flow state that

$$\nabla_3 \mathbf{q} = 0 , \tag{1a}$$

$$\mathbf{q} \cdot \mathbf{n} = 0 , \; z = H(x_1, x_2), \quad \mathbf{q} \cdot \mathbf{n} = 0 , \; z = -h(x_1, x_2), \tag{2b,c}$$

where ∇_3 denotes the gradient operator in 3D, and H denotes the mean set-down associated with the background current.
Eq. (2.c) is equivalently written in the form

$$W + U_1 \frac{\partial h}{\partial x_1} + U_2 \frac{\partial h}{\partial x_2} = 0 , \quad z = -h(x_1, x_2), \tag{2c'}$$

Also, using the fact that the current velocity has been assumed small, the mean set-down is negligible $(H \approx 0)$, and the kinematical free-surface boundary condition (2b) can be linearised as follows:

$$W = 0 , \; z = 0 . \tag{2b'}$$

The dynamics of the steady background current are expressed by conservation of energy. The latter is described by the corresponding Bernoulli equation, stating that total energy is conserved along the streamlines (Bachelor 1967),

$$Q = \frac{1}{2}|\mathbf{q}|^2 + \frac{P}{\rho} + gz = \text{const} = \frac{1}{2}|\mathbf{q}_\infty|^2 + \frac{P_\infty}{\rho} + gz_\infty , \tag{3}$$

where ρ is the density, g is the gravitational acceleration and z_∞ denotes the vertical position of each streamline at infinity. Taking P_∞ to be the static pressure of the fluid at rest, obtained by the superposition of the atmospheric (p_a) and the hydrostatic $(-\rho g z_\infty)$ pressure at infinity,

$$P_\infty = p_a - \rho g z_\infty , \tag{4}$$

we finally obtain from Eq.(3), the following equation for the background flow pressure

$$\frac{P}{\rho} = \frac{1}{2}\left(|\mathbf{q}_\infty|^2 - |\mathbf{q}|^2\right) + \frac{p_a}{\rho} - gz . \tag{5}$$

The total (current and wave) flow $\mathbf{u}$, as well as the background current flow $\mathbf{q}$, both satisfy Euler equations. Subtracting these equations by parts and omitting approximately the terms associated with the vorticity of the background flow $(\nabla_3 \times \mathbf{q})$, which is assumed weak, we finally obtain the following approximate Bernoulli equation

$$\frac{p}{\rho} + \frac{\partial \varphi}{\partial t} + \mathbf{q} \cdot \nabla_3 \varphi + \frac{1}{2}|\nabla_3 \varphi|^2 = 0 , \tag{6}$$

where $\varphi(x_1, x_2, z; t)$ denotes the potential associated with the wave disturbance flow,

$$\nabla_3 \varphi = \mathbf{u} - \mathbf{q} , \tag{7}$$

and p is the wave-disturbance pressure, which is defined as the difference between the total (P_u) and the background (P) pressure,

$$p = P_u - P . \tag{8}$$

Using the above equations, we finally obtain the following Bernoulli equation expressing the conservation of energy of the wave-current problem:

$$\frac{P_u - p_a}{\rho} = -\left[gz + \frac{\partial \varphi}{\partial t} + \mathbf{q} \cdot \nabla_3 \varphi + \frac{1}{2}|\nabla_3 \varphi|^2 \right] + \frac{1}{2}\left(|\mathbf{q}_\infty|^2 - |\mathbf{q}|^2\right) \text{, or}$$

$$P_u - p_a = -\rho\left[gz + \frac{\partial\varphi}{\partial t} + \frac{1}{2}\left(|\mathbf{u}|^2 - |\mathbf{q}_\infty|^2\right)\right]. \tag{9}$$

VARIATIONAL FORMULATION

The variational principle governing the total fluid motion is formulated using Luke's functional (1967), which is based on integration of pressure in the domain (see also Massel 1989, Ch.1.3),

$$\mathcal{L} = \int_t \int_{x_1} \int_{x_2} \int_{z=-h(x_1,x_2)}^{z=\eta(x_1,x_2;t)} \left(P_u - p_a\right) dz\ dx_2 dx_1 dt\,, \tag{10a}$$

where $\eta(x_1, x_2; t)$ denotes the free-surface elevation associated with the total wave and current flow. Using Eq. (9), the above equation is written as

$$\mathcal{L} = -\rho \int_t \int_{x_1} \int_{x_2} \int_{z=-h(x_1,x_2)}^{z=\eta(x_1,x_2;t)} \left[gz + \frac{\partial\varphi}{\partial t} + \frac{1}{2}\left(|\mathbf{u}|^2 - |\mathbf{q}_\infty|^2\right)\right] dz dx_2 dx_1 dt\,. \tag{10b}$$

Requiring the above functional to be stationary, $\delta\mathcal{L} = 0$, and after carying out the algebra, we finally obtain the following equation (see also Massel 1989, Eq.1.35)

$$\begin{aligned}
&-\rho \int_t \int_{x_1} \int_{x_2} \left\{ \left[\frac{1}{2}\left(|\mathbf{u}|^2 - |\mathbf{q}_\infty|^2\right) + \frac{\partial\varphi}{\partial t} + g\eta \right]_{z=\eta(x_1,x_2,t)} \delta\eta \right. \\
&- \left[\frac{\partial\eta}{\partial t} + \left(U_1 + \frac{\partial\varphi}{\partial x_1}\right)\frac{\partial\eta}{\partial x_1} + \left(U_2 + \frac{\partial\varphi}{\partial x_2}\right)\frac{\partial\eta}{\partial x_2} - \left(W + \frac{\partial\varphi}{\partial z}\right)\right]_{z=\eta(x_1,x_2,t)} \delta\varphi \\
&- \left[\frac{\partial\varphi}{\partial z} + \frac{\partial\varphi}{\partial x_1}\frac{\partial h}{\partial x_1} + \frac{\partial\varphi}{\partial x_2}\frac{\partial h}{\partial x_2}\right]_{z=-h(x_1,x_2)} \delta\varphi \\
&\left. - \left[\int_{z=-h(x_1,x_2)}^{z=\eta(x_1,x_2;t)} \left(\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial z^2}\right) dz \right] \delta\varphi \right\} dx_2 dx_1 dt = 0\,,
\end{aligned} \tag{11}$$

where also the continuity equation of the background current, Eq.(1), and the bottom boundary condition, Eq. (2c), have been used. The equations governing the fully non-linear wave-current problem are derived from the above form of the variational principle, Eq.(11), and will be studied in future work.

Using the fact that the current has been assumed essentially horizontal $(W << U_1, U_2)$ and slowly varying ($|\mathbf{q}|^2 \approx |\mathbf{q}_\infty|^2$), the quantity $|\mathbf{u}|^2 - |\mathbf{q}_\infty|^2$ appearing in the first term of Eq.(11) can be approximated as follows: $|\mathbf{u}|^2 - |\mathbf{q}_\infty|^2 \approx U_1 \frac{\partial\varphi}{\partial x_1} + U_2 \frac{\partial\varphi}{\partial x_2} + \left(\nabla_3\varphi\right)^2$.

Moreover, by assuming small wave amplitudes and using the expansions of various quantities with respect to mean water level $(H \approx 0)$, and keeping up to second-order terms in the left-hand side of Eq. (11), we finally obtain the following approximation:

$$\begin{aligned}
&-\rho \int_t \int_{x_1} \int_{x_2} \left\{ \left[U_1 \frac{\partial\varphi}{\partial x_1} + U_2 \frac{\partial\varphi}{\partial x_2} + \frac{\partial\varphi}{\partial t} + g\eta \right]_{z=0} \delta\eta \right. \\
&- \left[\frac{\partial\eta}{\partial t} + U_1\frac{\partial\eta}{\partial x_1} + U_2\frac{\partial\eta}{\partial x_2} - \frac{\partial\varphi}{\partial z} + \left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2}\right)\eta \right]_{z=0} \delta\varphi \\
&- \left[\frac{\partial\varphi}{\partial z} + \frac{\partial\varphi}{\partial x_1}\frac{\partial h}{\partial x_1} + \frac{\partial\varphi}{\partial x_2}\frac{\partial h}{\partial x_2}\right]_{z=-h(x_1,x_2)} \delta\varphi \\
&\left. - \int_{z=-h(x_1,x_2)}^{z=0} \left(\nabla^2\varphi + \frac{\partial^2\varphi}{\partial z^2}\right) dz\ \delta\varphi \right\} dx_2 dx_1 dt = 0\,.
\end{aligned} \tag{12}$$

In deriving the above equation, use was also made of the Taylor expansion of W at the free surface, in conjunction with Eq. (2b'),

$$W\big|_{z=\eta(x_1,x_2;t)} = W\big|_{z=0} + \left.\frac{\partial W}{\partial z}\right|_{z=0}\eta = -\nabla\mathbf{U}\cdot\eta\,, \tag{13}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ denotes the horizontal gradient operator and $\mathbf{U} = (U_1, U_2)$ are the horizontal components of the current flow. From Eq.(12), using the independence of the variations $\delta\varphi$ and $\delta\eta$, we obtain the formulation of the linearised wave-current problem:

$$\nabla^2\varphi + \frac{\partial^2\varphi}{\partial z^2} = 0\,, \qquad -h(\mathbf{x}) < z < 0\,, \tag{14a}$$

$$\frac{\partial\eta}{\partial t} + \mathbf{U}\nabla\eta - \frac{\partial\varphi}{\partial z} + (\nabla\mathbf{U})\cdot\eta = 0\,, \qquad z = 0\,, \tag{14b}$$

$$\frac{\partial\varphi}{\partial t} + \mathbf{U}\cdot\nabla\varphi + g\eta = 0\,, \qquad z = 0\,, \tag{14c}$$

$$\frac{\partial\varphi}{\partial z} + \nabla\varphi\nabla h = 0\,, \qquad z = -h(\mathbf{x})\,. \tag{14d}$$

where $\mathbf{x} = (x_1, x_2)$. The above equations are the ones given by Longuet-Higgins & Stewart (1961); see also Liu (1990, Eqs.4-7). Furthermore, by introducing the linear operator $\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)$, the free surface elevation can be easily obtained in terms of the wave potential on the free-surface, by using the dynamic free surface boundary conditions (14c) as follows:

$$\eta = -\frac{1}{g}\left(\frac{\partial}{\partial t} + \mathbf{U}\nabla\right)\varphi\,, \qquad z = 0\,. \tag{15}$$

Using the above equation to eliminate $\eta(x_1, x_2; t)$ in the variational equation (14), the latter is written as follows:

$$\begin{aligned}
&\rho \int_t \int_{x_1} \int_{x_2} \left\{ -\left[\frac{1}{g}\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)^2 \varphi + \frac{\partial\varphi}{\partial z} + \frac{1}{g}(\nabla\mathbf{U})\cdot\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\varphi \right]_{z=0} \delta\varphi + \right. \\
&\left. + \left[\frac{\partial\varphi}{\partial z} + \nabla\varphi\nabla h \right]_{z=-h(\mathbf{x})} \delta\varphi \quad + \int_{z=-h(\mathbf{x})}^{0} \left(\nabla^2\varphi + \frac{\partial^2\varphi}{\partial z^2}\right) dz\ \delta\varphi \right\} dx_2 dx_1 dt = 0\,.
\end{aligned} \tag{16}$$

Considering now the velocity field to be time harmonic with angular frequency ω, the flow field, associated with the wave disturbance flow, is described by means of a complex velocity potential

$$\varphi(x_1, x_2, z; t) = \mathrm{Re}\left\{\varphi(x_1, x_2, z)\exp(-i\omega t)\right\}. \tag{17}$$

In this case, Eq. (16) becomes,

$$\rho \int_{x_1}\int_{x_2}\left\{-\left[\frac{1}{g}\left(-i\omega+\mathbf{U}\cdot\nabla\right)^2\varphi+\frac{\partial\varphi}{\partial z}+\frac{1}{g}\left(\nabla\mathbf{U}\right)\cdot\left(-i\omega+\mathbf{U}\cdot\nabla\right)\varphi\right]_{z=0}\delta\varphi+\right.$$

$$\left.+\left[\frac{\partial\varphi}{\partial z}+\nabla\varphi\nabla h\right]_{z=-h(\mathbf{x})}\delta\varphi \qquad +\int_{z=-h(\mathbf{x})}^{0}\left(\nabla^2\varphi+\frac{\partial^2\varphi}{\partial z^2}\right)dz\,\delta\varphi\right\}dx_2 dx_1=0, \tag{18}$$

where, from now on, $\varphi=\varphi(x_1,x_2,z)$ is the complex wave potential. Furthermore, noting that the term

$$A=\left(-i\omega+\mathbf{U}\cdot\nabla\right)^2\varphi+\left(\nabla\mathbf{U}\right)\left(-i\omega+\mathbf{U}\cdot\nabla\right)\varphi \ , \tag{19a}$$

appearing in the integral on the mean free-surface (z=0) is written:

$$A=-\omega^2\varphi-2i\omega\left(\mathbf{U}\cdot\nabla\right)\varphi-i\omega\left(\mathbf{U}\cdot\nabla\right)\varphi+\nabla\cdot\left(\mathbf{U}\left[\left(\mathbf{U}\cdot\nabla\right)\varphi\right]\right), \tag{19b}$$

the variational principle for small-amplitude, time-harmonic wave motion, Eq. (18) is finally put in the form

$$\int_{x_1}\int_{x_2} dt\, dx_1\, dx_2\left\{\int_{z=-h(\mathbf{x})}^{z=0}\left(\nabla^2\varphi+\frac{\partial^2\varphi}{\partial z^2}\right)dz \quad + \right.$$

$$\left.\left[\frac{\partial\varphi}{\partial z}+\nabla\varphi\nabla h\right]_{z=-h(\mathbf{x})}-\left[\frac{A}{g}+\frac{\partial\varphi}{\partial z}\right]_{z=0}\delta\varphi\right\}=0\,. \tag{20}$$

THE COUPLED-MODE SYSTEM OF EQUATIONS

Following previous works (Athanassoulis & Belibassakis 1999, Belibassakis *et al* 2001), we introduce the following local-mode series expansion of the wave potential

$$\varphi(\mathbf{x};z)=\sum_{n=-1}\varphi_n(\mathbf{x})\cdot Z_n(z;\mathbf{x}), \tag{21}$$

in the variable bathymetry region. In the above expansion, the mode n=0 ($\varphi_0(\mathbf{x})Z_0(z;\mathbf{x})$) denotes the propagating mode and the remaining terms, $n=1,2,\dots$, are the evanescent modes. The additional term $\varphi_{-1}(\mathbf{x})Z_{-1}(z;\mathbf{x})$ is a correction term, called the sloping-bottom mode, properly accounting for the satisfaction of the Neumann bottom boundary condition on the non-horizontal parts of the bottom. In the present case, the functions $Z_n(z;\mathbf{x})$, $n=0,1,2...$, appearing in Eq. (21), are obtained as the eigenfunctions of local vertical Sturm-Liouville problems, formulated with respect to the local depth and the local intrinsic frequency $\sigma=\omega-\mathbf{U}\cdot\mathbf{k}$,

$$Z_n''+k_n^2 Z_n=0\,, \qquad -h(\mathbf{x})<z<0\,, \tag{22a}$$

$$Z_n'-\mu Z_n(0)=0\,, \qquad z=0\,. \tag{22b}$$

$$Z_n'(-h)=0\,, \qquad z=-h(\mathbf{x})\,, \tag{22c}$$

where the prime denotes differentiation with respect to z. The parameter μ in Eq. (22b) is taken to be the intrinsic frequency parameter, $\mu=\sigma^2/g$, where and ω is the absolute wave frequency. The definition of the intrinsic frequency depends on the vector wavenumber $\mathbf{k}$ associated with the wave kinematics, see e.g. Jonnson (1990), and thus, it is clearly dependent on the solution $\varphi(x_1,x_2;z)$. This fact introduces intrinsic nonlinearity to the problem (14), and iterations are necessary for its solution. The local vertical eigenfunctions are obtained from Eqs. (22) as follows:

$$Z_0=\frac{\cosh\left[k_0(z+h)\right]}{\cosh(k_0 h)}\,,\quad Z_n=\frac{\cos\left[k_n(z+h)\right]}{\cos(k_n h)}\,,\quad n=1,2,..\,, \tag{23}$$

where the wavenumbers k_n are obtained as a solution to the local dispersion relation, associated with the intrinsic frequency,

$$\sigma^2=k_0 g\tanh(kh)=-k_n g\tan(k_n h)\,. \tag{24}$$

We note here that (without loss of generality) the vertical eigenfunctions have been normalized, so that,

$$Z_n(z=0)=1\,,\ n=0,1,2,...\ . \tag{25}$$

Finally, as concerns the sloping-bottom mode $\varphi_{-1}Z_{-1}$, a specific convenient form of the function $Z_{-1}(z;\mathbf{x})$ is given by

$$Z_{-1}=h\ \left[(z/h)^3+(z/h)^2\ \right], \tag{26}$$

having the following properties: $Z_{-1}'(z=-h)=1, Z_{-1}(z=-h)=0,$ and $Z_{-1}'(z=0)=Z_{-1}(z=0)=0$ (see also the discussion by Athanassoulis & Belibassakis 1999). Using the representation (21), in conjunction with Eq. (22b) and the above properties of Z_{-1}, we obtain that,

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=0}=\mu\varphi=\frac{\sigma^2}{g}\varphi\,, \quad \text{on } z=0\,. \tag{27}$$

Introducing the above result in the last form of the variational principle, Eq. (20), we obtain

$$\int_{x_1}\int_{x_2} dt\, dx_1\, dx_2\left\{\int_{z=-h(x_1,x_2)}^{z=0}\left(\nabla^2\varphi+\frac{\partial^2\varphi}{\partial z^2}\right)dz \quad + \right.$$

$$\left.\left[\frac{\partial\varphi}{\partial z}+\nabla\varphi\nabla h\right]_{z=-h(\mathbf{x})}-\left[\frac{A}{g}+\frac{\sigma^2\varphi}{g}\right]_{z=0}\delta\varphi\right\}=0\,. \tag{28}$$

Using the local-mode representation (21) in the above equation, and its consequence,

$$\delta\varphi=\sum\delta\varphi_n Z_n\,, \tag{29}$$

in conjunction with the independence of the variations of the mode-amplitudes $\delta\varphi_n$, we finally obtain an equivalent reformulation of the linearised problem, Eqs. (14), in the form of a *coupled-mode system* of equations,

$$\sum_n\left\{a_{mn}\nabla^2\varphi_n+\left[\mathbf{b}_{mn}+\frac{2i\omega}{g}\mathbf{U}\right]\nabla\varphi_n+\left[c_{mn}+\frac{\omega^2-\sigma^2}{g}+\frac{i\omega}{g}(\nabla\cdot\mathbf{U})\right]\varphi_n+\right.$$

$$\left.-\frac{1}{g}\nabla\cdot\left(\mathbf{U}\left[\left(\mathbf{U}\cdot\nabla\right)\varphi_n\right]\right)\right\}=0, \qquad m=-1,0,1,...\ , \tag{30}$$

where the coefficients $a_{mn},\mathbf{b}_{mn},c_{mn}$ are defined as follows:

$$a_{mn}=\left\langle Z_m,Z_n\right\rangle,\ \mathbf{b}_{mn}=2\left\langle\nabla Z_n,Z_n\right\rangle+Z_n(-h)Z_m(-h)\nabla h\,, \tag{31a,b}$$

$$c_{mn}=\left\langle\nabla^2 Z_n+\frac{\partial^2 Z_n}{\partial z^2},Z_n\right\rangle+\left(\left.\frac{\partial Z_n}{\partial z}\right|_{z=-h}+\nabla Z_n\nabla h\right)Z_n(-h)\,, \tag{31c}$$

and the brackets denote the inner product in the vertical interval:

$$\left\langle f(z), g(z)\right\rangle = \int_{z=-h(\mathbf{x})}^{0} f(z) g(z) dz .$$

It is worth noticing here that, in the case of no current $(\mathbf{U}=0)$ the coupled-mode system (30) exactly reduces to the corresponding one derived by Athanassoulis & Belibassakis (1999) for wave propagation over variable bathymetry regions. Moreover, a significant simplification of this system is obtained by keeping only the propagating mode $(n=0)$ in the local-mode series expansion of the wave potential, which essentially describes the propagation features. In this case the above coupled-mode system is reduced to the following one-equation model on the horizontal plane,

$$a_{00}\nabla^2\varphi_0 + \left[\mathbf{b}_{00} + \frac{2i\omega}{g}\mathbf{U}\right]\nabla\varphi_0 - \frac{1}{g}\nabla\cdot\left(\mathbf{U}\left[\mathbf{U}\cdot\nabla\right]\varphi_0\right) +$$

$$+\left[c_{00} + \frac{\omega^2-\sigma^2}{g} + \frac{i\omega}{g}\left(\nabla\cdot\mathbf{U}\right)\right]\varphi_0 = 0 . \qquad (32)$$

The coefficients $a_{00}, \mathbf{b}_{00}, c_{00}$ are given by Eqs. (31), and after processing they become as follows:

$$a_{00} = \int_{z=-h}^{0} Z_0^2 dz = \frac{1}{2k_0}\tanh(k_0 h)\left(1+\frac{2k_0 h}{\sinh(2k_0 h)}\right) = \frac{1}{g}CC_g , \qquad (33a)$$

$$\mathbf{b}_{00} = \nabla a_{00} = \frac{1}{g}\nabla CC_g = 2\left\langle\nabla Z_0, Z_0\right\rangle + Z_0^2(-h)\nabla h , \qquad (33b)$$

$$c_{00} = k_0^2 a_{00} + \left\langle\nabla^2 Z_0, Z_0\right\rangle + \nabla Z_0 \nabla h\, Z(-h) = k_0^2 a_{00} + c_{00}^{(2)} . \qquad (33c)$$

The coefficient $c_{00}^{(2)}$ contains terms proportional to first and second horizontal derivatives of the depth function (proportional to bottom slope and curvature), as well as first and second horizontal derivatives of the horizontal current velocity components U_1 and U_2. Furthermore, denoting by $2\boldsymbol{\gamma} = \mathbf{b}_{00} - Z_0^2(-h)\nabla h$ the coefficient c_{00} can also be calculated as follows,

$$c_{00} = \nabla\cdot\boldsymbol{\gamma} - \int_{z=-h(x_1,x_2)}^{0} \left(\nabla Z_0\right)^2 dz . \qquad (33c')$$

Using the above expressions of the coefficients in the one-equation model (32) and multiplying by g, it is easily seen that the latter takes the form

$$\nabla\left(CC_g\nabla\varphi_0\right) - \nabla\cdot\left\{\mathbf{U}\left[\left(\mathbf{U}\cdot\nabla\right)\varphi_0\right]\right\} + 2i\omega\mathbf{U}\cdot\nabla\varphi_0 +$$

$$+\left[k_0^2 CC_g + g c_{00}^{(2)} + \omega^2 - \sigma^2 + i\omega\left(\nabla\cdot\mathbf{U}\right)\right]\varphi_0 = 0 , \qquad (34)$$

which will be called the *modified mild-slope equation* for wave scattering by ambient current in general bottom topography. The model (34), in the case of no current, exactly reduces to the modified mild slope equation derived by Massel (1993) and Chamberlain & Porter (1995). Furthermore, it is worth noticing here that if the term $c_{00}^{(2)}$ is approximately omitted, then Eq. (34) exactly reduces to the mild slope equation derived by Kirby (1984), in its time-harmonic form; see e.g., Chen *et al* (2005, Eq.8).

NUMERICAL RESULTS AND DISCUSSION

In this section we will present and discuss numerical examples, obtained by the solution of the present modified mild-slope equation (34), for two selected cases. Numerical solutions of the full coupled-mode system, Eq. (30), and discussion of the effects of the evanescent modes and the sloping bottom mode will be presented elsewhere. It is noted however, that similarly as in the no-current problem, the modified mild-slope equation is expected to produce reliable results for bottom slopes up to 1:3 or higher. The construction of the discrete system is obtained by using central, second-order finite differences to approximate the derivatives in Eq. (34). The modified mild-slope Eq. (34) is coupled with boundary conditions expressing incoming wave, transmitted outgoing wave, and lateral absorbing conditions, in the same form as they have been developed by Chen *et al* (2005) for Kirby's mild-slope equation. Second-order, forward and backward finite differences have been used for the discretisation of the derivatives involved in the above boundary conditions.

The case of a vortex-ring current: In this example we consider wave scattering by a vortex ring current in constant depth *h*=10m, examined also by Chen *et al* (2005). Such current structure is commonly seen in open sea and coastal areas, and has important impact on physical and biological processes, Mapp *et al* (1985). Following Yoon & Liu (1989), the background current flow associated with the vortex ring is defined by,

$$U_r = 0 , \quad U_\theta = \begin{cases} C_5\left(r/R_1\right)^N , & r \le R_1 \\ C_6 \exp\left[-\left(R_2-r\right)^2/R_3^2\right], & r \ge R_1 \end{cases} \qquad (35)$$

where U_r and U_θ denote the radial and tangential components of the horizontal flow $\mathbf{U}$, in a cylindrical-polar coordinate system ($R=|\mathbf{x}|$, $\theta=\tan^{-1}(x_2/x_1)$) with origin at the center of the vortex ring. Following Mapp *et al* (1985) the following values of the parameters have been selected to describe the vortex-ring:

$$C_5 = 0.9\,\text{m/sec}, \quad C_6 = 1.0\ \text{m/sec}, \quad N = 2 ,$$

$$R_1 = 343.706\,\text{m}, \quad R_2 = 384.881\,\text{m}, \quad R_3 = 126.830\,\text{m} .$$

The vortex ring creates a shearing current with maximum tangential velocity 1m/s, see Fig.2. We consider unit-amplitude harmonic waves of period *T*=20s, propagating along the x_1-axis, with phase velocity *C*=9.74m/s, and scattered by the above vortex-ring current. In the examined case, a 2km by 2km horizontal domain is considered, which is discretised by using 201 equidistant points along each horizontal direction. Numerical results obtained by the present model (34) are shown in Fig. 2 as concerns the real part of the wave field on the free surface, and in Fig.3 concerning the horizontal distribution of its amplitude, respectively. The focusing and defocusing of wave energy in the area downwave the vortex ring are well reproduced.

The case of rip current: Wave-induced rip currents, created by longshore currents converging into periodic rips and forming independent coastal circulation cells, play an important role in coastal morphodynamics. Also, rip currents, in conjunction with local amplification of wave energy, are responsible for many accidents in beaches. In order to illustrate the effects of a rip current, in conjunction with slow changes of the bathymetry, on the wave scattering, we examine a sloping beach of uniform slope 1/50. Using the expressions by Chen *et al* (2005), the structure of the rip current $\mathbf{U}=(U_1,U_2)$ considered here is modelled as follows:

$$U_1 = -0.072\ x_1\ F\left(x_1/76.2\right) F\left(x_2/7.62\right), \qquad (36a)$$

$$U_2 = -0.256\left[2-\left(x_1/76.2\right)^2\right] F\left(x_1/76.2\right) \int_{\tau=0}^{\tau=x_2/7.62} F(\tau)\, d\tau \,, \quad (36b)$$

where $F(\tau) = (2\pi)^{-1/2} \exp\left(-\tau^2/2\right)$. The structure of the current is shown in Figs. 4 and 5, and, its maximum value is selected about 0.3m/s. We consider unit-amplitude harmonic waves of period T=16s (corresponding to a swell) propagating along the x_1-axis (normally to the bottom contours). In this case, the phase velocity of the incident wave is C=6.2m/s. The scattered wave field by the above configuration as calculated by means of the solution of the present modified mild-slope equation (34) is plotted in Fig.4 (phase) and Fig.5 (amplitude). In the examined case, a 350m by 200m horizontal domain is considered, which is discretised by using 151 equidistant points along each horizontal direction. In this case, a significant focusing of wave energy in the shallow-end of the variable bathymetry region is observed, immediately shorewards the formation of the rip current, which agrees well with observations and results by other models (see, e.g., Chen *et al* 2005, Fig.6).

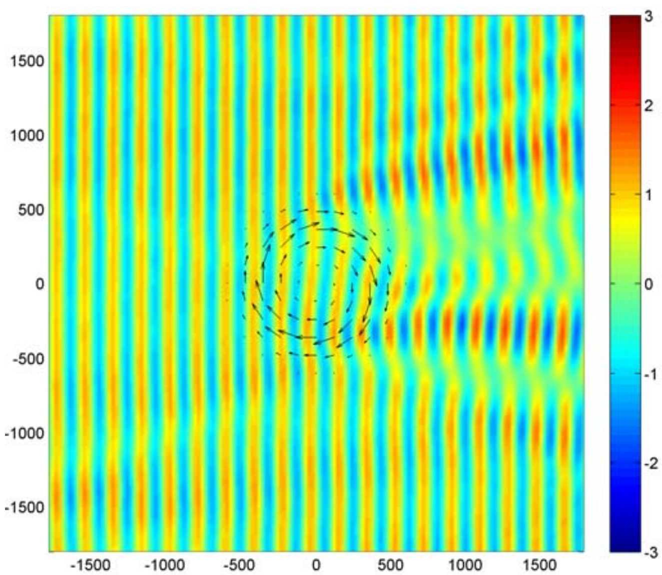

Fig.2. Real part of the scattered wave field on the free surface.

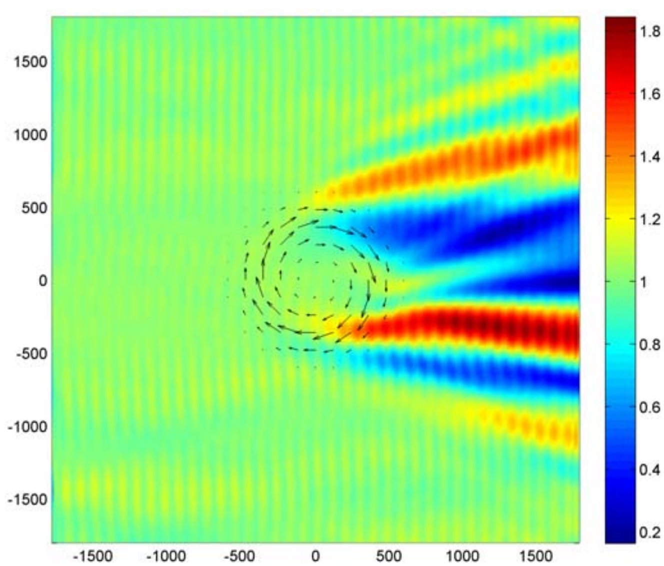

Fig. 3. Amplitude of the wave field in comparison with the incident wave amplitude, in constant depth region (h=10m).

Parallel architecture implementation: The discretized version of the present coupled-mode system of horizontal differential equations, Eqs. (30), leads to very large sparse systems of equations. The computation cost is expected to be huge, especially in realistic applications where the horizontal dimensions of the physical domain may be large, of the order of tens of kilometers. In order to efficiently implement the present method, a parallel code is currently under development, using the message passing paradigm (Wilkinson & Allen, 2005), open source compilers and libraries on a commodity computer cluster. The hardware configuration of each node contains two AMD Opteron 265 dual core processors at 1.8 GHz, 4Gb memory, and gigabit Ethernet. The nodes are interconnected with a Gigabit switch. The above programming model and hardware architecture has been selected because of its great portability, low cost of implementation in comparison with the performance, and great scalability. In order to maximize the portability of the code, the

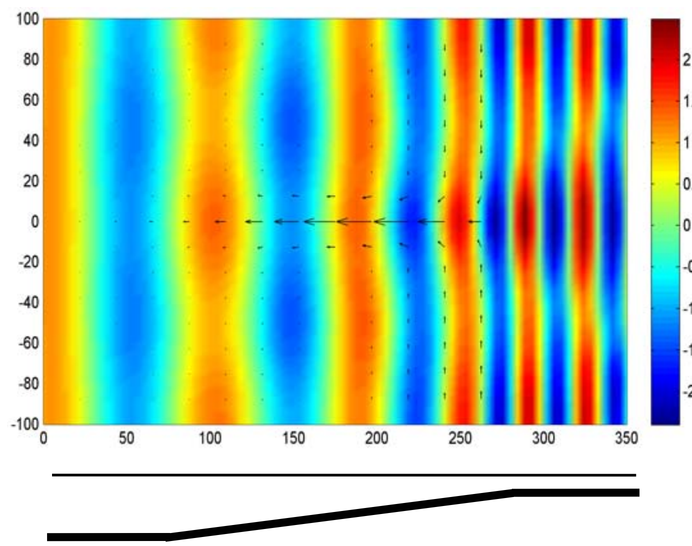

Fig.4. Real part of the scattered wave field on the free surface by a rip current in a sloping beach region

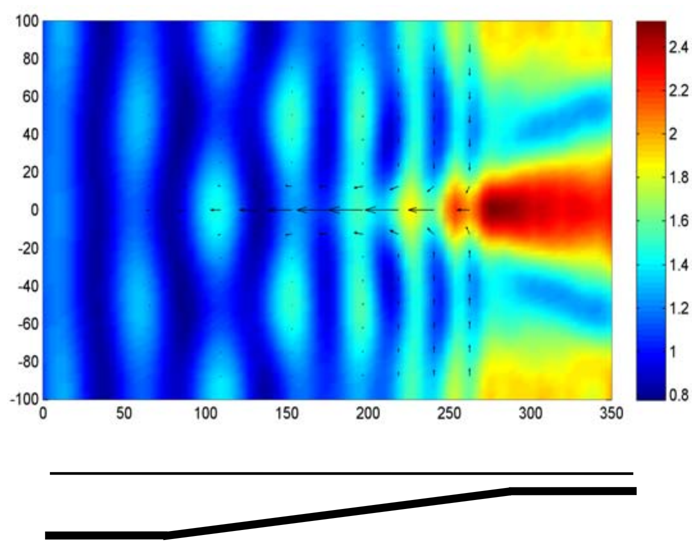

Fig.5. Amplitude of the scattered wave field by a rip current in a sloping (1/50) beach region, with depths ranging from 4m to 0.5m.

model is implemented using ANSI C++ programming language and the Message Passing Interface standard (MPI, 1994). The concurrency of the calculations is achieved by decomposition of the resulting very large systems of linear algebraic equations with complex coefficients, and, in the case of incident wave systems, by separation of the problems associated to different frequencies/ incident wave directions. The large sparse linear systems of equations are solved using distributed direct and iterative solvers, using the PETSc (Balay *et al* 2004) and SuperLU (Demmel *et al* 1999) libraries. These libraries are open-source, well-tested and can be compiled on different architectures (see, e.g., Gerostathis *et al*, 2005).

CONCLUSIONS

A coupled-mode technique for wave-current problems is presented, with application to wave scattering by ambient currents in variable bathymetry regions. Based on an appropriate variational principle, in conjunction with a rapidly-convergent local-mode series expansion of the wave potential in a finite subregion containing the current variation and the bottom irregularity, the present system can be considered as a generalization of the one derived by Athanassoulis & Belibassakis (1999) for the propagation of waves in variable bathymetry regions. The key feature of the present method is the introduction of an additional mode, describing the influence of the bottom slope, and accelerating the convergence of the local-mode series. If only the propagating mode is retained in the vertical expansion of the wave potential, the above coupled-mode system is reduced to the one-equation enhanced model called the *modified mild-slope equation for wave-current flow*, generalizing the mild-slope equation derived by Kirby (1984). Finally, the analytical structure of the present system facilitates its extension to various directions as, e.g., to non-linear wave-current problems and more general current profiles.

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