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A Coupled-Mode Technique for Wave-Current Interaction in Variable Bathymetry Regions

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ABSTRACT

A coupled-mode technique for wave-current interaction is presented, with application to the problem of wave scattering by steady currents in variable bathymetry regions, and current variations on various scales. We consider obliquely incident waves on a horizontally non-homogeneous current in a variable-depth strip, which is characterized by straight and parallel bottom contours. The flow associated with the current is assumed to be parallel to the bottom contours (along-axis current) and it is considered to be known. In a finite subregion containing the bottom irregularity we assume an arbitrary horizontal current structure. Outside this region, the current is assumed to be uniform (or zero). At a first-order of approximation the wave flow is assumed to be irrotational, i.e. the vorticity of the total field is the same with the vorticity associated with the current. Then, restricting ourselves to linear, monochromatic (harmonic) waves of absolute frequency ω, the wave potential, including the scattering effect by the current, is obtained as a solution to the modified Helmholtz equation, subject to the free-surface boundary condition formulated with respect to the intrinsic frequency, the bottom boundary condition, and the conditions at infinity. Based on an appropriate variational principle, in conjunction with a rapidly-convergent local-mode series expansion of the wave field in a finite subregion containing the current variation and the bottom irregularity, a coupled-mode system is obtained that can be considered as a generalization of the one derived by Athanassoulis & Belibassakis (1999). The present approach can be considered as an extension of the works by Smith (1983, 1987), and some of its main features are that it can be further elaborated to treat lateral discontinuities (e.g. vertical vortex sheets) and more general vertical current profiles with cross-jet component, and to include the effects of weak nonlinearity.

KEY WORDS: wave-current interaction, scattering of waves by current, variable bathymetry, coupled-modes.

INTRODUCTION

The characteristics of surface gravity waves could present significant variations as they propagate through non-homogeneous currents, and these variations are further modified by the effects of depth inhomogeneities that occur in variable bathymetry regions. For example, the wave amplitudes could present a significant enhancement within the streaks associated with 'Langmuir' circulation, which is generated by a pattern of alternating horizontal roll vortices and plays an important role in the kinematics and dynamics of air-sea interaction; see, e.g., Smith (2001). Furthermore, large amplitude waves can be produced in cases when obliquely propagating waves interact with opposing currents, see, e.g., Mei (1983, Ch.3.7). This situation could be further enhanced by inshore effects due to sloping seabeds, and has been reported to be connected with the appearance of "giant waves"; see, e.g., Faukner (2000), Dysthe (2000).

The study of spatial variation of waves and the investigation of scattering of realistic wave spectra over irregular sub-wavelength scale currents, with the effects of bottom irregularities, can be supported by theoretical models treating the simpler problem of monochromatic waves interacting with steady currents. Such kind of models have been developed for surface waves crossing weak current jets or steps with horizontal shear, see, e.g., Evans (1975), Smith (1983, 1987), McKee (1974, 1994).

In this work we consider the problem of scattering of obliquely incident waves on a horizontally non-homogeneous steady current, in a variable-bathymetry region, characterized by straight and parallel bottom contours. The flow associated with the current is assumed to be parallel to the bottom contours (along-axis current) and it is considered to be given. In a finite subregion we assume an arbitrary current structure, as, e.g., a monotonic one or a periodic one with characteristic width L. Outside this region, the current is assumed to be uniform (or negligible). We also assume (at a first-order of approximation) that the wave flow is irrotational. Then, restricting ourselves to linear, monochromatic (harmonic) waves, periodic in the y-direction, the wave potential, including the scattering effect by the current, can be obtained as a solution to the modified Helmholtz equation, subject to the free-surface boundary condition formulated with respect to the intrinsic frequency, the bottom boundary condition, and the appropriate conditions at infinity.

The above problem is very conveniently treated, for all scales concerning the width and depth of the current and the depth of the strip in comparison with the incident wavelength, by a generalisation of the coupled-mode model by Athanassoulis & Belibassakis (1999) for waves propagating over variable bathymetry regions.

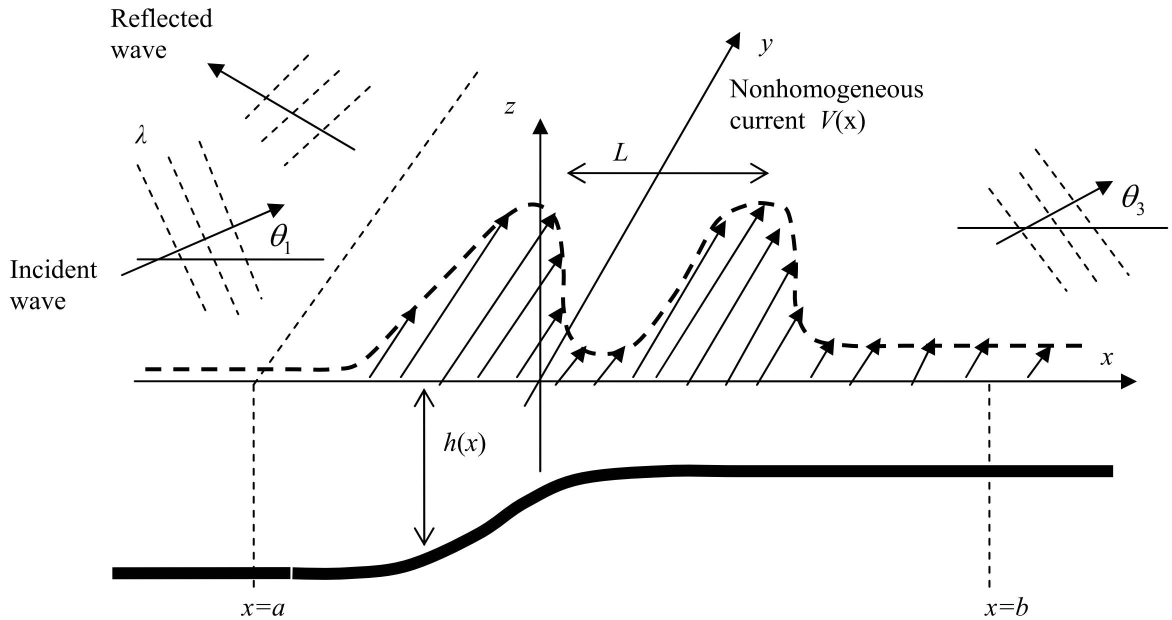

Figure 1. Geometrical configuration and basic notation

The latter model has been extended to 3D by Belibassakis *et al* (2001) and to the (weakly) non-linear case by Belibassakis & Athanassoulis (2002). The present derivation is based on an appropriate variational formulation of the problem, in conjunction with a rapidly convergent local-mode series expansion of the wave field in a finite subregion containing the current variation and the bottom irregularity. In order to illustrate the effects of the structure of the current and of the bottom slope on the wave characteristics, numerical results are presented and discussed in the case of waves propagating through a monotonic and a periodic current, in constant and in variable depth strips.

DIFFERENTIAL FORMULATION OF THE PROBLEM

The studied marine environment consists of a water layer D_{3D} bounded above by the free surface $\partial D_{F,3D}$ and below by a rigid bottom $\partial D_{II,3D}$. It is assumed that the bottom surface exhibits an arbitrary one-dimensional variation in a subdomain of finite length, i.e. the bathymetry is characterised by straight and parallel bottom contours lying between two regions of constant but possibly different depth, $h = h_1$ (region of incidence) and $h = h_3$ (region of transmission); see Fig. 1. A Cartesian coordinate system is introduced, with its origin at some point on the mean water level (in the variable bathymetry region), the *z*-axis pointing upwards and the *y*-axis being parallel to the bottom contours. The liquid domain is $D_{3D} = D \times R$, where D is the (two-dimensional) intersection of D_{3D} by a vertical plane perpendicular to the bottom contours, $D = \{(x,z) : x \in R, \, -h(x) < z < 0\}$, and $R = (-\infty, +\infty)$.

The function $h(x)$, appearing in the above definitions, represents the local depth, measured from the mean water level. It is considered to be a twice continuously differentiable function defined on the real axis *R*, such that

$$h(x) = h(a) = h_1, \text{ for all } x \le a, \quad h(x) = h(b) = h_3, \text{ for all } x \ge b. \tag{2.1a}$$

The strip D is further decomposed in three subdomains $D^{(i)}, i = 1,2,3,$ where $D^{(1)}$ and $D^{(3)}$ are constant-depth subdomains corresponding to $x < a$ and $x > b$, respectively, and $D^{(2)}$ is the variable bathymetry subdomain lying between $D^{(1)}$ and $D^{(3)}$. Without loss of generality, we assume $h_1 > h_3$. The same decomposition is also applied to the free-surface and the bottom boundaries. Finally, we define the vertical interfaces $\partial D_I^{(12)}$ and $\partial D_I^{(23)}$ separating the three subdomains. The latter are vertical segments (between the bottom and the mean water level) at $x = a$ and $x = b$, respectively.

In this work we consider a simple version of the scattering problem of monochromatic, obliquely incident plane waves, propagating with direction θ_1 with respect to the bottom contours in the region of incidence, under the combined effects of variable bathymetry and a horizontally non-homogeneous shear current, existing in $x > a$; see Fig.1. The flow associated with the shear current is considered to be steady and directed parallel to the bottom contours (i.e. along the *y*-axis). Moreover, the steady free-surface displacement associated with the current flow is assumed to be negligible. The horizontal current structure is described by the (given) continuous function *V*(*x*), which can be general in the intermediate region, $a \le x \le b$, as, e.g., a monotonic one or a periodic one with characteristic width *L*. Outside this region, the current is assumed to be uniform (or simply zero),

$$V(x) = V_1 = 0, \quad x \le a, \qquad \text{and} \qquad V(x) = V_3, \quad x \ge b. \tag{2.1b}$$

Assuming that the wave flow is irrotational, i.e. the vorticity of the total field is the same as the vorticity associated with the current $\boldsymbol{V}=\left(0,V\left(x\right),0\right)$, and restricting ourselves to linear, monochromatic (harmonic) waves of absolute frequency ω, periodic in the y-direction, the wave potential can be expressed in the form

$$\Phi\left(x,y,z;t\right)=Re\left(\phi\left(x,y,z\right)exp\left(-i\omega t\right)\right), \tag{2.2a}$$

where

$$\phi\left(x,y,z\right)=\varphi\left(x,z\right)exp\left(i\left(qy-\omega t\right)\right). \tag{2.2b}$$

In the above equations $\phi\left(x,y,z\right)$ is the complex wave potential and $\varphi\left(x,z\right)$ the reduced one on the vertical plane, q is the periodicity constant along the y-direction, and $i=\sqrt{-1}$.

Under the stated assumptions, the problem is governed by the Laplace equation on $\phi\left(x,y,z\right)$ or the modified Helmholtz equation on $\varphi\left(x,z\right)$, the no-entrance bottom-boundary condition and the free-surface condition,

$$\frac{\partial^2\varphi}{\partial^2 x}+\frac{\partial^2\varphi}{\partial^2 z}-q^2\varphi=0, \qquad \frac{\partial\varphi}{\partial z}+\frac{dh}{dx}\frac{\partial\varphi}{\partial x}=0,\ z=-h\left(x\right),$$

$$\frac{\partial\varphi}{\partial z}+\mu\left(x\right)\varphi=0,\ z=0\ , \tag{2.3}$$

where $\sigma=\sigma\left(x\right)=\omega-qV\left(x\right)$ is the local intrinsic frequency, $\mu\left(x\right)=\sigma^2/g$ the corresponding frequency parameter, and g is the acceleration due to gravity. In the examined case, the complex pressure p is proportional to the complex wave potential ($p=i\rho\omega\phi$, where ρ is the fluid density), and the above equations are in compatibility with Eqs. (6.17), (6.19) and (6.23) of Mei (1983, Ch. 3.6).

The problem of water-wave scattering by current, with the effects of variable bathymetry, can be formulated as a transmission problem in the bounded subdomain $D^{(2)}$, with the aid of the following general representations of the wave potential $\varphi\left(x,z\right)$ in the semi-infinite strips $D^{(1)}$ and $D^{(3)}$ (see, e.g., Smith 1983, 1987):

$$\varphi^{(1)}\left(x,z\right)=\left(A_0\exp\left(ik_0^{(1)}x\right)+A_R\exp\left(-ik_0^{(1)}x\right)\right)Z_0^{(1)}\left(z\right)+ \sum_{n=1}^{\infty}C_n^{(1)}Z_n^{(1)}\left(z\right)\exp\left(k_n^{(1)}\left(x-a\right)\right) \quad in\ D^{(1)}, \tag{2.4a}$$

$$\varphi^{(3)}\left(x,z\right)=A_T\exp\left(ik_0^{(3)}x\right)Z_0^{(3)}\left(z\right)+\sum_{n=1}^{\infty}C_n^{(3)}Z_n^{(3)}\left(z\right)\exp\left(k_n^{(3)}\left(b-x\right)\right), \quad in\ D^{(3)}. \tag{2.4b}$$

The terms $\left(A_0\exp\left(ik_0^{(1)}x\right)+A_R\exp\left(-ik_0^{(1)}x\right)\right)Z_0^{(1)}\left(z\right)$ and $A_T\exp\left(ik_0^{(3)}x\right)Z_0^{(3)}\left(z\right)$ in the series (2.4) are the *propagating modes*, while the remaining ones $\left(n=1,2,\ldots\right)$ are the *evanescent modes*. In the above expansions, the wavenumbers

$$k_0^{(i)}=\sqrt{\left(\kappa_0^{(i)}\right)^2-q^2},\quad k_n^{(i)}=\sqrt{\left(\kappa_n^{(i)}\right)^2+q^2},\ n=1,2,3,....,\ i=1,3, \tag{2.5a}$$

where $\left\{i\kappa_0^{(i)},\kappa_n^{(i)},n=1,2,\ldots\right\}$, $i=1,3$, are obtained as the roots of the following dispersion relations

$$\mu\ h_i=-\kappa^{(i)}h_i\tan\left(\kappa^{(i)}h_i\right),\qquad \mu=\sigma^2/g,\quad \sigma=\omega-pV_i,\quad i=1,3. \tag{2.5b}$$

The functions $\left\{Z_n^{(i)}\left(z\right),n=0,1,2,\ldots\right\}$ appearing in Eqs. (2.4) are given by

$$Z_0^{(i)}\left(z\right)=\frac{\cosh\left(\kappa_0^{(i)}\left(z+h_i\right)\right)}{\cosh\left(\kappa_0^{(i)}h_i\right)},\qquad Z_n^{(i)}\left(z\right)=\frac{\cos\left(\kappa_n^{(i)}\left(z+h_i\right)\right)}{\cos\left(\kappa_n^{(i)}h_i\right)},\quad n=1,2,\ldots,i=1,3. \tag{2.6}$$

Since the current is zero in $D^{(1)}$, $\sigma=\omega$, and thus, $p=\kappa_0^{(1)}\sin\theta_1$. The direction of the transmitted wave in $D^{(3)}$ is then given by

$$\theta_3=\sin^{-1}\left(\kappa_0^{(1)}\sin\theta_1/\kappa_0^{(3)}\right). \tag{2.7}$$

Given the representations (2.4), the problem can be re-formulated as a transmission boundary value problem in the bounded subdomain $D^{(2)}$, consisting of the following equations, boundary and matching conditions:

$$\nabla^2\varphi^{(2)}-p^2=0,\quad \left(x,z\right)\in D^{(2)}, \tag{2.8a}$$

$$\frac{\partial\varphi^{(2)}}{\partial n^{(2)}}-\mu\left(x\right)\varphi^{(2)}=0,\qquad \left(x,z\right)\in\partial D_F^{(2)}, \tag{2.8b}$$

$$\frac{\partial\varphi^{(2)}}{\partial n^{(2)}}=0,\qquad \left(x,z\right)\in\partial D_{II}^{(2)}, \tag{2.8c}$$

$$\varphi^{(2)}=\varphi^{(1)},\qquad \frac{\partial\varphi^{(2)}}{\partial n^{(2)}}=-\frac{\partial\varphi^{(1)}}{\partial n^{(1)}},\quad \left(x,z\right)\in\partial D_I^{(12)}, \tag{2.8d,e}$$

$$\varphi^{(2)}=\varphi^{(3)},\qquad \frac{\partial\varphi^{(2)}}{\partial n^{(2)}}=-\frac{\partial\varphi^{(3)}}{\partial n^{(3)}},\quad \left(x,z\right)\in\partial D_I^{(23)}, \tag{2.8f,g}$$

where $n^{(i)}=\left(n_x^{(i)},n_z^{(i)}\right)$ denotes the unit normal vector to the boundary $\partial D^{(i)}$ directed to the exterior of $D^{(i)}$, $i=1,2,3$. The normal derivative of the wave potential at the bottom $\partial D_{II}^{(2)}$ can also be expressed in the form:

$$\frac{\partial\varphi^{(2)}}{\partial n^{(2)}}=-\left(1+\left(\frac{dh}{dx}\right)^2\right)^{-1/2}\left(\frac{\partial\varphi^{(2)}}{\partial z}+\frac{dh}{dx}\frac{d\varphi^{(2)}}{dx}\right). \tag{2.9}$$

VARIATIONAL FORMULATION OF THE PROBLEM

The problem (2.8) admits an equivalent variational formulation, which will serve as the basis for the derivation of an equivalent coupled-mode system of horizontal equations. Consider the functional:

$$F\left(\varphi^{(2)}, A_R, \left\{C_n^{(1)}\right\}_{n\in N}, A_T, \left\{C_n^{(3)}\right\}_{n\in N}\right) = \frac{1}{2}\int_{D^{(2)}} \left(\left(\nabla\varphi^{(2)}\right)^2 + p^2\left(\varphi^{(2)}\right)^2\right) dV$$

$$-\frac{1}{2}\mu \int_{\partial D_F^{(2)}} \left(\varphi^{(2)}\right)^2 dS +$$

$$+\int_{\partial D_I^{(12)}} \left(\varphi^{(2)} - \frac{1}{2}\varphi^{(1)}\left(A_R, \left\{C_n^{(1)}\right\}_{n\in N}\right)\right) \cdot \frac{\partial\, \varphi^{(1)}\left(A_R, \left\{C_n^{(1)}\right\}_{n\in N}\right)}{\partial\, n^{(1)}} dS +$$

$$+\int_{\partial D_I^{(23)}} \left(\varphi^{(2)} - \frac{1}{2}\varphi^{(3)}\left(A_T, \left\{C_n^{(3)}\right\}_{n\in N}\right)\right) \cdot \frac{\partial\, \varphi^{(3)}\left(A_T, \left\{C_n^{(3)}\right\}_{n\in N}\right)}{\partial\, n^{(3)}} dS - A_0 A_R J^{(1)}, \tag{3.1}$$

where $J^{(1)} = 2k_0^{(1)} \int_{z=-h_1}^{z=0} \left(Z_0^{(1)}(z) \right)^2 dz$.

The function $\varphi^{(2)}(x,z),\ (x,z)\in D^{(2)}$ and the coefficients $A_R, \left\{C_n^{(1)}\right\}_{n\in N}$ and $A_T, \left\{C_n^{(3)}\right\}_{n\in N}$ constitute a solution of the problem, if and only if they render the functional F stationary, i.e.

$$\delta F\left(\varphi^{(2)}, A_R, \left\{C_n^{(1)}\right\}, A_T, \left\{C_n^{(3)}\right\}\right) = 0 \ . \tag{3.2}$$

To see this we calculate the first variation δF of the above functional (see also, Athanassoulis & Belibassakis, 1999). Making use of the Green's theorem and the properties of the modal representations (2.4) in the two constant-depth strips $D^{(i)}$, $i = 1,3$, the variational equation (3.2) takes the form:

$$-\int_{D^{(2)}} \left(\nabla^2\varphi^{(2)} - p^2\varphi^{(2)}\right) \delta\varphi^{(2)} dV + \int_{\partial D_{II}^{(2)}} \left(\frac{\partial\varphi^{(2)}}{\partial\, n^{(2)}}\right) \delta\varphi^{(2)} dS +$$

$$+\int_{\partial D_F^{(2)}} \left(\frac{\partial\varphi^{(2)}}{\partial\, z} - \mu\varphi^{(2)}\right) \delta\varphi^{(2)} dS +$$

$$-\int_{\partial D_I^{(12)}} \left(\frac{\partial\varphi^{(2)}}{\partial x} - \frac{\partial\, \varphi^{(1)}}{\partial x}\right) \delta\varphi^{(2)} dS \ + \int_{\partial D_I^{(23)}} \left(\frac{\partial\varphi^{(2)}}{\partial\, x} - \frac{\partial\, \varphi^{(3)}}{\partial\, x}\right) \delta\varphi^{(2)} dS +$$

$$+\int_{\partial D_I^{(12)}} \left(\varphi^{(2)} - \varphi^{(1)}\right) \delta\left(\frac{\partial\varphi^{(1)}}{\partial x}\right) dS - \int_{\partial D_I^{(23)}} \left(\varphi^{(2)} - \varphi^{(3)}\right) \delta\left(\frac{\partial\varphi^{(3)}}{\partial x}\right) dS = 0. \tag{3.3}$$

The functions $\varphi^{(i)}$ and their derivatives $\partial\varphi^{(i)} / \partial x, i = 1,3,$ appearing in the last terms of the left hand side of Eq. (3.3), are considered to be represented by means of their series expansions, Eqs. (2.4), and their horizontal derivatives, respectively. The proof of the equivalence of the variational equation (3.3) and the transmission problem (2.8) is obtained by using standard arguments of the *Calculus of Variations* (see, e.g., Rectorys 1977, ch.22).

THE COUPLED-MODE SYSTEM OF EQUATIONS

The problem on $\varphi^{(2)}(x,z)$ will be treated by an appropriate extension of the consistent coupled-mode theory developed by Athanassoulis & Belibassakis (1999), and is based on the following enhanced local-mode representation of the wave field (in the variable bathymetry region $D^{(2)}$ containing also the current variations):

$$\varphi^{(2)}(x,z) = \varphi_{-1}(x) Z_{-1}(z;x) + \varphi_0(x) Z_0(z;x) + \sum_{n=1}^{\infty} \varphi_n(x) Z_n(z;x) \tag{4.1}$$

In Eq. (4.1) the term $\varphi_0(x) Z_0(z;x)$ is the *propagating mode* of the wave field and the remaining terms $\varphi_n(x)\ Z_n(z;x),\ n = 1,2,\ldots$ are the *evanescent modes*. The additional term $\varphi_{-1}(x) Z_{-1}(z;x)$ is a correction term called the *sloping-bottom mode,* which accounts for the bottom boundary condition on the sloping parts of the bottom and which identically vanishes on the horizontal parts of the bottom. The function $Z_n(z;x)$ represents the vertical structure of the n-th mode. The function $\varphi_n(x)$ describes the horizontal pattern of the n-th mode and is called the *complex amplitude* of the n-th mode. The functions $Z_n(z;x)$, $n = 0,1,2\ldots$, appearing in Eq. (4.1) are obtained as the eigenfunctions of local vertical *Sturm-Liouville* problems, and are given by

$$Z_0(z;x) = \frac{\cosh\left[\kappa_0(x)\left(z + h(x)\right)\right]}{\cosh\left(\kappa_0(x) h(x)\right)}, \tag{4.2a}$$

$$Z_n(z;x) = \frac{\cos\left[\kappa_n(x)\left(z + h(x)\right)\right]}{\cos\left(\kappa_n(x) h(x)\right)}, \quad n = 1,2,\ldots, \tag{4.2b}$$

where the eigenvalues $\left\{ i\kappa_0(x), \kappa_n(x)\right\}$ are obtained as the roots of the local dispersion relation

$$\mu(x) h(x) = -\kappa(x) h(x) \tan\left[\kappa(x)\ \ h(x)\right], \qquad a \le x \le b\ . \tag{4.2c}$$

A specific convenient form of the function $Z_{-1}(z;x)$ is given by

$$Z_{-1}(z;x) = h(x) \left[\left(\frac{z}{h(x)}\right)^3 + \left(\frac{z}{h(x)}\right)^2\right], \tag{4.2d}$$

and all numerical results presented in this work are based on the above choice for $Z_{-1}(z;x)$. However, other choices are also possible. The main effect of the additional sloping-bottom mode $\varphi_{-1}(x) Z_{-1}(z;x)$ is that it makes the series (4.1) compatible with the bottom boundary condition (2.8c) on the sloping parts of the bottom surface and, at the same time, it significantly accelerates the convergence of the local-mode series. For more details about the role and significance of this term see Athanassoulis & Belibassakis (1999, Sec. 4).

By using the local-mode series representation (4.1) in the variational principle (3.3), and by following exactly the same procedure as in Athanassoulis & Belibassakis (1999), the following coupled-mode system is obtained:

$$\sum_{n=-1}^{\infty} a_{mn}(x)\varphi_n''(x)+b_{mn}(x)\varphi_n'(x)+\left(c_{mn}(x)-a_{mn}q^2\right)\varphi_n(x)=0, \tag{4.3}$$

$$a<x<b, \qquad m=-1,0,1,....$$

where a prime denotes differentiation with respect to *x*. The coefficients a_{mn}, b_{mn}, c_{mn} of the system (4.3) can be found in Table 1 of Athanassoulis & Belibassakis (1999). The system (4.3) is supplemented by the following decoupled end-conditions

$$\varphi_{-1}(a)=\varphi_{-1}'(a)=0\,, \qquad \varphi_{-1}(b)=\varphi_{-1}'(b)=0\,, \tag{4.4a,b}$$

$$\varphi_0'(a)+ik_0^{(1)}\varphi_0(a)=2ik_0^{(1)}\exp\left(i\;k_0^{(1)}a\right), \tag{4.4c}$$

$$\varphi_n'(a)-k_n^{(1)}\varphi_n(a)=0, \qquad n=1,2,..\,, \tag{4.4d}$$

$$\varphi_0'(b)-ik_0^{(3)}\varphi_0(b)=0\,, \tag{4.4e}$$

$$\varphi_n'(b)+k_n^{(3)}\varphi_n(b)=0, \qquad n=1,2,3,\dots\,, \tag{4.4f}$$

where the coefficients $k_n^{(1)}$, $k_n^{(3)}$, n=0,1,2,... are defined by Eqs. (2.5a). Furthermore, the reflection and transmission coefficients (A_R, A_T) appearing in Eqs. (2.4) are obtained from the solution of the coupled-mode system as follows:

$$A_R=\left(\varphi_0(a)-\exp\left(ik_0^{(1)}a\right)\right)\;\exp\left(ik_0^{(1)}a\right), \tag{4.5a}$$

$$A_T=\varphi_0(b)\exp\left(-ik_0^{(3)}b\right)\;. \tag{4.5b}$$

An important feature of the solution of the present scattering problem by means of the representation (4.1), is that it exhibits an improved rate of decay of the modal amplitudes $\left|\varphi_n(x)\right|$ of the order $O\left(n^{-4}\right)$. Thus, a small number of modes suffices to obtain a convergent solution to $\varphi(x,z)$, even for large bottom slopes.

NUMERICAL RESULTS AND DISCUSSION

The construction of the discrete system is obtained by truncating the local-mode series (4.1) to a finite number of terms (modes), retaining a number of evanescent modes, and by using central, second-order finite differences to approximate the derivatives in the coupled mode system (4.3). Discrete boundary conditions are obtained by using second-order forward and backward differences to approximate derivatives at the ends. Thus, the discrete scheme obtained in this way is uniformly of second order in the horizontal direction. The forcing appears only in one equation, at the left endpoint $x=a$ (see Eq. 4.4c).

(i) The case of a smooth underwater shoaling

In order to illustrate the combined effects of variable bathymetry and shear current on the calculated wave field, we examine the case of a smooth but steep underwater shoal, characterised by the following depth function

$$h(x)=\frac{h_1+h_3}{2}-\frac{h_1-h_3}{2}\tanh\left(3\pi\left(\frac{x-a}{b-a}-\frac{1}{2}\right)\right), \qquad a=0<x<b=20m, \tag{5.1}$$

with $h_1=15m$ and $h_3=5m$. This bottom profile has mean slope $s_{\text{mean}}=0.5$ and maximum slope $s_{\max}=2.40$. (A sketch of the bottom topography is shown in Fig. 2). The angular frequency of the incident wave is taken to be $\omega=1.62$ rad/sec (which corresponds to $\kappa_1 h_1=4$, implying almost deep water wave conditions in $D^{(1)}$), and its direction is taken to be $\theta_1=-30^o$. The phase speed of the waves in $D^{(1)}$ is then $c_1=6.06m/s$. We consider also an opposing shear current

$$V(x)=\frac{V_3}{2}+\frac{V_3}{2}\tanh\left(3\pi\left(\frac{x-a}{b-a}-\frac{1}{2}\right)\right), \qquad a=0<x<b=20m, \tag{5.2}$$

where $V_3=max V=0.5c_1$. In the region of transmission $D^{(3)}$ the wave characteristics are $\kappa_3 h_3=2.14$ $\left(k_3 h_3=1.48\right)$ and $\theta_3=-18.1°$. The effects of the shear current on the wave are shown in Fig. 2 on both the horizontal and vertical planes by using equipotential lines. (Only the real part of the wave field is plotted). The values of the wave potential on the free surface $\varphi(x,z=0)$ are also included in this figure, from which the free-surface elevation is obtained,

$$\eta(x,y;t)=\frac{\left(\omega-qV(x)\right)}{g}Re\left(i\varphi(x,z=0)exp\left(i\left(qy-\omega t\right)\right)\right). \tag{5.3}$$

In this case, the calculated values of the reflection and transmission coefficients are $\left|A_R\right|=0.14, \left|A_T\right|=0.71$. The previous result is to be compared with the corresponding one whitout the effect of the shear current shown in Fig.3, which is also obtained by the present method by using $V(x)=0$. In this case, $\theta_3=-26.8°$, and the values of the reflection and transmission coefficients are $\left|A_R\right|=0.014, \left|A_T\right|=0.91$. A small number of modes (totally 5 terms) have been retained in the modal series expansions, which has been proved enough for numerical convergence, even for such large gradients of the depth function and of the shear current. In all cases we observe that the equipotential lines intersect the bottom surface perpendicularly, which is evidence of satisfaction of the bottom boundary condition, both on the horizontal and on the sloping parts of the bottom.

By systematically varying the depth and the current profiles, using as shape functions Eqs. (5.1) and (5.2), respectively, we present in Fig. 4 numerical predictions concerning the reflection coefficient in the case of a steep shoal, for various depth and current variations. In this case, the incident wave conditions have been fixed to $\kappa_1 h_1=4$ and $\theta_1=-30°$, the shoaling ratio is taken to vary from $h_3/h_1=0.025\div 0.5$ (and thus, $\kappa_1 h_3=0.1\div 2$), and the strength of the current from $V_3/c_1=-2\div 2$. The situation closely resembles the conditions of waves crossing a step with horizontal shear for which results are available from Smith (1987). The present method provides compatible results with the ones presented in Fig. 6 of Smith (1987).

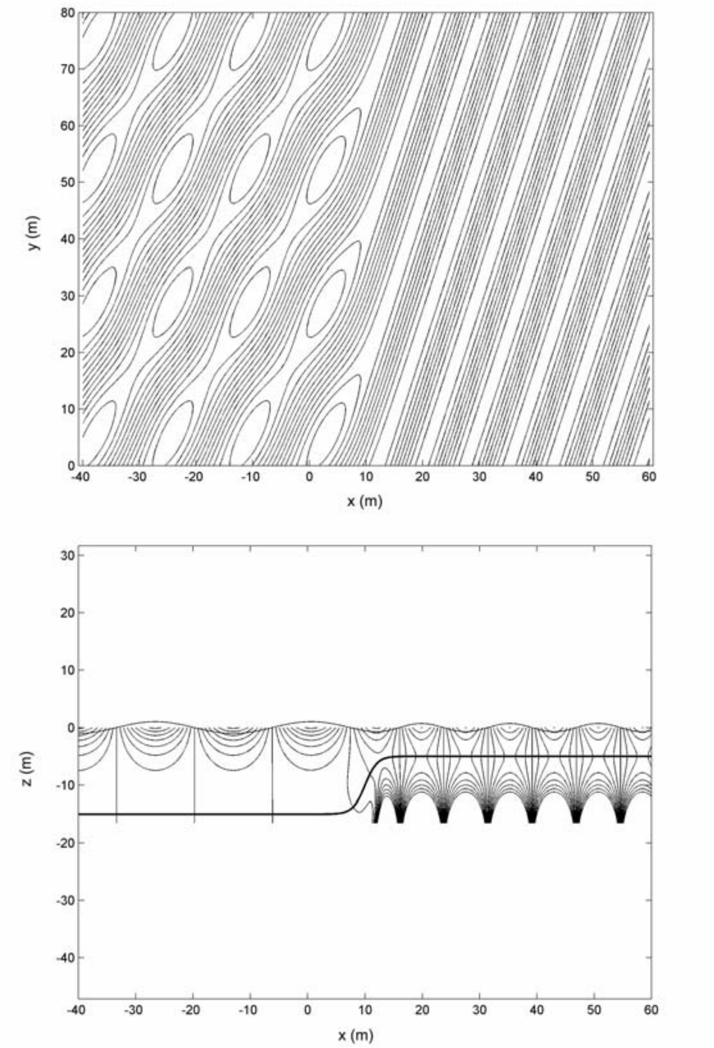

Figure 2 Calculated wave field in the case of oblique waves propagating with direction -30deg over a smooth and steep shoal and under the effects of an opposing shear current.

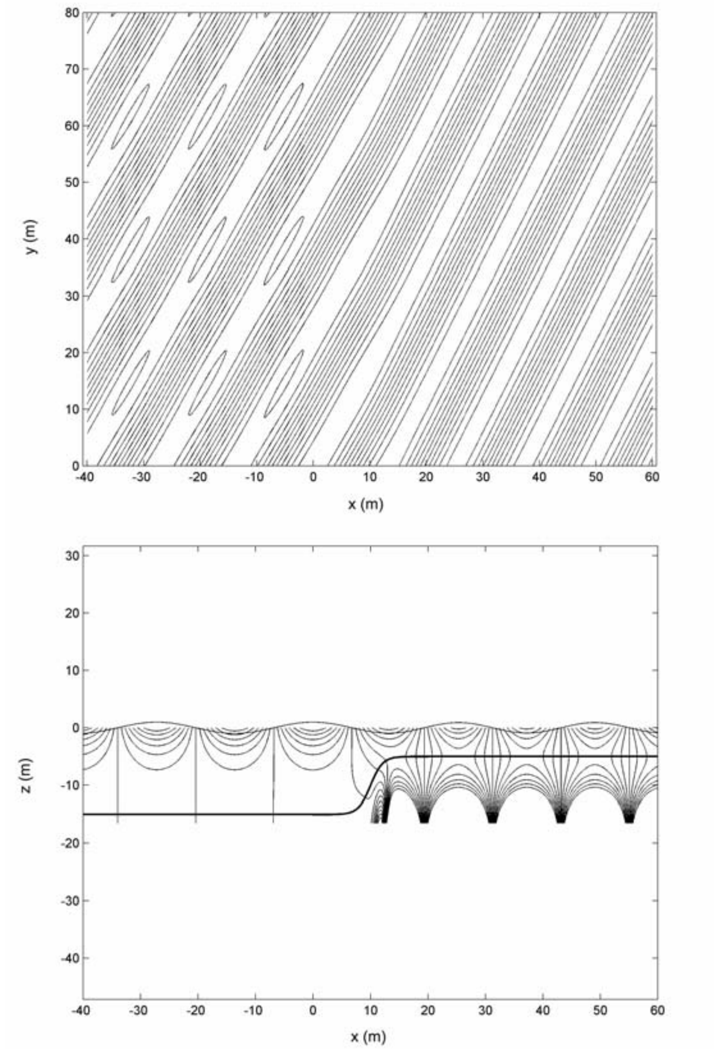

Figure 3 Same as in Fig.2, but without current. The extension of the equipotential lines below the bottom surface is maintained in order to better visualize the fulfillment of the bottom boundary condition.

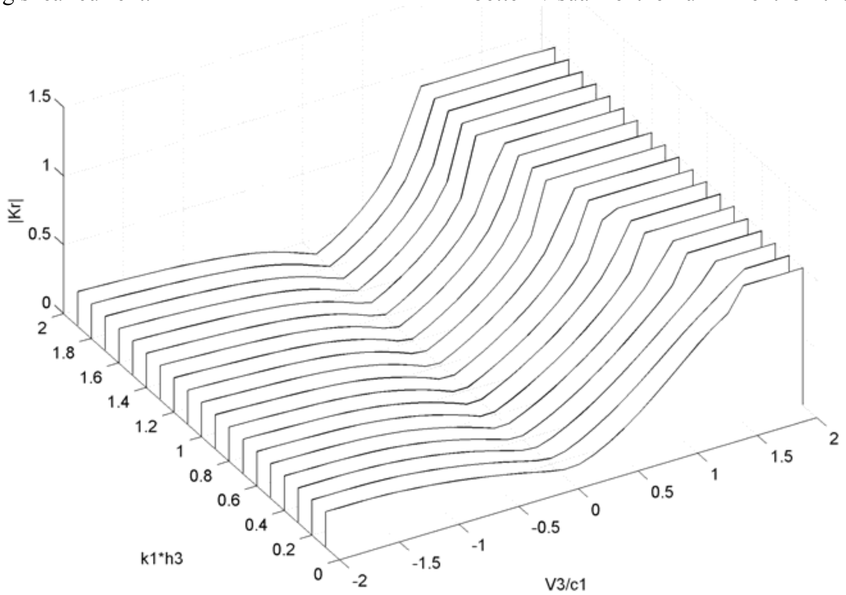

Figure 4 Calculated reflection coefficient in the case of a steep shoal, for various depth and current variations. Incident wave conditions: $\kappa_1 h_1 = 4$ and $\theta_1 = -30^o$

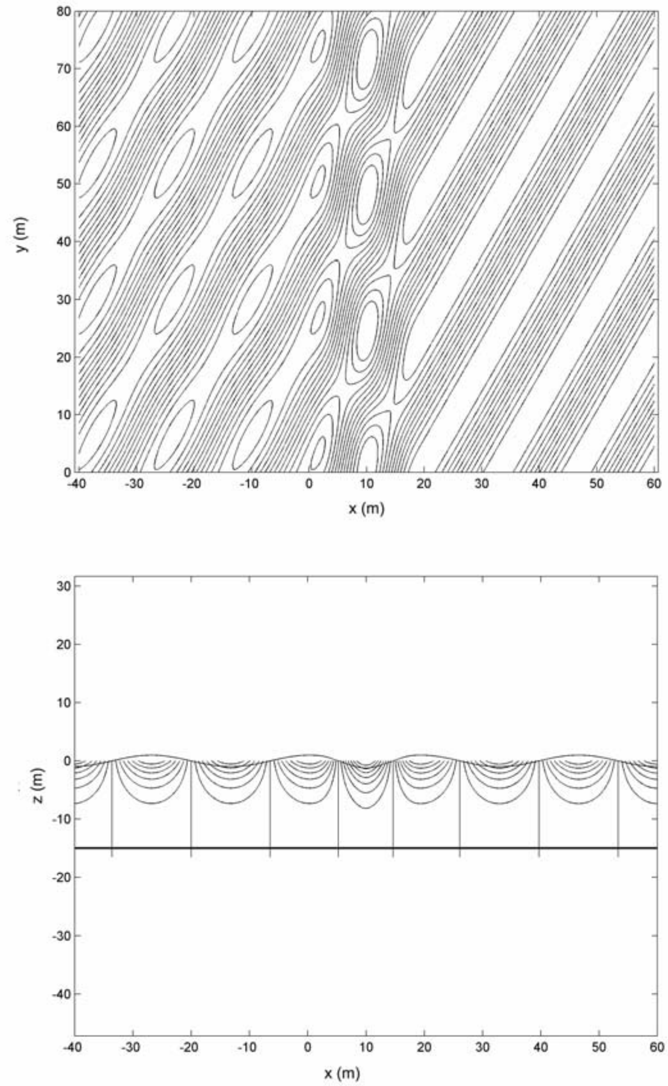

Figure 5 Calculated wave field in the case of oblique waves propagating with direction -30deg in constant depth under the effects of an opposing sinusoidal shear current.

(ii) The case of a sinusoidal current

As a final example, we consider the case of waves of angular frequency $\omega = 1.62$ rad/sec propagating with direction $\theta_1 = -30°$ in a constant depth strip *h*=15m, under the effects of an sinusoidal shear current with horizontal profile,

$$V(x) = \frac{c_1}{4}\left(1 - cos\left(2\pi\frac{x-a}{L}\right)\right), \quad L = \frac{b-a}{3}, \tag{5.4}$$

In this case, the shear current exists only in the region from $x=a$=0m to $x=b$=20m, it has a periodic structure with characteristic width L=6.67m, and its maximum is equal to the half of the phase speed of waves in $D^{(1)}$ and $D^{(3)}$, which is $c_1 = c_3 = 6.06 m/s$. The real part of the calculated wave field, as obtained by the present method using 5 modes, is shown in Fig. 5, and the values of the reflection and transmission coefficients are now $|A_R| = 0.08, |A_T| = 0.99$.

CONCLUSIONS

A coupled-mode technique for wave-current interaction in variable bathymetry regions is presented, with application to the problem of wave scattering by steady currents with current variations on various scales. The present method does not introduce any simplifying assumptions or other restrictions concerning the bottom slope and curvature, or the vertical structure of the wave field. All wave phenomena (refraction, reflection, diffraction) are fully modelled and, thus, the present method can serve as a useful tool for the analysis of the wave field in the whole range of parameters within the regime of linear theory.

Based on an appropriate variational principle, in conjunction with a rapidly-convergent local-mode series expansion of the wave field in a finite subregion containing the current variation and the bottom irregularity, the present coupled-mode system can be considered as a generalization of the one derived by Athanassoulis & Belibassakis (1999) for the propagation of waves in variable bathymetry regions. The key feature of the present method is the introduction of an additional mode, completely describing the influence of the bottom slope. It turns out that the presence of the additional mode in the local-mode series representation of the potential makes it consistent with the bottom boundary condition and, at the same time, substantially accelerates its convergence.

The analytical structure of the present model facilitates its extension to various directions as, e.g., to three-dimensional problems and to more complex wave-current systems, including the effects of lateral discontinuities (e.g. vertical vortex sheets), the effects of more general vertical current profiles with cross-jet component, and the effects of weak nonlinearity. In concluding, the present model, in conjunction with observations, can support further studies aiming to understand the dynamics of ocean mixing layer and the role of Langmuir circulation.

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