On the Nonlinear Theory for Gravity Waves on the Ocean's Surface. Part II: Interpretation and Applications

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ABSTRACT

In a previous paper (Weber and Barrick, 1977), a generalization of Stokes' perturbational technique permitted us to obtain solutions to higher orders for gravity-wave parameters for an arbitrary, two-dimensional periodic surface. In particular, the second-order wave-height correction and the third-order dispersion relation correction were derived there. In this paper, we interpret and apply those solutions in a variety of ways. First of all, we interpret the dispersion relation (and its higher order corrections) physically, as they relate to the phase velocity of individual ocean wave trains. Second, the validity of the two results derived previously is established by comparisons in the appropriate limiting cases with classical results available from the literature. It is shown how the solutions—derived for periodic surface profiles—can be generalized to include random wave fields whose average properties are to be specified. Then a number of examples of averaged higher order wave parameters are given, and in certain cases a Phillips' one-dimensional wave-height spectral model is employed to yield a quantitative feel for the magnitudes of these higher order effects. Both the derivations and the examples have direct application to the sea echo observed with high-frequency radars, and relationships with the radar observables are established and discussed.

1. Introduction

In a previous paper (Weber and Barrick, 1977) a generalization of Stokes' (1847) perturbation technique was presented which permitted the derivation of higher order corrections to the linear solution of the (nonlinear) hydrodynamic equations describing waves near the air-water interface. The generalization consisted in assuming a Fourier series expansion for the wave height for deep-water gravity waves. The method ignores the dynamics of energy transfer between waves, between the atmosphere and ocean and viscous damping effects. Hence the solution is expected to be valid over space and time scales less in extent than those over which energy transfer variations are important.

While the overall method was general, that paper concentrated upon the derivation of two results: 1) the second-order correction to the wave height (and velocity potential), and 2) the first nonzero correction to the lowest order dispersion relation obtained by carrying the perturbation analysis to third order. Bits and pieces of these derived quantities have appeared from time to time in the literature (e.g., height corrections for two waves at a time, dispersion-relation correction for colinear waves), but this is the first time to our knowledge that Stokes' techniques have been generalized to an infinite field of two-dimensional waves in such a way that both wave height and dispersion-relation corrections can be derived in the same analysis in a self-consistent manner.

It is the purpose of this paper to interpret and apply the solutions derived in the previous paper. In particular: 1) the dispersion relation will be interpreted physically; 2) the validity of the results for second-order wave height and the third-order dispersion-relation correction will be established by comparisons with classical solutions in the appropriate limits; 3) it will be shown how the results for periodic waves can be generalized to include random wave fields whose average properties are desired; and 4) specific sample applications for the average wave-height directional spectrum and dispersion relation mean and variances will be given. The rationale for wanting to know these latter two quantities derives from the interpretation of high-frequency radar echoes from the sea surface, and this application will be discussed briefly.

2. Interpretation of the dispersion relationship

a. Series simplification

The basic Fourier series expansion of the wave-height as given in Eq. (4) of W-B[1] shows the summation indices over $\mathbf{k}$ and ω (two-dimensional space and time) as seemingly independent. They are *not*, however. Having chosen $\mathbf{k}$ and $\eta_1(\mathbf{k},\omega)$ as the independent variables of the problem, all other quantities are dependent upon these variables. Thus to the lowest order $\omega \approx \omega_0$, where ω_0 was derived in Eq. (15) of W-B, and is seen to be a function of $k(\equiv|\mathbf{k}|)$ only. If one wishes to include the first nonzero correction to ω (i.e.,

[1] Hereafter, W-B refers to Weber and Barrick (1977), Part I.

ω_2 since $\omega_1=0$), this was shown in Eq. (29) of W-B to be a function of both $\mathbf{k}$ and $|\eta_1(\mathbf{k},\omega)|^2$. Hence having specified $\mathbf{k}$ and η_1, one is no longer free to choose ω independently. Therefore, η_1 is actually only a function of $\mathbf{k}$. Furthermore, the Fourier representation of wave height is no longer a triple series over $\mathbf{k}$ $(=k_x\hat{x}+k_y\hat{y})$ and ω, but is reduced to an explicit double series over $\mathbf{k}$, where $\hat{x}$ and $\hat{y}$ are unit vectors.

This dependence of ω on $\mathbf{k}$ can be used to simplify the series by the use of Kronecker-delta indicators. For example, suppose that the dispersion relation to the lowest order is adequate for a given purpose. Then the series Eq. (4) of W-B can be rewritten to first order as

$$\eta_1(\mathbf{r},t)=\sum_{\mathbf{k},\omega_0}\eta_1(\mathbf{k},\omega_0)\delta_{\omega_0}^{\pm\sqrt{gk}}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega_0 t)]$$
$$=\sum_{\mathbf{k}}\{\eta_1(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}-\sqrt{gk}t)]+\eta_1^*(\mathbf{k})\times\exp[-i(\mathbf{k}\cdot\mathbf{r}-\sqrt{gk}t)]\},\quad(1)$$

where we have defined and used the fact[2] that $\eta_1(\mathbf{k})\equiv\eta_1(\mathbf{k},\sqrt{gk})$ and $\eta_1^*(\mathbf{k})\equiv\eta_1^*(\mathbf{k},\sqrt{gk})=\eta_1(-\mathbf{k},-\sqrt{gk})$. A similar application of Kronecker deltas can be used to extend ω to its next nonvanishing order (i.e., $\omega=\omega_0+\omega_2$), except that now ω_2 is a function of wave heights $\eta_1(\mathbf{k})$, and hence a series in the exponential argument must be evaluated before the main series is summed. In a like manner, Kronecker-delta relationships between ω_0' (or ω') and k', as well as ω'' and $\mathbf{k}''$, can be used to simplify the series for the second-order wave height [Eq. (22) of W-B]. Hence one of the three summations can be dropped, with a single Kronecker-delta remaining in the double series. Finally, note that a single Kronecker-delta appears in the series for the second-order frequency correction, ω_2 [Eq. (29) of W-B]; this reduces the triple summation over $\mathbf{k}'$, ω' to a double summation over $\mathbf{k}'$, with the wave-height coefficients $\eta_1(\mathbf{k}',\omega')$, as before now only functions of $\mathbf{k}'$, i.e., $\eta_1(\mathbf{k}')$.

b. *Interpretation*

The gravity-wave dispersion relations have a simple interpretation in terms of the phase velocity of a wave train of wave vector $\mathbf{k}$. This quantity is the speed with which the crests of waves of length $2\pi/|\mathbf{k}|$ pass a given point, i.e., $v_{ph}=\omega(\mathbf{k})/k$ (where $k\equiv|\mathbf{k}|$). Since successive terms in the perturbation expansion for ω are small, we can write (to second order)

$$\omega(\mathbf{k})=\omega_0+\omega_2=\omega_0\left(1+\frac{\omega_2}{\omega_0}\right)=\sqrt{gk}\left(1+\frac{\omega_2}{\omega_0}\right),\quad(2)$$

and hence

$$v_{ph}(\mathbf{k})=\sqrt{\frac{g}{k}}\left(1+\frac{\omega_2}{\omega_0}\right)=\sqrt{\frac{g}{k}}[1+\Delta v_{ph}(\mathbf{k})].\quad(3)$$

Therefore, the correction term ω_2/ω_0 to the dispersion equation derived as (29) of W-B represents the correction to the phase velocity of an ocean wave of length $2\pi/k$. The general form of Eq. (29) indicates that this change in phase velocity comes about not only as a result of the existence of that wave alone, but as a result of the presence of all the other waves. Stokes (1847) showed that the phase velocity of a *solitary* wave train tended to increase slightly as its height increased. Longuet-Higgins and Phillips (1962) showed that a wave of length $2\pi/k$ whose own amplitude is infinitesimally small is affected by a second wave moving parallel to it. Our solution contains both of these two results as limiting cases, but also applies to an arbitrary number of waves moving in arbitrary directions.

In order to understand this effect physically, let us specialize (29) to the following two cases. We shall study the normalized phase velocity correction of a sinusoidal wave of length $2\pi/k$ due to both itself and a second wave of wavelength $2\pi/k'$ moveing 1) parallel to the first wave and 2) perpendicular to the first wave. Thus (29) becomes[3]

Parallel waves

$$\Delta v_{ph}(\mathbf{k})=\omega_2/\omega_0=2k^2|\eta_1(\mathbf{k})|^2\pm4\frac{\omega_0'k}{\omega_0}|\eta_1(\mathbf{k}')|^2\times\begin{cases}k & \text{for } k'>k\\ k' & \text{for } k'<k\end{cases}\quad(4)$$

Perpendicular waves

$$\Delta v_{ph}(\mathbf{k})=\omega_2/\omega_0=2k^2|\eta_1(\mathbf{k})|^2+\frac{k^2\omega_0'^2}{\omega_0^2\omega_\perp^2}F_\perp|\eta_1(\mathbf{k}')|^2.\quad(5)$$

In the first equation, the upper/lower sign is used if the second wave (with period $2\pi/k'$) is traveling in the same/opposite direction as the first wave (with period $2\pi/k$), respectively. In the second equation,

$$F_\perp\equiv\omega_0^2+\omega_0'^2-2\omega_\perp^2+\frac{2(\omega_0^2+\omega_0'^2-\omega_\perp^2)(\omega_0^2\omega_0'^2-2\omega_0^4-2\omega_0'^4)}{[\omega_\perp^2-(\omega_0-\omega_0')^2][\omega_\perp^2-(\omega_0+\omega_0')^2]},\quad(6a)$$

where

$$\omega_\perp^2\equiv\sqrt{\omega_0^4+\omega_0'^4}.\quad(6b)$$

[2] While we required that $\eta_1^*(\mathbf{k},\omega)=\eta_1(-\mathbf{k},-\omega)$ we make no similar requirement on $\eta_1(\mathbf{k})$. Therefore, $\mathbf{k}$ is taken to lie in the direction of travel, and $|\eta_1(\mathbf{k})|\neq|\eta_1(-\mathbf{k})|$, since in general we want to allow waves traveling in opposite directions to have different amplitudes. Note, however, that the *total* spatial coefficient (dropping time), i.e., $\eta_1(\mathbf{k})+\eta_1^*(-\mathbf{k})$ *does* satisfy the complex conjugate relationship required for real wave fields.

[3] Note that because of the complex Fourier series representation used here for wave height, the actual amplitude of the sinusoid is $a(\mathbf{k})=2|\eta_1(\mathbf{k})|$.

Examining (4) and (5), we can represent the phase speed change as two separate effects: the self effect (first term) and the mutual effect (second term). Thus the nonzero height of the original wave tends to increase its speed slightly (just as predicted by Stokes, 1847). The second wave acting on the first may increase or decrease its speed, depending upon its direction with respect to the first wave. To obtain some feel for this "mutual" interaction effect, let us consider two lengths and three directions for the second wave (with wavenumber k'). First we rewrite the second wave in terms of its slope [i.e., $s'=2k'|\eta_1(\mathbf{k}')|$], since in a fully developed sea it is the slope which is more nearly maintained at a constant value than the wave height. In Table 1 we present the change in the phase velocity of the first wave with wave vector $\mathbf{k}$ due to the second wave with wave vector $\mathbf{k}'$ when the second wave is i) twice as long ($k'=\frac{1}{2}k$), and ii) half as long ($k'=2k$), and traveling in i) the same direction, ii) the perpendicular direction, and iii) the opposite direction with respect to the first wave.

As a simple summary of these interaction effects (expressed in terms of constant wave slope), the longer, higher second wave produces a greater phase velocity change on the first wave (by a factor of 4) than the shorter, lower second wave; when the wave trains are parallel, the velocity change is the same in magnitude, but its sign depends on whether the waves are moving in the same or opposite directions, entirely as one would expect. (Had the result been expressed in terms of the height of the second wave $2|\eta_1(\mathbf{k}')|$, the magnitude of the phase velocity change would *not* have depended upon the wavelength of the second wave train for any of the directions considered.).

It is interesting to note that there *is* a mutual interaction even when the second wavetrain is moving orthogonally to the first wave train, and this interaction is such as to *increase* the speed of the first wave train. However, this interaction velocity change is small compared with cases when they are aligned or colinear (i.e., it is only 2.78% of the value for colinear alignment), and hence for many purposes the perpendicular interaction might be considered negligible.

3. Comparison with Stokes (1847) and Longuet-Higgins and Phillips (1962)

As discussed previously, Stokes (1847) employed a technique—valid only for a solitary periodic wavetrain—which permitted him to obtain the second-order Fourier correction to the wave-height profile and the higher order correction to the dispersion equation. He showed that the second spatial harmonic (traveling at the same phase speed as the fundamental cosine wave) is also a cosine wave with amplitude $a_2=\frac{1}{2}ka_1^2$, where a_1 is the amplitude of the fundamental cosine wave. If we reduce the exponential series (1) to two terms representing a cosine wave [where $\eta_1^*(\mathbf{k})=\eta_1(\mathbf{k})$],

TABLE 1. Phase velocity change in wave with wave vector $\mathbf{k}$ due to wave with wave vector $\mathbf{k}'$.

	$k'=2k$ (Second wavelength = $\frac{1}{2}\times$first wavelength)	$k'=\frac{1}{2}k$ (Second wavelength = $2\times$first wavelength)
Same directions	$+\frac{\sqrt{2}}{4}s'^2$	$+\sqrt{2}s'^2$
Perpendicular directions	$+\frac{(\sqrt{5}-2)}{24}s'^2$	$+\frac{(\sqrt{5}-2)}{6}s'^2$
Opposite directions	$-\frac{\sqrt{2}}{4}s'^2$	$-\sqrt{2}s'^2$

we obtain from Eq. (22) of W-B $\eta_2(2\mathbf{k},2\omega)=k\eta_1^2(\mathbf{k})$. Since $a_1=2\eta_1$ and $a_2=2\eta_2$, we see that the two results are identical. Furthermore, our second-order wave is seen to move with a phase velocity $\Omega/K=2\omega/2k=\omega/k$, which is exactly the phase velocity of the fundamental or first-order wave; hence the total wave-train profile "stays together." However, the frequency $\Omega=2\omega$ and wavenumber $K=2k$ of the second-order wave do not satisfy the first-order dispersion relation (15) of W-B. That is, $\Omega_0^2=4\omega_0^2=4gk=2gK$, rather than $\Omega_0^2=gK$; hence the second-order component is *not* freely propagating, but is tied to or trapped by the fundamental, which is all consistent with Stokes' analyses.

The first term of (4) represents the normalized correction to the dispersion relation when only one wave is present. This should agree with the normalized phase-velocity increase derived by Stokes (1847) for a single periodic wave train. His normalized phase velocity correction is given as $\frac{1}{2}k^2a_1^2$, which is seen to be identical to the first term when one notes that $a_1=2|\eta_1(\mathbf{k})|$.

The second term of (4) is the normalized phase velocity (or dispersion-relation) correction to the first wave of wavenumber $\mathbf{k}$ due to a second periodic wave which is colinear with the first and having wavenumber $\mathbf{k}'$. Longuet-Higgins and Phillips (1962) specifically analyzed this case, and found the actual phase velocity correction to be $\omega_0'k'a_1'^2$ for $k'<k$ and $\omega_0'ka_1'^2$ for $k'>k$; they also note that the phase velocity correction has a negative sign if the waves are traveling opposite to each other. By using the identity $a_1'=2|\eta_1(\mathbf{k}')|$ in the second term of (4) and multiplying by $\sqrt{g/k}$ to convert from a normalized to an actual phase velocity correction, we see that the two results are identical.

A curious but important fact should be noted about the two terms in (4): if one attempts to make the second wave (with wavenumber k') merge into and replace the first [i.e., let $\eta_1(\mathbf{k})\rightarrow 0$, but allow $\mathbf{k}'\rightarrow\mathbf{k}$] so that only the second term remains, it is *twice* as large as the first. In other words, mutual and self effects on the phase velocity are different, and one cannot predict one effect starting from the other. Thus Stokes analysis gave only the self-effect term, while

Longuet-Higgins and Phillips' (1962) analysis gave only the mutual interaction term. Our generalized analysis, however, gives both terms, and each agrees properly with its respective classical predecessor. One point in common between all three analyses is the initial formulation of the problem in terms of periodic waves (i.e., Fourier series). We note in passing that Huang and Tung (1976), using instead a Fourier-Stieltjes approach, obtained a general expression for the phase velocity correction due to mutual interaction effects for colinear waves [the second term of their Eq. (22)] that does not agree with either our result [Eq. (4)] or Longuet-Higgins and Phillips' (1962) solution. Both of the latter results show that one must switch factors from k to k' depending upon whether $k \lesseqgtr k'$.

4. Generalization to random surfaces

This section will show how the Fourier series representation can be converted to describe a random surface in which averages rather than deterministic descriptions are desired; in the process summations become integrals (in the Riemann sense).

In any practical experiment, one is interested in a description of the sea surface only over some finite area of space, $L_x \times L_y$ and a finite interval of time, T. We assume here that the sea surface is statistically stationary over these intervals. If we are permitted the further assumption that these intervals are much larger than the dominant gravity-wave spatial and temporal periods of interest in the analysis, then Rice has shown (Davenport and Root, 1958) that the real and imaginary parts of the Fourier series coefficients, $\eta(\mathbf{k},\omega)$, describing the sea can be taken to be random variables which become mutually uncorrelated when the spatial/temporal intervals greatly exceed the spatial/temporal wavelengths of the dominant waves. Since we are interested in describing the surface only within these limits, we can take L_x, L_y, T to be the fundamental periods of the Fourier series and allow the surface to be periodic (i.e., repeat itself) for other areas and times. Finally, it has been shown in many places (Kinsman, 1965) that the *first-order* sea-surface height coefficients can be allowed to be zero mean *Gaussian* random variables for most practical purposes.

With these assumptions, we can define the surface wave-height spectrum in terms of the height coefficients (for each order) as

$$\langle \eta_n(\mathbf{k},w)\eta_n^*(\mathbf{k}',w')\rangle = \begin{cases} \dfrac{(2\pi)^3}{L_xL_yT} S_n(\mathbf{k},w) & \text{for } \mathbf{k}'=\mathbf{k} \text{ and } w'=w \\ 0 \text{ for other } \mathbf{k}',w', \end{cases} \tag{7}$$

where $\langle f\rangle$ denotes an ensemble average of f, and $S_n(\mathbf{k},w)$ is the nth order directional wave-height spatial/temporal spectrum for arbitrary spatial wave vector $\mathbf{k}$ and arbitrary temporal wavenumber w. By the zero mean assumption of the preceding paragraph, we have $\langle\eta_1(\mathbf{k},w)\rangle \equiv 0$ for all $\mathbf{k},w$.

If one is interested in the first-order wave-height spatial/temporal spectrum evaluated using the dispersion equation to lowest order, the following relation can be used:

$$\langle \eta_1(\mathbf{k})\eta_1^*(\mathbf{k}')\rangle = \begin{cases} \dfrac{(2\pi)^2}{2L_xL_y} S_1(\mathbf{k}) & \text{for } \mathbf{k}'=\mathbf{k} \\ 0 \text{ for other } \mathbf{k}', \end{cases} \tag{8}$$

along with the last form of (1) to establish the following identity:

$$S_1(\mathbf{k},\omega_0) = \tfrac{1}{2}S_1(\mathbf{k})\delta(\omega_0-\sqrt{gk}) + \tfrac{1}{2}S_1(-\mathbf{k})\delta(\omega_0+\sqrt{gk}), \tag{9}$$

where $\delta(x)$ is the Dirac-delta function.

Finally, if one is concerned with only one-dimensional (colinear) ocean waves so that (1) is a function of only x and t, then we have

$$\langle \eta_n(k,w)\eta_n^*(k',w')\rangle = \begin{cases} \dfrac{(2\pi)^2}{L_xT} S_n(k,w) & \text{for } k'=k,\ w'=w \\ 0 \text{ for other } k',w' \end{cases} \tag{10}$$

and

$$\langle \eta_n(k)\eta_n^*(k')\rangle = \begin{cases} \dfrac{2\pi}{2L_x} S_n(k) & \text{for } k'=k \\ 0 \text{ for other } k' \end{cases} \tag{11}$$

with identity (9) holding here also.

These wave-height spectra are defined with the following normalization[4] with respect to root-mean-square (rms) wave height h:

$$h^2 = \langle \eta^2(\mathbf{r},t)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\mathbf{k} \int_{-\infty}^{\infty} dw S(\mathbf{k},w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\mathbf{k} S(\mathbf{k}). \tag{12}$$

One can also define a temporal (only), nondirectional wave-height spectrum, a quantity which is readily measured with buoys and/or wave staffs as

$$S(w) \equiv \int_{-\infty}\int_{-\infty} d^2\mathbf{k} S(\mathbf{k},w),$$

where

$$h^2 = \int_{-\infty}^{\infty} dw S(w). \tag{13}$$

[4] Attention is again called to the fact that, as per our definitions after (1) and (8), $S(\mathbf{k})$ is nonsymmetric so that it can represent wave fields traveling over 360° of space with arbitrary amplitudes.

One further identity will be used in subsequent higher order averaging processes; this identity, valid for Gaussian random variables such as η_1, is established in elementary statistics texts (e.g., Davenport and Root, 1958)

$$\langle\eta_1(\mathbf{k}_1)\eta_1^*(\mathbf{k}_2)\eta_1(\mathbf{k}_3)\eta_1^*(\mathbf{k}_4)\rangle = \langle\eta_1(\mathbf{k}_1)\eta_1^*(\mathbf{k}_2)\rangle\langle\eta_1(\mathbf{k}_3)\eta_1^*(\mathbf{k}_4)\rangle+\langle\eta_1(\mathbf{k}_1)\eta_1(\mathbf{k}_3)\rangle \times\langle\eta_1^*(\mathbf{k}_2)\eta_1^*(\mathbf{k}_4)\rangle+\langle\eta_1(\mathbf{k}_1)\eta_1^*(\mathbf{k}_4)\rangle\langle\eta_1^*(\mathbf{k}_2)\eta_1(\mathbf{k}_3)\rangle. \quad (14)$$

By employing (8), one can see that the first term on the right side can be nonzero only when $\mathbf{k}_2=\mathbf{k}_1$ and $\mathbf{k}_4=\mathbf{k}_3$. The second term can *never* be nonzero. The third term is nonzero only when $\mathbf{k}_4=\mathbf{k}_1$ and $\mathbf{k}_2=\mathbf{k}_3$. Under these conditions the right side of (14) becomes $[8\times(2\pi)^4/(L_xL_y)^2]S_1(\mathbf{k}_1)S_1(\mathbf{k}_3)$.

We will now apply these definitions and identities to show how the results derived in W-B for periodic, nonrandom, Fourier series descriptions of waves can be converted to integrals representing average spectra, etc. In this process, we form products, take ensemble averages [in the sense of (7) or (8)], interchange averaging and summation processes and finally convert remaining Fourier sums to integrals. This latter process is done in a Riemann sense, where, for example, $(2\pi)^2/L_xL_y\equiv d\mathbf{k}$ and $2\pi/T\equiv dw$, L_x, L_y and T being the observation periods. Let us illustrate this process on (22) of W-B, where our purpose is to derive an expression for the second-order wave-height spatial/temporal spectrum $S_2(\mathbf{K},\Omega)$ in terms of first-order wave-height spectra. First, we express the second-order wave height as follows [using (4) and (22) of W-B and the lowest-order dispersion relation]:

$$\eta_2(\mathbf{r},t)=\sum_{\mathbf{K}}\eta_2(\mathbf{K})\exp\{i[\mathbf{K}\cdot\mathbf{r}-(\omega_0+\omega_0')t]\}, \quad (15a)$$

where

$$\eta_2(\mathbf{K})=\sum_{\mathbf{k},\omega_0}\sum_{\mathbf{k}',\omega'_0}A(\mathbf{k},\omega_0,\mathbf{k}',\omega_0')\eta_1(\mathbf{k},\omega_0)\eta_1(\mathbf{k}',\omega_0') \times\delta_{\mathbf{K}}^{\mathbf{k}+\mathbf{k}'}\delta_{\omega_0}^{\pm\sqrt{gk}}\delta_{\omega_0'}^{\pm\sqrt{gk'}}. \quad (15b)$$

Therefore, we perform averaging on the spatial second-order coefficient, convert these sums to integrals, and then take the temporal Fourier integral of the result:

$$S_2(\mathbf{K},\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\eta_2(\mathbf{K})\eta_2^*(\mathbf{K})\rangle\cdot\left(\frac{L_xL_y}{(2\pi)^2}\right) \times\exp[-i(\omega_0+\omega_0')t+i(\omega_0''+\omega_0''')(t+\tau)-i\Omega\tau]d\tau. \quad (16)$$

By employing (14) along with the Kronecker-delta functions, we can show that all of the summations contained in (16) reduce to a single vector wavenumber sum, which is then converted to an integral in the Riemann sense. Finally, the integral in (16) over τ becomes a Dirac-delta function, leaving as the final result (written in symmetrical form)

$$S_2(\mathbf{K},\Omega)=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^2\boldsymbol{\kappa}\sum_{u,l}\sum_{u,l}A^2(\mathbf{k},\pm\sqrt{gk},\mathbf{k}',\pm\sqrt{gk'}) \times S_1(\pm\mathbf{k})S_1(\pm\mathbf{k}')\delta(\Omega\mp\sqrt{gk}\mp\sqrt{gk'}), \quad (17)$$

where $\mathbf{k}\equiv\frac{1}{2}\mathbf{K}+\boldsymbol{\kappa}$, $\mathbf{k}'\equiv\frac{1}{2}\mathbf{K}-\boldsymbol{\kappa}$, and the summation indices refer to the upper and lower signs in the equation. The quantity $A(\mathbf{k},\omega_0,\mathbf{k}',\omega_0')$ is as derived and given in (23) of W-B. The Dirac-delta function permits one of the two integrations to be performed, leaving one which must (in general) be done numerically for a given form of the wave-height spatial spectrum.

An identical expression holds for one-dimensional spatial spectra due to colinear wave trains, where the double integral is replaced by a single integral (i.e., $d\kappa$ instead of $d^2\boldsymbol{\kappa}$), and where the spatial wavenumber in all spectra is a scalar rather than a vector quantity; this case will be treated separately later.

Next we apply these statistical techniques to calculations of the mean and variance of the correction to the dispersion relation (or phase velocity), ω_2/ω_0, as given by Eq. (29) of W-B. So that the algebraic expressions will not be so cumbersome as to detract from the main points to be illustrated by the example, we restrict our attention here to colinear waves. Then the $C(\mathbf{k},\omega,\mathbf{k}',\omega')$ of (29) of W-B reduces to the factor multiplying the second term of (4), i.e., we have

$$\frac{\omega_2(k)}{\omega_0(k)}=\pm\frac{4k}{\omega_0}\sum_{k'>0}\omega_0'|\eta_1(k')|^2\cdot\begin{cases}k' & \text{for } k'<k\\ k & \text{for } k'>k\end{cases} \quad (18)$$

where, as before, $\omega_0=\sqrt{g|k|}$ and $\omega_0'=\sqrt{g|k'|}$. The upper sign is used *except* when the wave whose desired phase velocity correction (with wavenumber k) is moving opposite to the direction of the waves with wavenumbers k'. Note that this expression is valid for all waves *except* the one at wavenumber k; at this wavenumber the corresponding term in (18) must be divided by two, as discussed after (4). Where many waves are present, however, this "self-induced" phase velocity correction is negligible compared to the mutual interaction effects; hence it will be ignored here as we proceed to a spectrum of many waves.

The average value of (18) is readily taken using the techniques discussed above:

$$\left\langle\frac{\omega_2(k)}{\omega_0(k)}\right\rangle=\pm\frac{k}{\omega_0}\sum_{k'=-\infty}^{\infty}\omega_0'S_1(k')\cdot\frac{2\pi}{L_x}\begin{cases}k' & \text{for } k'<k\\ k & \text{for } k'>k\end{cases}$$

or

$$\left\langle\frac{\omega_2(k)}{\omega_0(k)}\right\rangle=\pm\frac{2k}{\omega_0}\int_0^k\omega_0'k'S_1(k')dk' \pm\frac{2k^2}{\omega_0}\int_k^{\infty}\omega_0'S_1(k')dk'. \quad (19)$$

This agrees with the expression given by Longuet-Higgins and Phillips (1962) in their Eq. (4.1).

Finally, we take the variance of ω_2/ω_0, inasmuch as this is a measure of the expected spread in the frequencies or phase velocities of first-order waves with wavenumber k. Of the three terms resulting from the use of (14), one cancels with the square of the mean. Here we initially allow the sums to run over negative wavenumbers and divide by 2:

$$\mathrm{Var}\left[\frac{\omega_2}{\omega_0}(k)\right]=\left\langle\left(\frac{\omega_2(k)}{\omega_0(k)}\right)^2\right\rangle-\left\langle\frac{\omega_2(k)}{\omega_0(k)}\right\rangle^2$$

$$=\frac{4k^2}{\omega_0^2}\sum_{k'=-\infty}^{\infty}\sum_{k''=-\infty}^{\infty}\omega_0'\omega_0''\begin{Bmatrix}k'\\k\end{Bmatrix}\begin{Bmatrix}k''\\k\end{Bmatrix}$$

$$\times\langle\eta_1(k')\eta_1^*(k')\eta_1(k'')\eta_1^*(k'')\rangle$$

$$-\frac{4k^2}{\omega_0^2}\sum_{k'=-\infty}^{\infty}\sum_{k''=-\infty}^{\infty}\omega_0'\omega_0''\begin{Bmatrix}k'\\k\end{Bmatrix}\begin{Bmatrix}k''\\k\end{Bmatrix}\langle\eta_1(k')\eta_1^*(k')\rangle$$

$$\times\langle\eta_1(k'')\eta_1^*(k'')\rangle=\frac{2k^2}{\omega_0^2}\sum_{k'=-\infty}^{\infty}\left[\omega_0'\begin{Bmatrix}k'\\k\end{Bmatrix}S_1(k')\right]^2\cdot\left(\frac{2\pi}{L_x}\right)^2$$

or

$$\mathrm{Var}\left[\frac{\omega_2}{\omega_0}(k)\right]=\frac{8\pi k^2}{\omega_0^2L_x}\left[\int_0^k\omega_0'^2k'^2S_1^2(k')dk'+k^2\int_k^\infty\omega_0'^2S_1^2(k')dk'\right].\quad(20)$$

It is curious that in (20) one of the $2\pi/L_x$ factors is *not* used up as an integration increment dk', as in all other cases considered. Recall that initially we described L_x as the spatial increment over which the Fourier series with random coefficients was valid. Outside of this increment of space the surface profile repeats itself. Hence L_x physically corresponds to the region of space over which observations relating to this particular statistic of the sea are either made or desired. As one can see, the larger this observation window, the smaller the variance.

5. Examples of the second-order wave height and its spectrum

a. *Two-wave interactions and diffraction grating analogies*

Eq. (22) of W-B shows that the spatial wavenumber of the second-order wave, $\mathbf{K}$, is the vector sum of the wavenumbers of the first-order waves present. The same holds true for the temporal wavenumbers. In other words

$$\mathbf{K}=\mathbf{k}+\mathbf{k}'\quad\text{and}\quad\Omega_0=\omega_0+\omega_0'\qquad(21)$$

(to lowest order, where $\omega_0=\sqrt{gk'}$ and $\omega_0'=\sqrt{gk'}$). These relationships have been referred to as representing second-order Bragg "scatter" or a second-order Feynman interaction (Hasselmann, 1966; Barrick, 1972).

To obtain a clearer physical picture of this interaction, let us consider the case of two first-order sinusoidal wave trains, where $\mathbf{k},\mathbf{k}'=\pm\mathbf{k}_a,\pm\mathbf{k}_b$. Eq. (22) of W-B shows that there will be several sinusoidal second-order wave trains whose Fourier coefficients $\eta_2(\mathbf{K},\Omega_0)$ are determined by the products of terms in the sum. Neglecting the second-order coefficients at zero wavenumber (which only redefine the mean sea level), we have four sets of second-order waves:

1) The self-generating second-harmonic waves:

Wavenumbers $\mathbf{K}_{aa}=2\mathbf{k}_a$; $\mathbf{K}_{bb}=2\mathbf{k}_b$ (22a)

Frequencies $\Omega_{0aa}=2\omega_{0a}$; $\Omega_{0bb}=2\omega_{0b}$ (22b)

Phase Speeds $V_{aa}=\Omega_{0aa}/K_a=\omega_{0a}/k_a=v_a$; $v_{bb}=\Omega_{0bb}/K_b=\omega_{0b}/k_b=v_b$. (22c)

2) The mutual cross-coupling waves:

Wavenumbers $\mathbf{K}_{s,d}=\mathbf{k}_a\pm\mathbf{k}_b$ (23a)

Frequencies $\Omega_{0s,d}=\omega_{0a}\pm\omega_{0b}$ (23b)

Phase Speeds $v_{s,d}=\Omega_{0s,d}/K_{s,d}=(\sqrt{gk_a}\pm\sqrt{gk_b})/K_{s,d}$. (23c)

The first set (second harmonics) originates from the analysis of Stokes (1847) where one periodic wave train alone is considered. The second set originates because of the nonlinear (square-law) interaction between two separate sets of first-order waves, and cannot arise from Stokes analysis by mere superposition of his results.

An interesting interpretation of these second-order waves is obtained by analogy with Moirè patterns in diffraction gratings. Let us represent the peaks (+) and troughs (−) of two sets of first-order, arbitrarily oriented sinusoidal waves $(\mathbf{k}_a,\mathbf{k}_b)$ as shown at the top of Fig. 1. Where the peaks of one set reinforce with the peaks of the other set, we have a linear superposition (or "piling up") of water as shown by the circles with the plus signs (with a similar situation for negative reinforcement). These line "intersections" would appear as a separate dense pattern, known as the Moirè effect, if two diffraction gratings were overlain. Now, if each first-order wave train moves with its own characteristic phase velocity, these dense spots (circles) will also move as a solid pattern, but in a different direction than either of the two first-order gratings.

The second-harmonic waves are easy to describe: they lie parallel to the first-order wavesets, have half the spatial period of the fundamental, but move at the same phase velocity as the fundamental; they are shown at the lower left of Fig. 1. The "cross-coupling" waves, however (shown at the lower right), have their crests aligned along lines joining the dense spots at the corner of a triangle, the two sides of which lie along the first-order crestlines, one full period on each side. These second-order crestlines appear to stay attached to the dense spots or circles as they move, carried along

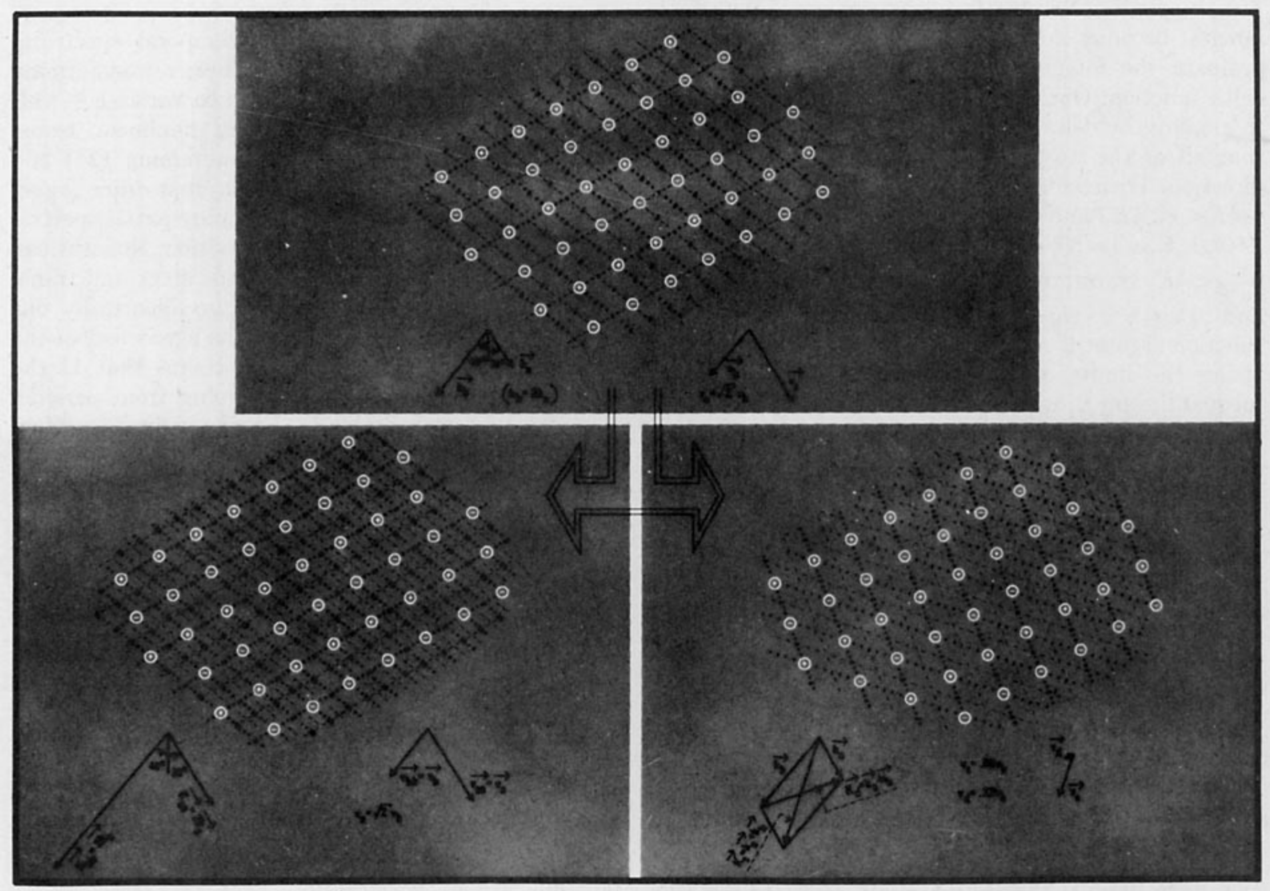

FIG. 1. Sketch showing crest (+)/trough (−) patterns for two sets of first-order sinusoidal wave trains (upper figure) and the second-order wave trains they produce. Lower left shows the "second-harmonic" wave trains, while the lower right shows the "cross-coupling" wave trains. Relative scales on wave vector wave trains reinforce positively ⊕ and negatively ⊖.

by the first-order waves. Their wavelengths and directions of propagation are of course predicted by the vector triads, coming from (21). Their phase speeds can be predicted trigonometrically by following the Moirè pattern temporally, and this identically corresponds to those derived rigorously and given in (23c). If Fig. 1 were indeed a double diffraction grating in which each grating moved in the direction and at the speed indicated, one would in fact noticė these second-order "mutual" waves (when viewed from a distance), due to second-order optical nonlinearities (i.e., the abrupt change from transparent to opaque).

For the wavepatterns shown (and also in general), the heights of the "second-harmonic" second-order waves are of the same order as the heights of the "mutual" second-harmonic waves. For the example shown [letting $k_a=1$ and $k_b=2$, we have $\eta_2(\mathbf{K}_{aa},\Omega_{0aa}) =\eta_1^2(\mathbf{k}_a)$, $\eta_2(\mathbf{K}_{bb},\Omega_{0bb})=2\eta_1^2(\mathbf{k}_b)$, but $\eta_2(\mathbf{K}_s,\Omega_{0s})=0.324\,\eta_1(\mathbf{k}_a)\eta_1(\mathbf{k}_b)$ and $\eta_2(\mathbf{K}_d,\Omega_{0d})=2.637\,\eta_1(\mathbf{k}_a)\eta_1(\mathbf{k}_b)$]. The actual heights of these second-order waves are small compared to the first-order waveheights. Since the half-heights of the first-order waves *at maximum* (i.e., when breaking occurs) must be such that $|\eta_1(\mathbf{k})| \leq \pi/(14k)$, we have (upon using the equality and allowing η_1 to be pure real) $\eta_1(\mathbf{k}_a)=0.2244$, $\eta_1(\mathbf{k}_b) =0.1122$, $\eta_2(\mathbf{K}_{aa},\Omega_{0aa})=0.0504$, $\eta_2(\mathbf{K}_{bb},\Omega_{0bb})=0.0252$, $\eta_2(\mathbf{K}_s,\Omega_{0s})=0.00816$, and $\eta_2(\mathbf{K}_d,\Omega_{0d})=0.0664$. Therefore, even with the maximum possible first-order wave heights, the second-order wave heights are small in terms of them, and the perturbation analysis of W-B used to derive these results is justified.

As one generalizes from the case of two first-order waves to a spectrum of very many waves (say N), it is obvious that the number of second-order "cross-coupling" waves far exceeds the number of "second-harmonic" waves; the latter goes as N whereas the former goes as $N(N-1)\approx N^2$. Hence, the total second-order sea waveheight for many first-order waves for all practical purposes consists only of the "cross-coupling" waves; this becomes especially true in the limit of an infinite spectrum of first-order waves, as seen from (17).

b. Average second-order spectrum for colinear waves

Following (17), it was mentioned that in the case of colinear waves, that equation has vector wavenumbers replaced by scalar wavenumbers, and the double integral becomes a single integral. This permits us to evaluate the integral exactly because of the Dirac-delta function. One must transform variables in order to employ the delta function, however. Let us assume that all of the waves are propagating along the $+x$ direction. Then we can break k space into two regions: $-\frac{1}{2}K<\kappa<\frac{1}{2}K$, and $\kappa<-\frac{1}{2}K$, $\kappa>\frac{1}{2}K$. Note first that $A^2(k,\omega_0,k',\omega_0')=K^2/4$. Now, considering the region $0<\kappa<\frac{1}{2}K$, transform variables first to $u\equiv\sqrt{g(\frac{1}{2}K+\kappa)}$, and to $w\equiv u+\sqrt{gK-u^2}$. The upper signs in the delta-function argument must be used within the first region. Using the limits of the integral to define required inequalities for Ω, we arrive at the following expression (for $K>0$ and for the total region $-\frac{1}{2}K<\kappa<\frac{1}{2}K$):

$$S_2(K,\Omega)=\begin{cases}\dfrac{K^2}{4g}\dfrac{(\Omega^2-gK)}{\sqrt{2gK-\Omega^2}}S_1\left[\dfrac{1}{2}\left(K+\dfrac{\Omega}{g}\sqrt{2gK-\Omega^2}\right)\right] \\ \qquad\times S_1\left[\dfrac{1}{2}\left(K-\dfrac{\Omega}{2g}\sqrt{2gK-\Omega^2}\right)\right] \\ \qquad\qquad \text{for}\quad \sqrt{gK}<\Omega<\sqrt{2gK} \\ 0 \quad \text{for} \quad \Omega>\sqrt{2gK}.\end{cases} \tag{24}$$

Likewise, the transformations required for $\kappa>\frac{1}{2}K$ are $u\equiv\sqrt{g(\frac{1}{2}K+\kappa)}$ and $\omega\equiv u-\sqrt{u^2-gK}$. Opposite signs in the delta-function argument must be used in this region. Adding in the contribution from the region $\kappa<-\frac{1}{2}K$, we obtain (for $K>0$)

$$S_2(K,\Omega)=\begin{cases}\dfrac{K^2[(gK)^2-\Omega^4]}{8g\Omega^3}S_1\left[\dfrac{1}{4g\Omega^2}(gK-\Omega^2)^2\right] \\ \qquad\times S_1\left[\dfrac{1}{4g\Omega^2}(gK-\Omega^2)^2\right] \quad \text{for} \quad 0<\Omega<\sqrt{gK} \\ 0 \quad \text{for} \quad \Omega<0.\end{cases} \tag{25}$$

Hence (24) and (25) define the second-order one-dimensional spatial-temporal spectrum in terms of the one-sided first-order spatial spectrum. It is seen that, for positive K, the region over Ω in which the spectrum exists is bounded between 0 and $\sqrt{2gK}$; a square-root-type singularity occurs at $\Omega\rightarrow\sqrt{2gK}$, but the area is finite under this singularity. Since the waves are required to be real quantities, half the energy in the first and second-order spectra lies in the region $K<0$. Hence (24) and (25) also apply, when $K<0$, and in this case Ω lies in the region $-\sqrt{2g|K|}<\Omega<0$.

Tick (1959) used a Fourier integral approach and statistical methods on colinear wave trains to obtain expressions involving second-order wave heights. While he never specifically derived an expression for the second-order wave-height spatial-temporal spectrum, one can obtain this quantity by Fourier transforming his Eq. (36) over the spatial distance variable ξ; this involves Dirac-delta functions and nonlinear transformations such as those used in obtaining (24) and (25). Furthermore, he works with first-order wave-height temporal spectra instead of our spatial spectra, which necessitates another transformation. Nonetheless, if one performs the required algebraic steps and transformations on his result, one obtains identically our (24) and (25) for $S_2(K,\Omega)$. Hence the agreement of the two results lends credence to our claims that 1) the techniques we demonstrated for going from periodic wave trains, representable by Fourier series to random wave fields whose average properties are sought, are generally valid; 2) our higher order expressions involving two-dimensional wave fields are also valid because they agree with Tick's (1959) and Longuet-Higgins and Phillips (1962) results in the limit of one-dimensional (colinear) wave fields.

c. Example of spatial spectra for colinear Phillips model

To show an example of how (22), (24) and (25) can be applied to indicate the shape, magnitude and distribution of second-order wave-height spectral energy, we employ the commonly used Phillips (1969) spectral model for fully developed seas, converted to a one-dimensional spatial form

$$S_1(k)=\begin{cases}\dfrac{B}{2k^3} & \text{for} \quad k_{c0}<k<k_\gamma \\ 0 & \text{for} \quad -\infty<k<k_{c0}\end{cases} \tag{26}$$

where $k_{c0}=g/v^2$ (v=wind speed in m s^{-1}, g=gravitational constant $=9.81$ m s^{-2}), and where $k_\gamma=\sqrt{\rho g/\gamma}$ is the upper cutoff at the capillary wave region, with ρ=water density ($=10^3$ kg/m^{-3}) and γ=surface tension ($=0.073$ N m^{-1}). B is a dimensionless constant whose equilibrium value has been determined experimentally to be $\sim$0.005. This spectrum is normalized such that the rms slope of the colinear waves is the same as that for a two-dimensional semi-isotropic spectrum (Phillips, 1969). This yields a first-order two-sided spatial-temporal spectrum whose temporal frequency ω_0 (to lowest order) is uniquely related to the spatial wavenumber as follows:

$$S_1(k,\omega_0)=\begin{cases}\dfrac{B}{4|k|^3}\delta(\omega_0+\sqrt{g|k|}) & \text{for} \quad -k_\gamma<k<-k_{c0}, \\ 0 & \text{for} \quad -k_{c0}<k<k_{c0}, \\ \dfrac{B}{4k^3}\delta(\omega_0-\sqrt{gk}) & \text{for} \quad k_{c0}<k<k_\gamma.\end{cases} \tag{27}$$

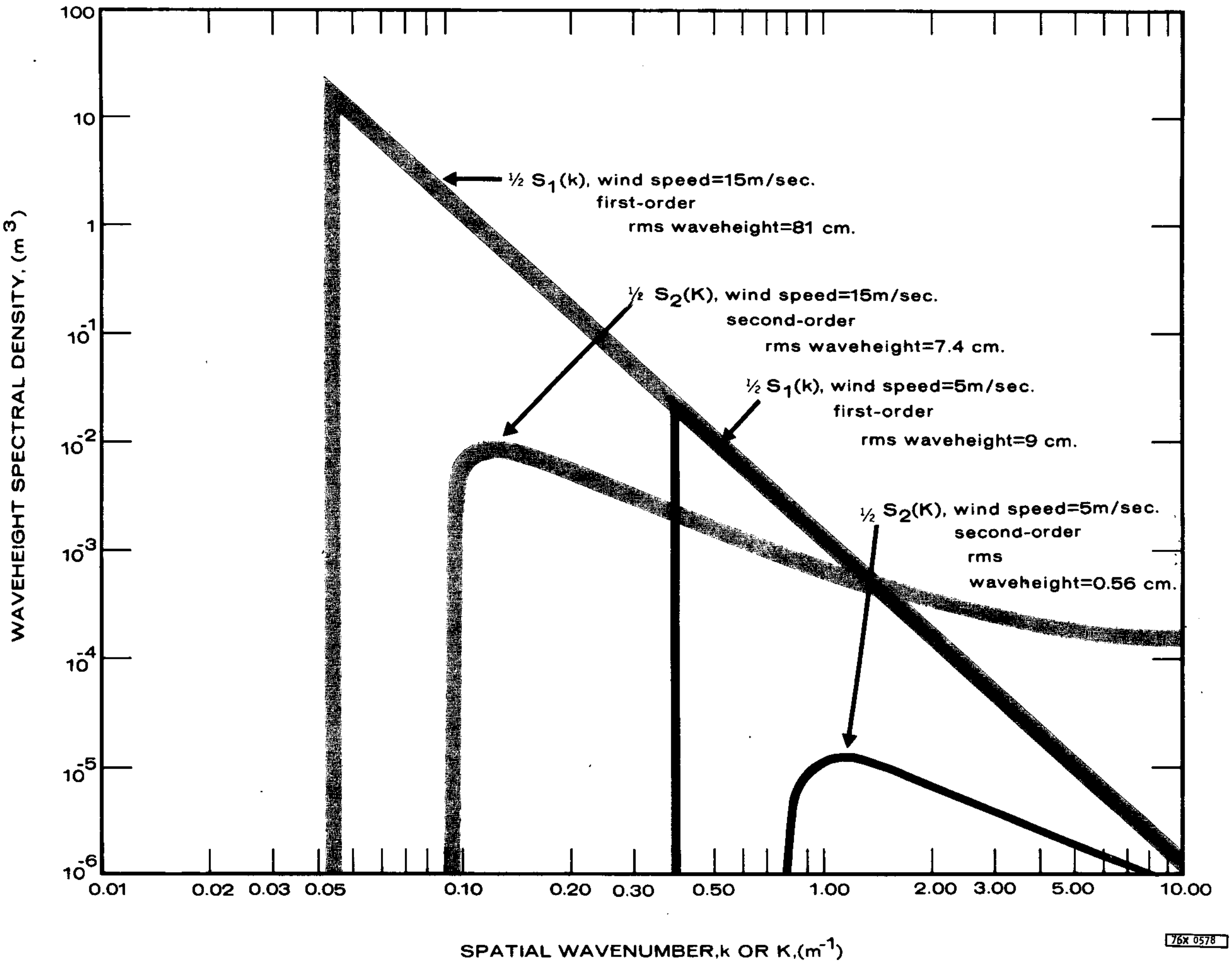

FIG. 2. Spatial wave-height spectra using Phillips' one-dimensional (colinear) first-order model.

This form defines waves traveling in the $+x$ direction (where k is positive).

Upon substituting (26) into (24) and (25), simplifying and converting the inequalities in (26) into appropriate temporal form, we arrive at the following result for the second-order spatial-temporal wave-height spectrum for $K>0$:

$$\Omega_F S_2(K,\Omega) = \begin{cases} \dfrac{2^7B^2\mu^9}{K^3(1-\mu^4)^5} & \text{for } 0<\mu^2<1-2\mu_{c0}\sqrt{1-\mu_{c0}^2} \\ \dfrac{2^2B^2}{K^3\sqrt{2-\mu^2}(\mu^2-1)^5} & \text{for } 1+2\mu_{c0}\sqrt{1-\mu_{c0}^2}<\mu^2<2 \\ 0 & \text{for other } \mu \end{cases}$$

$$\text{and } 2\mu_{c0}^2>1 \quad (28)$$

where μ is a normalized temporal frequency $\mu\equiv\Omega/\Omega_F$, $\Omega_F\equiv\sqrt{gK}$; $\mu_{c0}\equiv\Omega_{c0}/\Omega_F$ and $\Omega_{c0}=g/v$; the spectrum is identical for negative K and Ω.

Whereas the first-order spatial-temporal spectrum appears at a discrete temporal frequency ($\omega_0=\sqrt{gk}$), the second-order spectrum is distributed over a continuum of frequencies around $\sqrt{gK}$ between $\Omega=0$ and $\Omega=\sqrt{2}\times\sqrt{gK}$; an integrable singularity occurs at the latter limit.

The one-sided second-order spatial spectrum $S_2(K)$ can be found by integrating (28) over positive Ω, the temporal wavenumber and dividing by two. This can be done from the tables, and the result expressed in closed form; it is not given here for lack of space. Rather, we give curves showing $S_1(k)$ and $S_2(K)$ over the gravity wave region for the above Phillips model and two different values of wind speed ($v=5$ and 15 m s^{-1}) in Fig. 2. Also, the first-order and second-order rms wave heights h_1 and h_2 corresponding to these spectra are given

$$\left[\text{i.e., } h_{1,2}^2\equiv\int_0^\infty S_{1,2}(\kappa)d\kappa\right].$$

Note that even though the mean-square second-order wave height (i.e., the area under the curve) is always less than the mean-square first-order wave height (as required in the perturbation theory), the second-order spectral power can exceed the first-order power at certain higher wavenumbers. This may at first appear strange, for the Phillips model measured by oceanographers (following a κ^{-4} or ω^{-5} law) should more realistically be thought of as the *total* spectrum, including all perturbation orders. Thus it may have been more meaningful in calculating the curves of Fig. 2 if we had "iterated" until the *sum* of the first and second-order curves satisfied the model given in (26). Our purpose here was only to provide an illustrative example of a second-order spectrum, given a first-order spectrum. In addition, Fig. 2 illustrates that the second-order portion does not begin to dominate until one has gone over four orders of magnitude down from the first-order spectral peak. Nearly all wave-height spectral measurements reported in the literature cover a dynamic range of only two orders of magnitude. An exception to this are very precise measurements by Mitsuyasu and Honda (as reported by Pierson, 1976), which in fact do show a departure from ω^{-5} law some three orders of magnitude down from the spectral peak; this departure is an *increase* from the inverse fifth-power law, commonly assumed to hold everywhere in the gravity-wave region.

6. Examples of first-order phase velocity mean and variances

Because Section 2 showed that the greatest correction to the dispersion equations occurs for parallel (colinear) rather than perpendicular wave trains, we will employ as an example the Phillips colinear spectral model (26) in (19) and (20). Using this model, we obtain for the mean

$$\left\langle\frac{\omega_2(k)}{\omega_0(k)}\right\rangle = 2B(\sqrt{k/k_{c0}}-\tfrac{2}{3})\,(\text{assuming } k>k_{c0}), \quad (29)$$

where $k_{c0}=g/v^2$.

The phase-velocity standard deviation (assuming an average over an infinite ensemble) using (20) is

$$\sqrt{\mathrm{Var}\left[\frac{\omega_2}{\omega_0}(k)\right]} = B\sqrt{\pi/(kL_x)}\cdot\sqrt{k^2/k_{c0}^2-\tfrac{1}{2}}. \quad (30)$$

Note again that this result depends upon L_x, the length of the area under observation; as this quantity becomes very large compared to the water wavelength $2\pi/k$, the above standard deviation for an infinite ensemble average approaches zero.

A more sensible quantity than the infinite ensemble variance of phase velocity is the variance for a finite sample size. If, for example, one formed a sample average of ω_2^2 (consisting of the sum of N independent samples of ω_2^2 divided by N) and called it $N:\overline{\omega_2^2}$, one can show that the sample phase velocity standard deviation correction is given by

$$\sqrt{\mathrm{Var}_N\left[\frac{\omega_2}{\omega_0}(k)\right]} = \frac{1}{\omega_0}[\langle(N:\overline{\omega_2^2})^2\rangle - \langle N:\overline{\omega_2^2}\rangle^2]^{\frac{1}{2}}$$

$$= \left(\frac{2}{N}\right)^{\frac{1}{2}}\left\langle\frac{\omega_2(k)}{\omega_0(k)}\right\rangle = \left(\frac{2}{N}\right)^{\frac{1}{2}} 2B(\sqrt{k/k_{c0}}-\tfrac{2}{3}). \quad (31)$$

To illustrate the magnitudes of these various quantities, let us consider an HF radar application and assume the following parameters: wind speed $v=15$ m s^{-1}; $2\pi/k=5$ m (e.g., 5 m long ocean waves would be observed with an HF backscatter radar at 30 MHz having a wavelength of 10 m); the number of independent samples $N=12$; the length of observed ocean patch $L_x=3$ km; $B=0.005$. For 5 m ocean waves, the phase velocity (to lowest order) is 2.794 m s^{-1}. The normalized phase velocity correction mean, standard deviation and 12-sample standard deviation for this example are then 0.04702, 0.00045345 and 0.03004, respectively. The actual phase velocity correction mean, standard deviation and 12-sample standard deviation corresponding to these numbers for the 5 m long ocean wave component are then 13.14, 0.13 and 8.9 cm s^{-1}, respectively.

7. Discussion and conclusions

In this series of two papers, we have presented a general perturbational formulation in which all desired higher order corrections to deep-water gravity wave parameters can be obtained at the same time; the approach is valid over temporal and spatial scales sufficiently small that energy exchange processes can be neglected. In particular, the technique was used to obtain the second-order wave height, velocity potential, and the first nonzero correction to the dispersion relationship (a third-order quantity). These results were interpreted physically and shown to agree with special limiting cases treated in the classical literature. It was shown how the solutions—based upon periodic two-dimensional wave trains—are readily converted to a form suited to random descriptions of the sea wave height. Finally, several examples were presented, primarily based upon colinear (one-dimensional) random wave fields and a Phillips spectral model for fully developed seas; these two simplifications led to closed-form solutions which are not possible in the general two-dimensional case, permitting one to obtain an insight into these higher order quantities.

The basic techniques leading to the derivation of these quantities were outlined in the literature over a decade ago (Tick, 1959; Phillips, 1960; Hasselmann, 1962, 1963a, b). In certain cases these authors indicated solutions in a formalistic manner, but did not complete the details of the algebra. Possibly this failure to complete and expand upon these solutions at that time was due to the fact that experimental techniques were neither available nor of sufficient accuracy to permit confirmation of these higher order wave parameters. Two prospects in recent years have altered this picture, however: 1) the interest in energy transfer between different regions of the wave-height spectrum via nonlinear wave-wave interactions (Hasselmann *et al.*, 1973); and 2) the application of high-frequency radars (as a remote sensing tool) to the measurement of ocean-wave statistics and near-surface currents (along with the concomitant verification of theoretical models explaining this interaction; Barrick *et al.*, 1974; Stewart and Joy, 1974). It is the latter application which has led us to formulate and complete the steady-state derivations presented here.

In particular, the second-order wave-height spectrum and the higher order phase velocity correction for first-order waves are directly observable with HF radar systems. Theory (Barrick, 1972) and experiment (Barrick *et al.*, 1974) have shown that the average signal power spectral density (expressed as an average radar backscattering cross section per unit area per radian per second bandwidth for vertical polarization at grazing incidence) can be written

$$\sigma_{1,2}{}^{0}(\omega)=2^{7}\pi k_0{}^{4}S_{1,2}(\boldsymbol{\kappa}_r,\,\omega-\omega_0),\qquad(32)$$

where ω_0 is the radar carrier frequency (rad s^{-1}), k_0 is the magnitude of the incident and scattered radio wavenumbers (i.e., $k_0=\omega_0/c$, c being the radiowave free-space velocity); $\boldsymbol{\kappa}_r=\mathbf{k}_i-\mathbf{k}_s$ or $\boldsymbol{\kappa}_r=2\mathbf{k}_i=2k_0\hat{k}_i$ for backscatter, since $\mathbf{k}_s=-\mathbf{k}_i$. Thus $\omega-\omega_0$, the radian frequency at which the ocean wave-height spectrum is being observed, appears in the radar receiver as the Doppler shift of the sea-scattered signal from the carrier. The spatial vector wavenumber $\boldsymbol{\kappa}_r$ indicates that the Bragg (or diffraction-grating) effect is giving rise to the scatter.[5]

We saw that $S_1(\mathbf{k},\omega)$ is an impulse function from the lowest order dispersion equation (centered at $\omega\approx\omega_0\pm\sqrt{gk}$). Any finite width to this normal infinitesimally narrow impulse function in the (Doppler) frequency domain is therefore a measure of the variance of the dispersion equation for first-order waves (neglecting any frequency broadening due to system limitations, nonscatter-related mechanisms, current shears within the scattering patch). Hence the variance of ω_2, as discussed earlier, manifests itself here as the width of the first-order sea echo Doppler peak.

The second-order portion of the sea echo Doppler spectrum is related in its magnitude and shape to the wave-height spatial spectrum in a nonlinear manner via the integral (17). The fact that this second-order echo energy is more sensitive to the longer, higher ocean waves than is the first-order echo energy at useful HF radar frequencies is generating considerable interest in utilizing this portion of the echo to remotely sense sea state. While the simple colinear wave models examined in detail here can provide some feeling for the general distribution of sea-echo energy, the complete solution to the two-dimensional model (17) must be pursued in order to study the effect of wave directionality on the echo spectral shape. Because of the complexity of solving (17) numerically for two-dimensional wave-height spectra, this topic must be undertaken separately.

[5] To second order, there is another term proportional to the second-order waveheight spectrum which originates from double radio-wave scatter (from two sets of ocean waves). This term—neglected here—is generally smaller than the "hydrodynamic" contribution considered here; it is derived elsewhere (Barrick, 1972) and given there.

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