Unexpected wave group behaviour challenges use of Stokes' theory for ocean waves

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Abstract

A key result of Stokes' water wave theory is that deep-water gravity waves of larger amplitude travel faster than those of lower amplitude at fixed wavelength. Recent observations, however, suggest that maximally-steep breaking wave crests actually travel significantly slower than expected, calling into question the predictions of Stokes' theory and its impact on diverse areas of ocean-wave physics ranging from rogue wave generation to the role of wave breaking in climate modelling. Here we report our discovery of a *generic* wave-crest slowdown mechanism that occurs within unsteady, propagating wave groups, which modifies the phasing of individual wave crests. Our numerical and observational studies show that just prior to reaching its maximum height, each wave crest slows down significantly. It either breaks at this reduced speed, or accelerates forward unbroken. Implications for oceanic and other natural wave systems are described.

Strong wind forcing over the sea surface generates waves, which can steepen and break conspicuously as whitecaps. These breaking waves play a leading role in the air-sea exchange of many fundamental quantities, including greenhouse gases. This has stimulated strong interest in measuring whitecap properties in relation to the wavelength of the underlying wave. However, accurately measuring individual wavelengths is difficult, whereas measuring the *speed* of the attached whitecap offers an indirect but more convenient measure. Since the whitecap remains attached to crest of the underlying wave during active breaking, routine use of Stokes' classical deep water wave theory[1,2] determines the wavelength from the observed whitecap speed. Here the crest is understood to be the local elevation maximum of the dominant waves, after high wavenumber ripples have been filtered out.

In this study, we set out to investigate how accurately Stokes' theory[1,2] estimates breaking crest speeds. Here we found that it over-predicts the speed of *every* wave crest as it reaches its maximum height, whether it breaks or not, and by a surprisingly wide margin of up to 37%. This discovery represents a *generic* departure of natural ocean waves from Stokes' theory behaviour, as described below.

Stokes' theory for a steady, uniform train of two-dimensional (2D) non-linear deep-water waves of small-to-intermediate mean steepness ak (=π×waveheight/wavelength) predicts their intrinsic wave speed c increases slowly with ak:

$$c = c_0[1+(ak)^2+\text{higher order terms in } (ak)]^{1/2} \qquad (1)$$

where c_0 is the wave speed for linear (very small steepness) waves. Equation (1) was extended computationally to maximally-steep steady waves[3], for which c reaches close to $1.1c_0$. Thus, increased wave steepness has long been associated with *higher* wave speeds. These results are used routinely in offshore engineering design.

The sea surface comprises waves interacting on different scales, which are described conventionally using a wave spectrum. This interaction results in evolving wave-group patterns rather than trains of uniform waves[4]. Each spectral component has an amplitude and phase, and the group structure arises from constructive interference - individual spectral component crests combining to produce a locally taller crest at a given point in space and time (the focal point) because their phases align there. Here we concentrate on the 'dominant' ocean waves, i.e. those with the largest amplitudes within the wave spectrum. Within a group, each dominant wave moves through the group, gradually changing its height and shape, characterised by slow forward and backward leaning of the wave crests[5]. Each dominant wave grows as it approaches the centre of the group, where it is transiently the tallest wave. Subsequently, the tallest wave may break, or else decrease in height as it progresses unbroken towards the front of the group.

Historically, propagation of the individual waves within groups has been quantified using 'academic' approximations, namely 'uniform' or 'slowly-varying' wavetrain theory[6]. Figure 1 shows typical 'academic' evolutions of waveheight time series (measured at a fixed location) for uniform wavetrains of different steepness *ak* and a 'slowly varying' wavetrain. A representative 'real' ocean waveheight record is included for comparison.

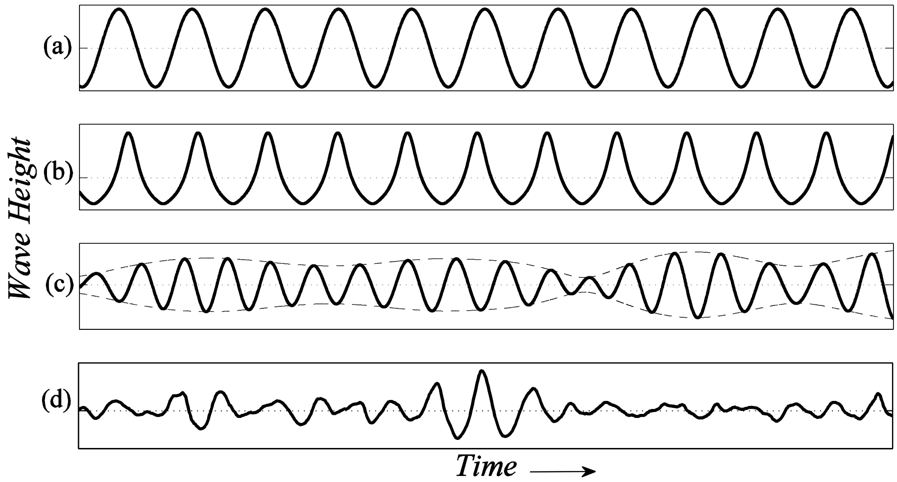

Figure 1. Representative deep-water waveheight time series (a) Stokes linear wave train (b) 5th order Stokes wave train (c) narrow spectral bandwidth wavetrain (a wavetrain composed of a limited range of wave frequencies) (d) representative record from the open ocean. Dashed lines in (c) show the wave envelope, which highlights the group structure, which is also clearly evident in (d).

Results from several laboratory studies of deep water wave groups have suggested[7,8,9] that breaking-crest speeds are typically 20% lower than expected from linear-wave theory, contrary to the expectation from (1) that breaking waves should travel faster than their linear wave speeds. It was essential to verify and understand this paradoxical behaviour, which is central to both refining present knowledge on deep-water waves and optimal utilisation of the spectral framework for breaking waves due to Phillips[10,11,12,13].

Concise account of the findings

We focussed our investigation on whether breaker speeds are indeed systematically lower, and if so, whether the wave decelerates before or after breaking starts. Thus, it was crucial to track changes in the dominant wave crest speeds within evolving wave groups, up to the point of breaking initiation.

We used a fully 3D numerical wave code[14] complemented by analytical studies, experiments in a laboratory wave basin and observation of ocean waves in their natural environment. This multi-faceted approach has identified striking new properties on how deep-water waves actually propagate when the theoretical constraints of steady, uniform wave behavior or slowly-varying wavetrain behaviour are relaxed. A major finding is that a previously undetected generic slowdown of dominant wave crests occurs as they transition through their maximum height within the wave group. This slowdown can be as large as about 20% of the corresponding theoretical linear wave speed. We now illustrate and describe in detail this surprisingly complex behaviour, which confirms and explains the unexpectedly slow speeds observed for breakers.

Methodology and observed wave behaviour

In our numerical wave basin study, wave groups are generated by a wave paddle. Its motion is programmed to produce a specific group structure comprising a prescribed number of carrier waves with given initial amplitudes, wavenumbers and frequencies. This shapes the spatial and temporal bandwidths that characterize the wave group structure and its spectrum.

Figure 2 shows the detailed shape of representative dominant waves evolving within a two-dimensional nonlinear wave group in the absence of breaking. The initial steepest wave decays and is replaced by the following growing wave, which grows modestly, then slows down and is replaced by the annotated faster-growing crest, which evolves to its maximum height and decays. This sequence illustrates the complex growth behaviour experienced by all crests in the group. As each new crest develops, it grows then slows down and attenuates as it advances through the front of the group.

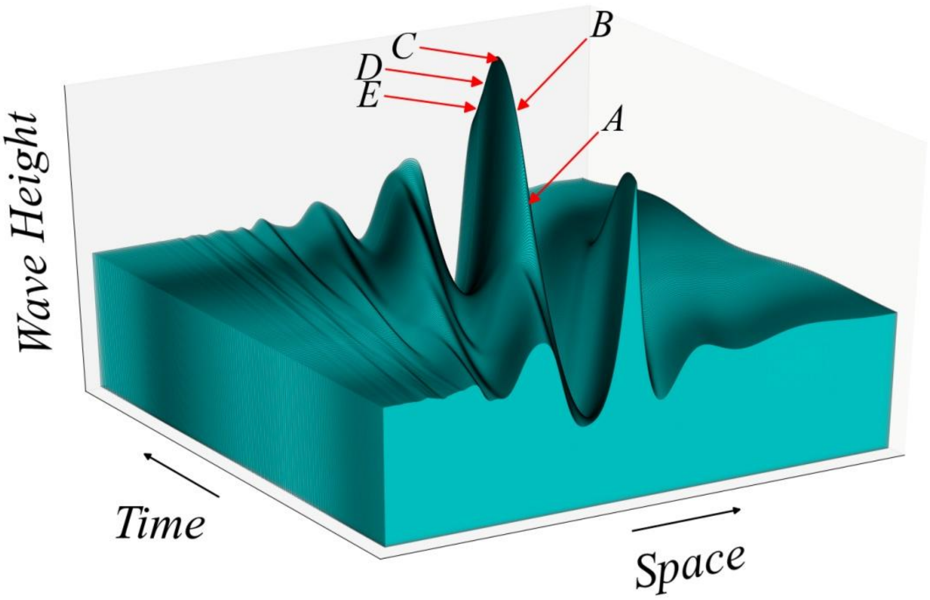

Figure 2. Space-time evolution diagram (not a 3D snapshot) of a non-breaking 2D wave group, moving towards the right. This shows the decay of the initial tallest crest, the growth of the following tallest crest and the complex transitions of other developing crests. Wave properties at the annotated times A-E are described in the text and in Figure 3.

In greater detail, Figure 3(a) shows the spatial wave profile at the different evolution times A-E indicated in Figure 2. The dominant wave is seen to grow asymmetrically, initially leaning forward as it steepens within the group. In the absence of breaking, it is seen that the steepest wave undergoes a continuous progression from leaning forward, relaxing back to symmetry at close to maximum height (the focal point), followed by leaning backwards past the maximum elevation. When a given crest leans forward, the adjacent troughs lean backward, and vice-versa. While geometrical asymmetry of real waves in nonlinear deep-water groups is well-established[5], the existence of a simultaneous appreciable change to the *crest* speeds of the waves has not been previously reported. This exciting discovery is now described in detail.

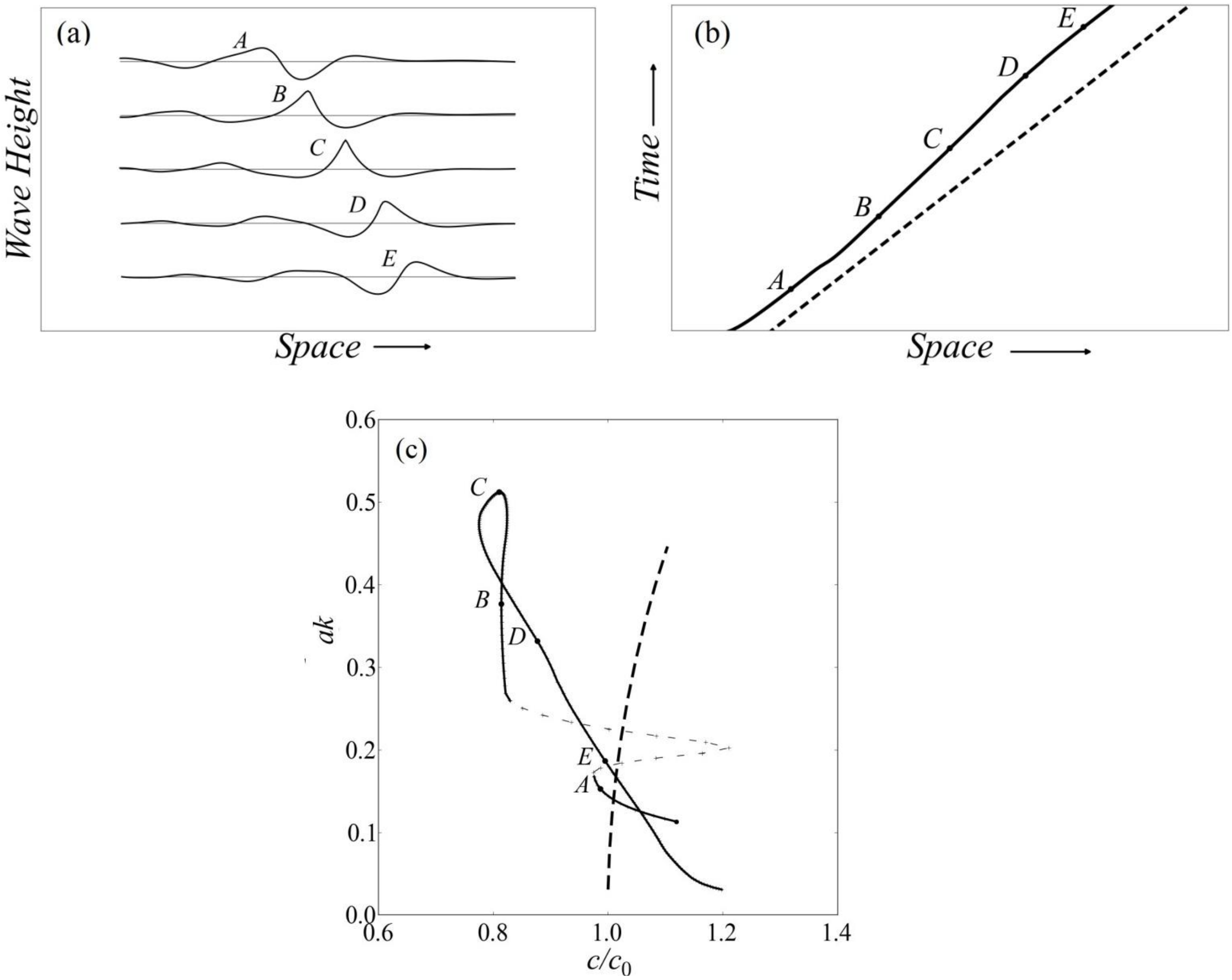

Figure 3. Tallest crest shapes and speeds for the numerically-simulated wave group in Figure 2 showing the five different evolution times A-E (a) waveheight profiles transitioning from forward-leaning through symmetry to backward-leaning (b) tallest crest location versus time. The steeper slope between B and D shows the crest-speed reduction relative to the linear wave speed c_0 (dotted line) (c) trajectory of the corresponding tallest crest speed c, normalized by c_0 as a function of the local wave steepness ak. The Stokes theory prediction (Equation 1) is shown (dashed line) for comparison. The apparent crest speed surge at ak ~ 0.2 is explained in the text.

Accompanying the leaning are significant crest (and trough) speed changes relative to the speed of a classical symmetrical (Stokes) wave. These can be measured by tracking the (horizontal) speed of a given wave crest profile in space and time. During the forward-leaning phase relative to the underlying waveform, the crest speed increases. Then, as the crest transitions to leaning backwards, the crest speed tends to decrease. Identifying a unique crest speed for such unsteady leaning waves may not be possible.

In any event, this leaning sequence is a *generic* feature of focussing in unconstrained, unsteadily-evolving wave groups. The generic crest speed slowdown can be identified in Figure 3(b) by the steeper slope of the displacement-time curve between B and D relative to the *linear* wave trajectory (speed c_0), shown as a dashed line. The actual speed reduction relative to c_0 is 18%. This lasts about one wave period and has a spatial extent of about one wavelength. Further, a typical hysteresis pattern is seen in Figure 3(c) when the crest speed is plotted against the local wave steepness *ak*.

The departure of the actual crest speed versus slope trajectory from the classical Stokes wave speed prediction (1) included in Figure 3(c) is striking and represents the main finding in this study.

In this case, the slowest crest speed marginally precedes the maximum waveheight, while other cases show the reverse of this sequence. Note that the peak in crest speed between A and B at ak~0.2 is an artefact associated with tracking the speed during complex crest transition behaviour seen in Figure 2.

The above discussion was for two-dimensional waves. Curvature (three-dimensionality) in the wave fronts does not significantly change the leaning or crest slowdown behavior. We investigated a number of cases of three-dimensional wave packets in both our fully nonlinear wave model study and in our complementary wave-basin investigation (described below). Both the model and wave-basin results confirmed the occurrence of leaning and crest slowdown, with the subsequent breaker crest speed remaining close to $0.8c_0$. As a consequence of the asymmetry of the dominant wave shape as it reaches its maximum height within the group, growing and decaying crests for waves of the same steepness *ak* do not have the same speed. This is evident in the example shown in Figure 3b.

To place these new results in the context of the classical wave phase speed, we found that if the motions of the adjacent left-hand and right-hand zero-crossing points are tracked within the numerical model, the average of these two speeds remains close to the linear theory wave speed c_0, but with significant local fluctuations. We also computed the mean energy flux speed for the wave packet and found it to be within 2% of the group speed for linear waves. Therefore aside from the strong unsteady leaning crest and trough motions, the waves, on the average, propagate largely as expected from Stokes theory.

Breaking onset and speed

Should the steepest wave proceed to break rather than recur, our numerical study found that breaking onset occurs when the tallest wave in the group attains its maximum steepness and when its crest speed is close to its minimum. This explains why breaking wave crests have been observed to travel at about 80% of the speed of the linear speed of the carrier wave[7,8,9]. This behaviour was found in all our computations and was verified in our laboratory measurements, which also confirmed directly that the subsequent mean breaker speed was close to $0.8c_0$ (see Figure 5b).

Is the slowdown a nonlinear effect?

Is the appearance of leaning modes associated only with steep waves? To resolve this, we carried out a theoretical study of a *linear* (small steepness) stochastic wave group that attains its maximal crest height at the focal point by a constructive superposition of a large number of elementary waves that form the wave energy spectrum[15,16]. Here this is assumed to have the conventional JONSWAP form characteristic of real ocean waves[17]. This component of the study revealed the important finding that the slowdown and leaning modes can be significant even during a linear wave-focusing crest event, provided the range of frequencies in the spectrum is sufficiently large.

Historically, in the theoretical analysis of dispersive linear wave groups, crest speed accelerations were noted after the maximum steepness was reached but the author[18] did not investigate the crest kinematics prior to the focal point. Our study found that for very narrow spectral bandwidths, there is negligible modulation of the wavetrains and slowdown approaching the focal point. For a modest range of spectral frequencies, the crests vary slowly along the group, and slight leaning and slowdown occurs around a point of maximum crest steepness. Figure 4a shows typical waveforms and the variation of the crest speed c/c_0 with local wave steepness. Figure 4b shows evidence of slowdown and weak hysteresis. However, for a broader range of spectral frequencies, significant leaning and slowdown are observed around a point of maximum crest steepness, as seen clearly in Figures 4c and 4d.

Further, analysis of the classical example of a steady wave group resulting from the superposition of two almost identical linear waves with the same amplitudes but slightly different wave numbers and frequencies did not show any leaning or slowdown. This is consistent with the very modest slowdown observed for a narrow range of spectral frequencies and with our companion laboratory experimental results for very low steepness (nearly linear) wave groups composed of two discrete frequencies.

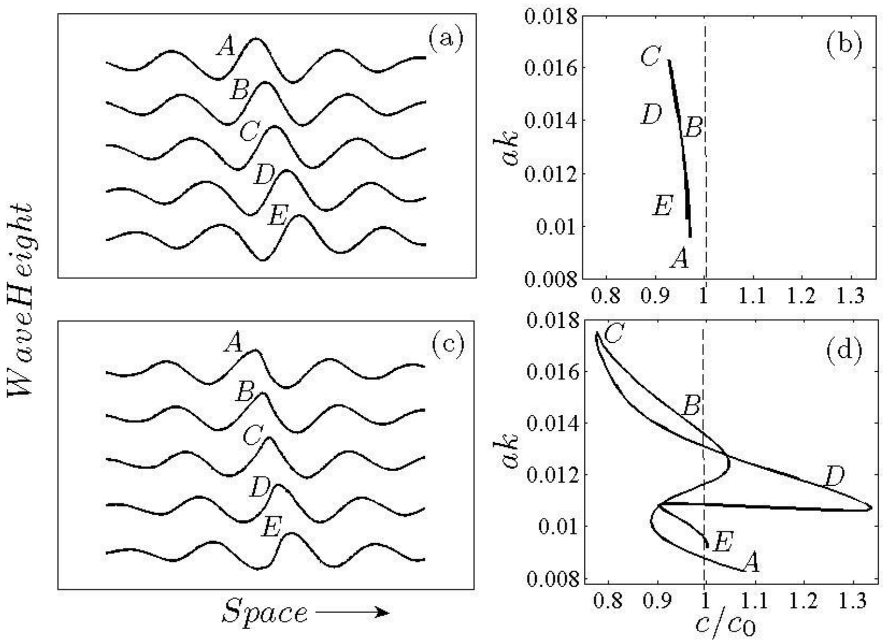

Figure 4. Surface profiles and their kinematic properties (a) water surface profiles of a linear wave group composed of a narrow range of frequencies from a JONSWAP spectrum (b) crest speed c/c_0 as a function of local steepness of the tallest wave shown in Figure 4(a) (c) water surface profiles of a linear wave group composed of a broad range of frequencies from a JONSWAP spectrum (d) crest speed c/c_0 as a function of local steepness of the tallest wave shown in Figure 4(c). For these linear simulations, Stokes' theory (Equation 1) predicts the crest speed $c/c_0 = 1$.

Wave basin measurements

Validation experiments to verify our model computations were performed in a wave basin 27m long, 7.75m wide and 0.55m mean water depth. Wave groups were generated at one end of the basin by a computer-controlled wave generator, comprising 13 bottom-cantilevered flexible plate segments. Wave directionality was achieved by varying systematically the wave phase of each lateral segment. An absorbing beach at the opposite end of the basin ensured that reflected wave energy remained less than 5% of the incident waves. Instantaneous water surface elevations were measured to within ±0.5 mm by an array of nine in-line wave probes, spanning approximately one wavelength. The array was movable enabling wave conditions to be monitored, from the paddle to slightly upstream of the absorbing beach.

Wave groups were generated to match those in the nonlinear wave computations, and their surface displacement evolution was measured in great detail using the wave probe array. Wave crests were identified, tracked between the wave probe signals and their motion interpolated using cubic splining. Crest speeds were determined by the time derivative of the local horizontal crest displacement. Wavenumbers were determined from the distance between the two zero-crossings adjoining a given crest.

The breaking crests were captured at 100 frames per second by an overhead 1024×1024 pixel digital video camera. The video imagery was corrected for lens and mounting distortion, and transformed onto a regular grid for analysis. The lateral extent and leading edge were identified manually for each individual breaker in the corrected video sequence. The velocity of the advancing breaker was determined by the time derivative of the displacement of the leading edge. Figure 5b shows the resulting mean trajectory for the ensemble of 240 breakers measured in this study. This highlights the breaker slowdown that was the primary motivation for this investigation.

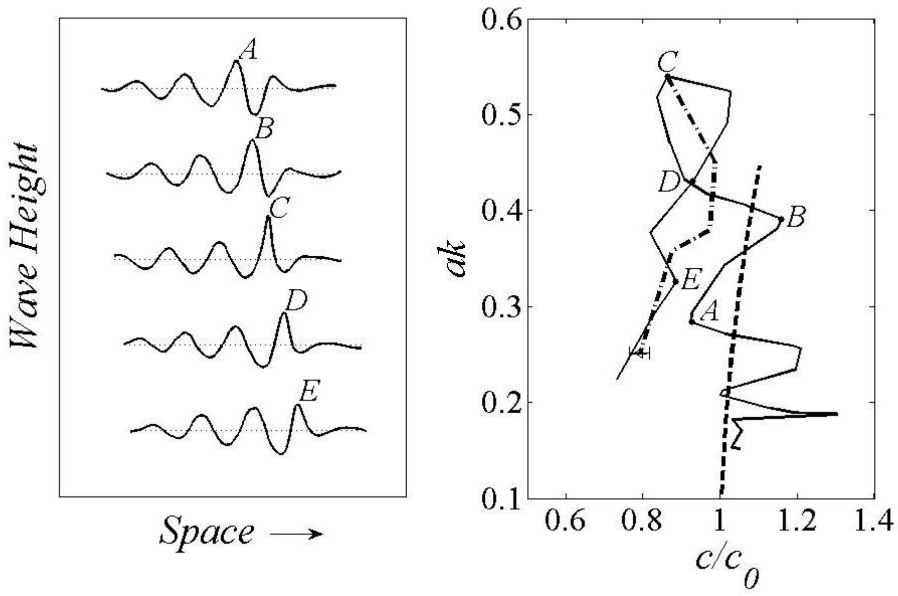

Figure 5. Surface profiles of a laboratory wave group and their kinematic properties. (a) measured water surface profiles captured at different evolution times A-E, with breaking initiation at C. (b) corresponding evolution trajectory of the corresponding crest speed c of the tallest wave normalized by the linear wave speed c_0 against its wave steepness *ak*. The ensemble-mean breaker speed trajectory and standard error are shown by the dot-dash line. The black dashed line shows Stokes' theoretical wavetrain prediction (Equation (1)).

Open ocean observations

Our stereo video ocean wave observation system *(WASS)* further validated these findings. *WASS* was deployed at the oceanographic tower *Acqua Alta* in the Northern Adriatic Sea, 16 km off the coastline of Venice in 16 m water depth. Stereo video measurements were acquired in three experiments during 2009-2010 to investigate space-time and spectral properties of ocean waves[19,20]. To maximize the stereocamera overlap area, *WASS* was deployed 12.5 m above the mean sea level at a 70^{o} depression angle. This provided a trapezoidal overlap area with sides of 30 m and 100 m, a width of 100 m and an imaged area of approximately 1100 m^2. We describe results from data acquired during Experiment 2, using 21,000 frames captured at 10 Hz. The mean wind speed was 9.6 m/s with a fetch of approximately 110 km.

Analyses performed included estimating the speed *c* of crests reaching maximum local steepness as they travelled through the imaged area, using the crest-tracking methodology described in the wave basin measurements. As the imaged area was too small to capture the entire wave group around the tallest wave, the local reference wave speed c_0 was calculated from the spectral peak frequency of a 120-second time series centred at the crest event using linear wave theory. The data were filtered to remove short wavelength waves and only waveheights greater than 25% of the maximum waveheight of 2.1 m were included in the analysed ensemble of 12,000 wave crests. Figure 6 shows the probability density function

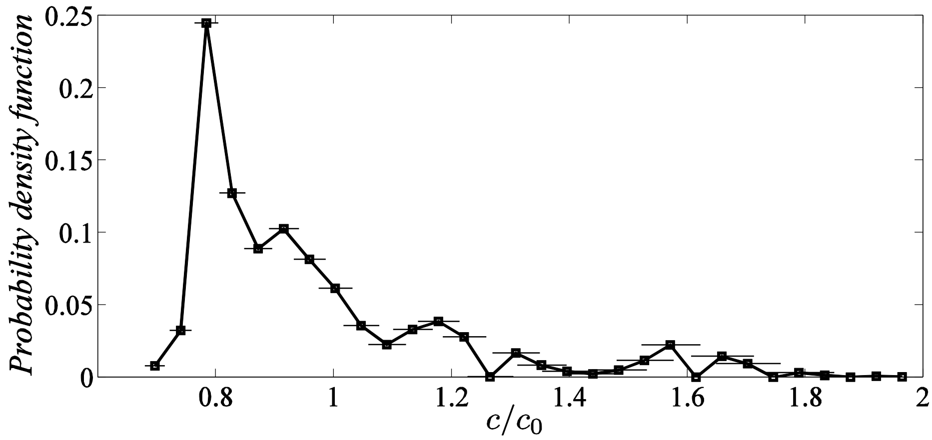

Figure 6. Probability density function of the normalized crest speed c/c_0 for all crests transitioning through a maximum local steepness in the field. Measurements were obtained from a 35 minute stereo video sequence recorded from an open ocean tower by the *WASS*. Note the tall peak at $c/c_0 \sim 0.8$. Local standard error bounds are indicated.

of c/c_0 which peaks at close to $0.8c_0$. This highlights the observed slowdown, which is consistent with that from the nonlinear simulations and experiments described above.

Discussion and conclusions

This study provides fundamental new insights into the speed and behaviour of dominant deep-water ocean waves in groups. The principal finding here is that *steady* nonlinear Stokes theory, the historical cornerstone of ocean wave physics, does not describe accurately the crest speeds of dominant ocean waves. Whereas Stokes theory predicts a weak increase with steepness, our study finds that the crest speeds of dominant waves can show a strong (up to order 20%) deceleration as they reach their maximum steepness within groups. The underlying physics is associated intrinsically with the *unsteady* wave group behaviour and accompanying wave energy convergence that occurs in the focal region near the centre of these groups, irrespective of whether the dominant wave evolves to breaking or not. It can even occur for a superposition of linear waves around around a point of maximum crest steepness, provided the range of frequencies in the spectrum is sufficiently broad.

This discovery has important consequences. Here we show that in oceanography, it explains and quantifies the puzzling generic slowdown of breaking wave crests, which is central to correctly assimilating breaking wave observations into sea-state forecast models. This discovery should provide new insights into the formation of the unusually large waves that appear from nowhere in the middle of the ocean, the so-called rogue waves, and their breaking. Key air-sea fluxes of momentum and energy, which depend respectively on the square and cube of the sea-surface velocity, may be modified appreciably. Internal waves in the atmosphere and ocean[21] have unsteady group structure and may experience similar effects to those described here. We also found that this slowdown effect occurs in wave computations using the Nonlinear Schrödinger (NLS) equation. This is a weakly-nonlinear equation commonly used to describe important wave phenomena in large-scale and small-scale geophysical flows[22], as well as in nonlinear optics[23]. Exploring the importance of this present discovery in these diverse contexts offers the potential for new insights in these allied fields.

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