Experiments on a gravity-free dispersion of large solid spheres in a Newtonian fluid under shear

By R. A. Bagnold, F.R.S.

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Dispersions of solid spherical grains of diameter $D = 0{\cdot}13$ cm were sheared in Newtonian fluids of varying viscosity (water and a glycerine-water-alcohol mixture) in the annular space between two concentric drums. The density σ of the grains was balanced against the density ρ of the fluid, giving a condition of no differential forces due to radial acceleration. The volume concentration C of the grains was varied between 62 and 13 %.

A substantial radial dispersive pressure was found to be exerted between the grains. This was measured as an increase of static pressure in the inner stationary drum which had a deformable periphery. The torque on the inner drum was also measured. The dispersive pressure P was found to be proportional to a shear stress T attributable to the presence of the grains.

The linear grain concentration λ is defined as the ratio grain diameter/mean free dispersion distance and is related to C by $\lambda = \dfrac{1}{(C_0/C)^{\frac{1}{3}} - 1}$, where C_0 is the maximum possible static volume concentration.

Both the stresses T and P, as dimensionless groups $T\sigma D^2/\lambda\eta^2$ and $P\sigma D^2/\lambda\eta^2$, were found to bear single-valued empirical relations to a dimensionless shear strain group $\lambda^{\frac{1}{2}}\sigma D^2(\mathrm{d}U/\mathrm{d}y)/\eta$ for all the values of $\lambda < 12$ ($C = 57$ % approx.) where $\mathrm{d}U/\mathrm{d}y$ is the rate of shearing of the grains over one another, and η the fluid viscosity.

This relation gives
$$T \propto \sigma(\lambda D)^2\,(\mathrm{d}U/\mathrm{d}y)^2$$
and
$$T \propto \lambda^{\frac{3}{2}}\eta\,\mathrm{d}U/\mathrm{d}y,$$
according as $\mathrm{d}U/\mathrm{d}y$ is large or small, i.e. according to whether grain inertia or fluid viscosity dominate. An alternative semi-empirical relation $\mathscr{T} = (1+\lambda)\,(1+\tfrac{1}{2}\lambda)\,\eta\,\mathrm{d}U/\mathrm{d}y$ was found for the viscous case, when $\mathscr{T}$ is the whole shear stress. The ratio T/P was constant at 0·3 approx. in the inertia region, and at 0·75 approx. in the viscous region.

The results are applied to a few hitherto unexplained natural phenomena.

1. Introduction

The effect of a dispersion of solid grains on the shear resistance of a fluid was calculated by Einstein (1906). Though the scope of the reasoning has since been extended, the conditions have always been limited to small grains at such a wide dispersion that the mutual effects of one grain on another can be neglected. The condition when these effects greatly predominate has been investigated by many workers in the rheological case of the pseudo-viscous flow of non-Newtonian systems containing a dispersion of macro-molecules or finely ground solids. But here complications arise from electro-chemical interactions.

The simple case of a high concentration of large solid spheres in a Newtonian fluid appears to have been neglected on the frontier between rheology and hydrodynamics.

Such a case must be of basic importance in the difficult problem of what happens close to the gravity bed of a stream of fluid which is transporting granular matter. The present work arose out of a conclusion by the writer that the phenomena observed here cannot be explained unless a dispersive grain pressure is postulated, of such a magnitude that an appreciable part of the moving grains is in equilibrium between it and the force of gravity, whether or not fluid turbulence also plays a part.

In the static case of closest contact packing a grain mass behaves as a deformable solid. But, as first noticed by O. Reynolds (1885), a shear stress causes a tendency to dilate. If this is resisted by a compressive stress the ratio shear stress-compressive stress is near unity. But if the dilation is allowed to proceed to the static limit, the ratio falls to that given by the tangent of the angle of repose. It will be shown that these relations appear to have their counterparts in the dynamic case of continuing shear strain when the limit of static dilation has been far exceeded and the grain mass has become a dispersion.

In view of the possible application of the results to hydraulic problems and to ore-dressing techniques the choice of appropriate symbols for new quantities has needed thought. In the general case the grains are surrounded by a fluid. Quantities pertaining only to the grains such as pressure, shear stress and strain, velocity, and to some extent viscosity, are capable of co-existence, in different values at the same point, with the mean values of the same quantities pertaining only to the fluid. Hence a distinction by mere suffixes seems inadequate and likely to confuse. The system has therefore been adopted whereby the capital form of the accepted symbol is used for the grain quantity and the usual small form for the fluid quantity. Mixed quantities are denoted by capital script symbols, e.g., $\mathscr{T}$ for total shear stress. Conflict already exists over the use of η and μ for viscosity. Since the subject essentially involves discrete particles whose smaller sizes are often expressed in μ the symbol η current in physics is used here for the fluid viscosity.

2. Theory

Suppose a mass of rigid spheres of uniform diameter D and density σ to be arrayed in contact in the tetrahedral-rectangular piling which gives minimum pore-space (Reynolds's 'normal' or 'cannon-ball' piling), and suppose the mass now to be dispersed uniformly so that the distance D between centres is increased to bD. If the resulting free distance is s, we have

$$b = s/D + 1 = 1/\lambda + 1, \tag{1}$$

where $\lambda = D/s$ represents the 'linear concentration'.

The ratio C of the grain-occupied space to the whole space, i.e. the volume concentration, is given by

$$C = \frac{C_0}{b^3} = \frac{C_0}{(1/\lambda + 1)^3}, \tag{2}$$

where C_0 is the maximum possible concentration when $\lambda = \infty$ $(s = 0)$ and is equal to $\pi/3\sqrt{2} = 0{\cdot}74$ for spheres.

The normal piling array is such that parallel planes can include the centres of adjacent spheres either as a square, or as a triangular, pattern. The distance between centres of adjacent 'triangular' grain layers is $\sqrt{\tfrac{2}{3}}\,bD$. In order that one of these layers may just slide over another, the value of λ cannot exceed 22·5. The distance between 'square' layers is $\tfrac{1}{2}\sqrt{2}\,bD$, and for sliding to take place between those 'square' layers, the maximum value of λ is 8·3. If the spheres are not perfectly matched for size, or if shearing were to take place along parallel curved surfaces, we might expect general shearing to become possible at some intermediate value of λ, say 12 to 14. At lower values of λ the grains should pass one another with progressively greater freedom. Alternatively, it might be said that the chances of shear stress being transmitted by continuous and simultaneous contact between many grains should decrease rapidly with decreasing λ.

The following general assumptions are now made:

(*a*) The dispersion is in a state of uniform shear strain $\mathrm{d}U/\mathrm{d}y$, and the mean relative velocity between the grains and the intergranular fluid is everywhere zero.

(*b*) The shear strain $\mathrm{d}U/\mathrm{d}y$ is somehow maintained constant through time by externally applied forces in the x-direction acting on all the grains.

(*c*) The kinetic energy per unit volume of the system is maintained constant by frictional losses.

(*d*) The motion of the grains consists, in addition to a drift in the x-direction, of oscillations in all three directions, involving approaches to, and recessions from, neighbouring grains.

Case I. When the effects of grain inertia dominate

The oscillations are here supposed due to successions of glancing collisions as the grains of one layer overtake those of the next slower layer. In figure 1*a* the grains of layer B are sheared over those of layer A at a mean relative velocity $\delta U = kbD\,\mathrm{d}U/\mathrm{d}y$, where k has some value between $\sqrt{\tfrac{2}{3}}$ and $\tfrac{1}{2}\sqrt{2}$, and collisions result between A and B grains. For simplicity the phasing of the collisions is supposed such that all

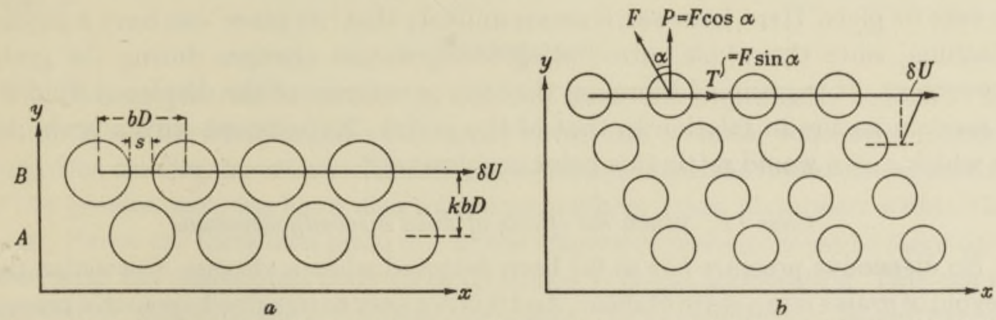

FIGURE 1. *a*, Cross-section of equidistant grain arrangement (three-dimensional). (Grains of alternate layers displaced in *z*-direction.) *b*, Two-dimensional sketch of possible statistically preferred grain arrangement (non-equidistant) which might allow of dispersive pressure proportional to shear stress, in a viscous fluid.

collisions occur at the ends of mutual approaches. In this way the A grains can be replaced by a rigid granular surface of infinite inertia, and we can consider the B grains alone. Apart from this artifice, however, the collisions are supposed to occur at random, and it is assumed that this results in a uniform mean distribution of the grains in space.

Let each B grain make $f(\lambda)\,\delta U/s$ collisions with an A grain in unit time, where $f(\lambda)$ is unknown. The number of grains in unit area in the xz-plane in each layer is $1/b^2D^2$, and at each collision each B grain experiences a total change of momentum $2m\,\delta U\cos\alpha_i$ (for elastic grains) in the y-direction, where α_i is some unknown angle determined by the collision conditions, including grain rotation, in this inertia state of grain motion.

It follows that a repulsive pressure P_y should exist between the grains of the two layers, of magnitude

$$P_y = \frac{1}{b^2D^2}\frac{f(\lambda)\,\delta U}{s}2m\delta U\cos\alpha_i$$

$$= a_i\sigma\lambda f(\lambda)\,D^2\,(\mathrm{d}U/\mathrm{d}y)^2\cos\alpha_i, \tag{3}$$

together with a proportional grain shear stress T_{xy} of magnitude

$$T_{xy} = P_y\tan\alpha_i, \tag{4}$$

the shear stress T being additional to any residual stress τ' due to momentum transfer within the intergranular fluid.

When $\lambda \ll 1$, $f(\lambda)$ should become proportional to λ, i.e. the stresses should decrease as the square of the distance of separation, but there seems no *a priori* reason why this should be so at the high grain concentrations considered here.

Equation (3) gives the dispersive grain pressure normal to the direction of shear. There should also be pressure components in other directions, but their magnitudes are uncertain. Since random grain movement has been assumed, together with a mean free distance s which is uniform from one direction to another, it seems likely that the grain stresses will be unaffected by the persistence of a residual turbulence in the combined grain fluid.

It will be noted that the density ρ of the intergranular fluid does not enter into (3). If a single solid body is moved through a fluid, the total rate of momentum transfer is measured by $(\sigma-\rho)$ because the fluid tends to flow back round the body to take its place. Here, however, it seems unlikely that 'its place' can have a physical meaning, since the whole surrounding configuration changes during the grain's movement. It is assumed therefore that the movement of the displaced fluid is of a random nature in relation to that of the grains. Experiment with a grain fluid in which $\sigma = \rho$ would settle this point conclusively.

Case II. When the effects of fluid viscosity dominate

No dispersive pressure has so far been detected when a viscous Newtonian fluid devoid of grains is in a state of shear. And it is not easy to imagine how such a pressure can arise if a dispersion of grains in the fluid remains isotropic. If, however, the passage of one grain over another involves a temporary reduction of the x-component of the shear velocity from its mean value δU during approach, followed by an increase during recession, a statistically preferred order must result, such as that sketched in figure 1*b*. This should give rise to a transverse pressure analogous to that which occurs in the static 'arching' of grains in contact. A close analogy is also afforded by the arrangement and transverse dispersive tendency of a dense overtaking motor traffic along a one-way road.

Suppose the amplitude ζ of the shear velocity fluctuation bears some ratio $f'(\lambda)$ to the mean shear velocity δU, so that $\zeta = \delta U f'(\lambda)$. If the total instantaneous shear velocity is ξ, the value of the instantaneous shear stress $\mathscr{T}$ is given by

$$\mathscr{T} = \frac{\eta}{s}\xi.$$

And if the velocity fluctuation is assumed to be approximately harmonic

$$\xi = \delta U + \zeta \sin\theta = \delta U(1 + f'(\lambda)\sin\theta).$$

The work done during a displacement of one grain layer over another which covers one complete fluctuation is given by

$$\int_0^{2\pi} \mathscr{T}\xi \, \mathrm{d}\theta,$$

and the mean shear stress $\overline{\mathscr{T}}$ during this displacement will be

$$\overline{\mathscr{T}} = \frac{\eta}{2\pi s \delta U}\int_0^{2\pi} \xi^2 \, \mathrm{d}\theta = \frac{\eta \delta U}{2\pi s}\int_0^{2\pi} (1 + f'(\lambda)\sin\theta)^2 \, \mathrm{d}\theta.$$

Since $$\frac{\delta U}{s} = \frac{bD}{s}\frac{\mathrm{d}U}{\mathrm{d}y} = (1+\lambda)\frac{\mathrm{d}U}{\mathrm{d}y},$$

whence $$\overline{\mathscr{T}} = \eta(1+\lambda)\left(1 + \frac{f'(\lambda)}{2}\right)\frac{\mathrm{d}U}{\mathrm{d}y}. \tag{5}$$

It will be noted that in the inertia case I the shear stress T is purely a grain stress, to be regarded as additive to any fluid shear stress τ' which may persist by virtue of a residual fluid turbulence, and as vanishing when $\lambda = 0$ and τ' become τ. So in the inertia case we can write $\mathscr{T} = T + \tau'$ (replacing $\overline{\mathscr{T}}$ by $\mathscr{T}$). But in the viscous case II, $\mathscr{T}$ is a mixed shear stress due to the effect of fluid viscosity as modified by the presence of grains, and cannot legitimately be split into grain and fluid elements T and τ'.

3. Experiment

The nearest practical approach to uniform shear strain probably occurs in the annular space between an inner stationary drum and an outer concentric rotating drum. The smaller the ratio radial difference/mean radius, the nearer the approach. In the present case we have also to ensure uniform grain dispersion within this space. Hence the force on a grain due to the dispersive pressure must be large compared with the differential body force on it due to radial acceleration. This can be achieved by balancing the grain density σ against the fluid density ρ. But an experimental range of grain density is thereby precluded. To restrict the grains to the annular space the free mechanical clearances outside the edges of the inner drum cannot exceed, say, half the grain diameter, and this sets a lower practical limit to grain size. Finally, the annular space must clearly be a reasonable number of grain diameters in radial width; and this restricts the range of grain size it is possible to use in one size of apparatus.

The grains used were spherical droplets of a nearly 50 % mixture of paraffin wax and lead stearate with a very constant diameter of 0·132 cm. Their density differed from that of water by less than 0·001. The measured dispersive grain pressure was of the order of 50 to 700 times the differential pressure due to rotation according to the grain concentration. So this differential pressure could be entirely neglected. The advantage of the wax material over anything harder is that a fragment which would otherwise jam the clearances disintegrates before any damage is done and before a drag reading is affected. No troubles were encountered from this cause, or from any general degradation of the material during use, provided a safe maximum of collision velocity is not exceeded. At excessive speeds there is an appreciable falling off in the drag presumably due to contact melting.

The apparatus shown in figure 2 was designed to give preliminary confirmation of the reasoning outlined in case 1. The existence of a dispersive grain pressure in the linear flow region of a Newtonian fluid was not contemplated at the time.

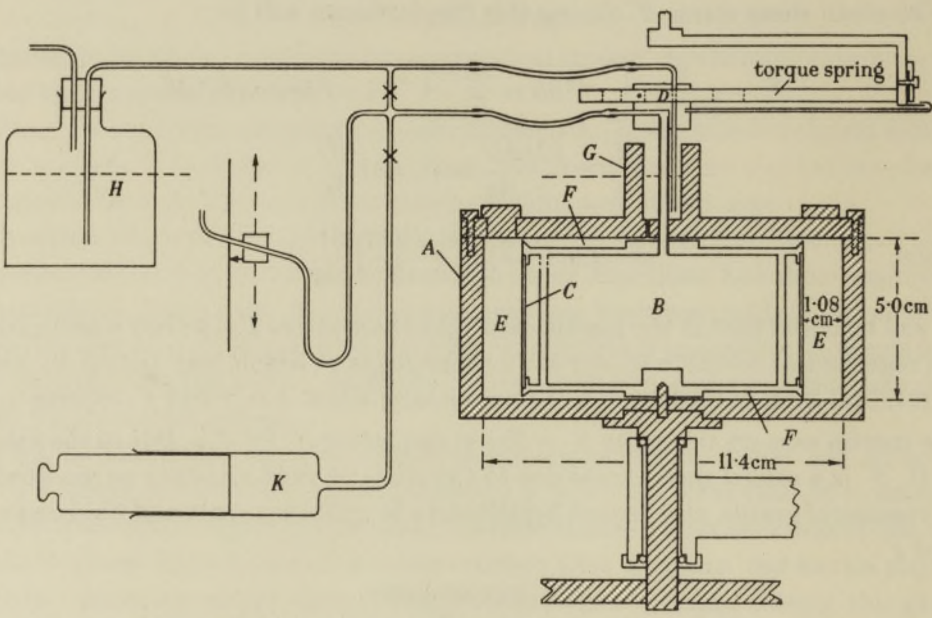

FIGURE 2. The apparatus (rotating parts shown hatched).

The outer drum A, solidly built of Perspex and brass, could be rotated through change-wheels at a series of known speeds ranging from 0·25 to 8·6 rev/sec. The periphery C of the inner drum B consisted of sheet rubber sealed to the drum flanges, the only access to the inside of B being through the hollow upper spindle and the head D. To this head a calibrated spring was attached for torque measurement. The flanges of B extended to make free but grain-proof clearances with the end-plates of the outer drum. The flanges were bevelled off to a knife edge to reduce the likelihood of jamming by grain fragments.

The whole of the outer system, including the end-spaces FF, was initially filled with water up to a convenient level in the open stand-pipe G. By means of a stationary syphon held in place by the head D this level was reproduced in the wide reservoir H. The inner drum was permanently filled with water, and was connected through the head D to a constant-volume manometer. A short-circuit between the two systems enabled any static pressure difference to be eliminated before use. A pump K enabled air bubbles to be driven out of the whole apparatus when needed.

On rotating A the manometer indicated the excess pressure in the annular space E due to the rotation of the fluid in the end-spaces FF, from the axis out to the radius of the inner drum periphery. This was recorded over the range of speeds. The torque on the inner drum was also recorded. This consisted of contributions from (*a*) spindle friction, (*b*) fluid drag on the ends, (*c*) drag on the periphery due to the fluid in the annular space. The plain-fluid values of (*c*) alone were found from the difference between the torques when the liquid filled the end-spaces and when air was locked in them. Both the plain-fluid pressures and the plain-fluid torque contributions (*a*) and (*b*) were assumed to remain unchanged when grains were added to the annular space.

A measured volume of grains was now put into the annular space, the outer system was topped up with fluid and the pressures were equalized by opening and closing the short-circuit. Pressures and torques were again measured over the same range of speeds as before. This process was repeated over a range of grain concentrations between 62 and 13 % ($\lambda = 17$ to 1·3).

The excess pressure with grains in the annular space over that with the plain fluid was assumed to give the dispersive grain pressure P directly. To arrive at the corresponding grain drag T it was first necessary to deduct from the gross measured torque the instrumental torque due to (*a*) and (*b*) above. But the difficulty remained that the contribution (*c*) clearly does not stay unchanged as the grain concentration is increased from zero. In the case of the plain fluid ($\lambda = 0$) this contribution τ was due almost wholly to turbulence. And while at very low concentrations the residual value of τ' must be nearly equal to τ, increasing the grain concentration not only decreased the amount of fluid present but also suppressed the turbulence in what was left. As an approximation therefore it was assumed that the residual fluid drag contribution τ', to be deducted in order to give T, was equal to $\tau/(1+\lambda)$. The effect of this adjustment was negligible at high grain concentrations and was only appreciable at low concentrations at the highest speeds, when grain inertia was dominant, as will be seen from figure 3.

Subsequent measurements of P and T were made at one selected grain concentration, corresponding to $\lambda = 11$ at higher fluid viscosities. For this a density-balanced mixture of glycerine and methylated spirit was added to the intergranular water.

Objections might be raised that (i) an excess pressure on the fluid in the inner drum might arise not from any dispersive grain pressure outside the rubber wall but from a tendency for the wall to shrink owing to increased tangential stress on it; and (ii) both pressure and drag deadings might be seriously affected by slip at the peripheries of the annular space. Objection (i) was ruled out by the use of pure glycerine as a fluid. In the absence of grains the speed of rotation was made such as to give a torque on the inner drum of twice the maximum value obtaining in the main experiments. This speed was too low to give any measurable pressure whatever in the inner drum. Grains were now added, and the speed was greatly reduced to give approximately the same torque as before. A large pressure was immediately indicated. Objection (ii) was to some extent met as follows. The main experiments were duplicated. At first the cylindrical boundaries to the annular space consisted of polished Perspex and smooth rubber sheet at the outer and inner walls respectively. Both boundaries later consisted of rough rubber sheet. When correction

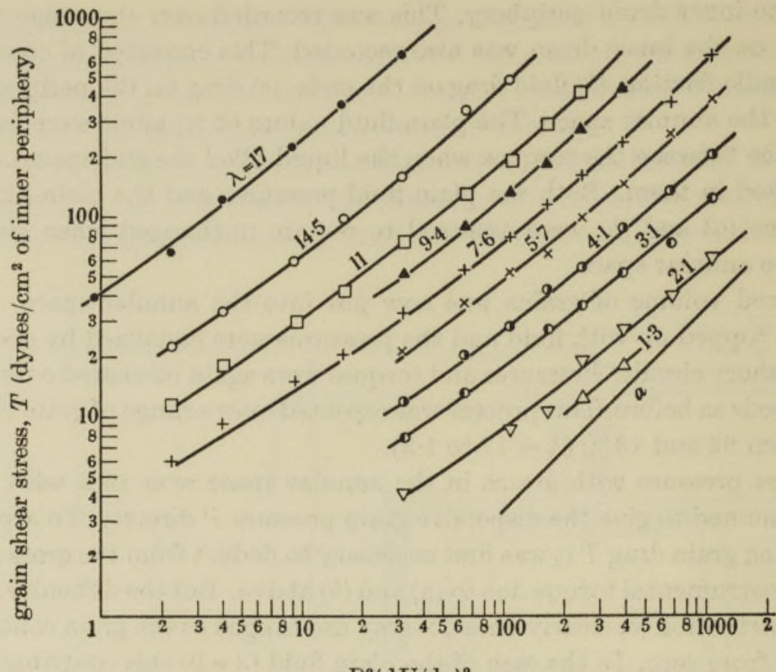

FIGURE 3. T is obtained by converting into dynes/cm² of inner periphery the gross measured torque *less* the torque due to plain-fluid drag on drum ends and to spindle friction *and less* the residue, estimated as $\tau/(1+\lambda)$ of the measured plain-fluid drag on the inner drum periphery. Values of λ (linear concentration = grain diameter/free linear dispersion) are indicated on the curves. Curve a, plain fluid shear stress τ ($\lambda = 0$) at inner periphery. $\sigma = 1$, $\rho = 1$, $D = 0{\cdot}132$ cm, $\eta = 0{\cdot}01$ (water).

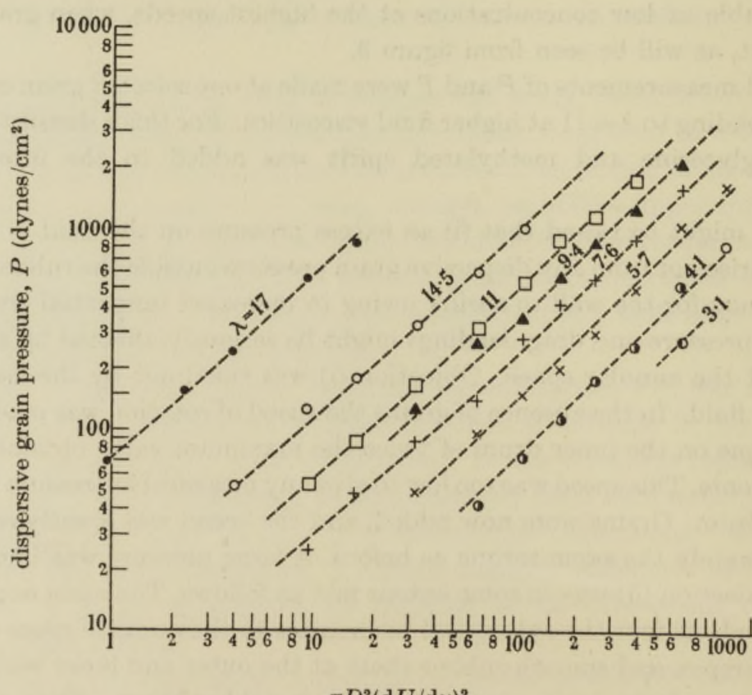

FIGURE 4

was made for the small change in the thickness δR of the space, the two sets of results were the same.

Finally, to gauge the effects of rotation on the results, the torque spring was transferred to the outer drum, and the inner drum was rotated. Though the pressures could not now be measured, the drag values, adjusted for τ' in the same way as before, were not appreciably different, in spite of the greatly increased values of R.

4. Experimental results and their discussion

(i) *General*

Errors due to non-uniform shear strain at the ends of the annular space were unavoidable and difficult to assess. The high local value at the inner corners of the space must to some extent have been offset by correspondingly low values at the outer corners, but this source of potential error remains. Error may also arise from the assumption of uniformity of shear strain along the radius of the annular

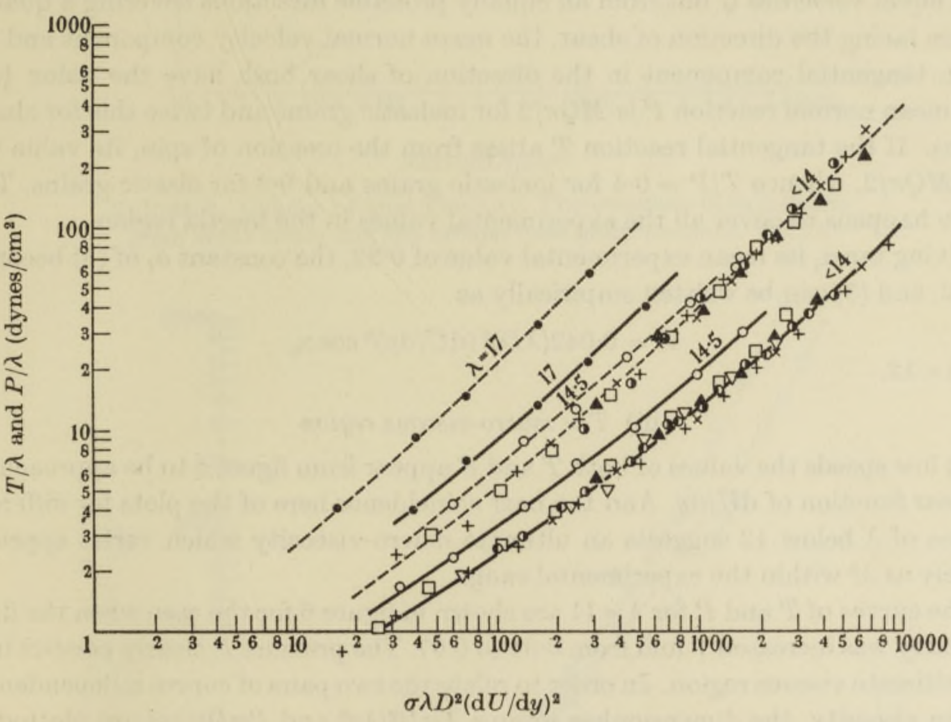

Figure 5. The complete lines indicate shear stress and the broken lines pressure.

	●	○	□	▲	+	×	◑	◐	▽	△
λ	17	14·5	11	9·4	7·6	5·7	4·1	3·1	2·1	1·3
C (%)	62·3	60·6	55·5	53·2	49·5	44·5	37·3	30·8	22·2	13·5

space. An appreciable error may exist in the highest values of λ owing to the very rapid rate at which this quantity increased with C by which it must be measured. The error in the value of 17 for λ may amount to 10%, but it should decrease rapidly with decreasing λ.

The values of shear stress and grain pressure are plotted in figures 3 and 4 against the corresponding values of $\sigma D^2(\mathrm{d}U/\mathrm{d}y)^2$, $\mathrm{d}U/\mathrm{d}y$ being the nominal value $R_m\omega/\delta R$

of the shear strain rate. In figure 3 the plain water shear stress τ is also shown by way of contrast. The general pattern of both figures at once suggested replotting in terms of T/λ and P/λ against $\lambda\sigma D^2/(\mathrm{d}U/\mathrm{d}y)^2$ as shown in figure 5.

(ii) *The grain-inertia region*

It will be seen from figure 5 that for all values of λ within the experimental range both T and P become proportional to $(\mathrm{d}U/\mathrm{d}y)^2$ at sufficiently high speeds.

In this high-speed or inertia region both T and P become proportional to λ^2 for all values of λ less than 12 approx.

Hence $f(\lambda)$ in (3) becomes empirically proportional to λ.

The ratio $T/P = \tan\alpha_i$, for $\lambda < 12$, seems to be approaching a constant value of about 0·32. For $\lambda > 12$ when $f(\lambda)$ is some higher power of λ than unity, $\tan\alpha_i$ increases to about 0·4. It is of interest, as Sir Geoffrey Taylor, F.R.S., has pointed out to me, that if spherical grains having no initial spin strike a plane boundary with equal velocities Q but from all equally probable directions covering a quarter sphere facing the direction of shear, the mean normal velocity component and the mean tangential component in the direction of shear both have the value $\frac{1}{2}\pi Q$. The mean normal reaction P is $MQ\pi/2$ for inelastic grains and twice this for elastic grains. If the tangential reaction T arises from the creation of spin, its value will be $\frac{2}{5}MQ\pi/2$, whence $T/P = 0{\cdot}4$ for inelastic grains and 0·2 for elastic grains. This range happens to cover all the experimental values in the inertia region.

Giving $\tan\alpha_i$ its mean experimental value of 0·32, the constant a_i of (3) becomes 0·042, and (3) can be written empirically as

$$P = 0{\cdot}042(\lambda D)^2(\mathrm{d}U/\mathrm{d}y)^2\cos\alpha_i \tag{6}$$

for $\lambda < 12$.

(iii) *The macro-viscous region*

At low speeds the values of both T and P appear from figure 5 to be approaching a linear function of $\mathrm{d}U/\mathrm{d}y$. And the near coincidence here of the plots for different values of λ below 12 suggests an ultimate macro-viscosity which varies approximately as $\lambda^{3/2}$ within the experimental range.

The curves of T and P for $\lambda = 11$ are shown in figure 6 for the case when the fluid viscosity was increased 7-fold from 0·01 to 0·07. The pressure P clearly persists into the ultimate viscous region. In order to relate the two pairs of curves independently of the viscosity, the dimensionless groups $T\sigma D^2/\lambda\eta^2$ and $P\sigma D^2/\lambda\eta^2$ are plotted in figure 7 against the dimensionless shear strain group

$$\frac{\text{inertia stress}}{\text{viscous stress}} = \frac{\lambda^2\sigma D^2(\mathrm{d}U/\mathrm{d}y)^2}{\lambda^{3/2}\eta\,\mathrm{d}U/\mathrm{d}y} = \frac{\lambda^{1/2}\sigma D^2\,\mathrm{d}U/\mathrm{d}y}{\eta} = N.$$

It will be seen that the curves of both T and P for the two different viscosities join up satisfactorily.

The ratio $T/P = \tan\alpha$ increases progressively from 0·32 through the transition region, till it finally reaches another constant value at about 0·75.

The shear-stress curve in the ultimate viscous range gives an empirical relation

$$T = 2{\cdot}25\lambda^{3/2}\eta\,\mathrm{d}U/\mathrm{d}y. \tag{7}$$

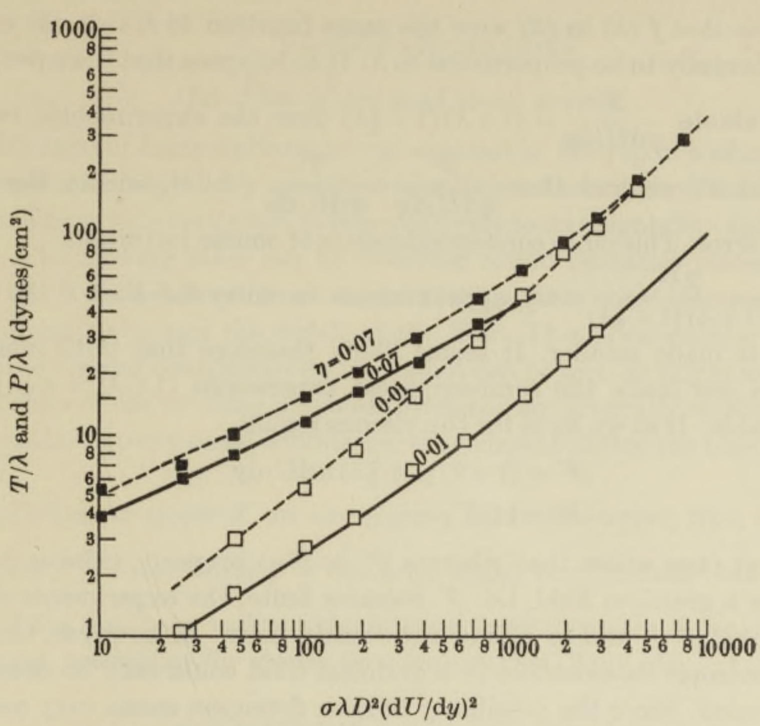

Figure 6. Change of fluid viscosity. The complete lines indicate grain shear stress and the broken lines grain pressure. $\lambda = 11$.

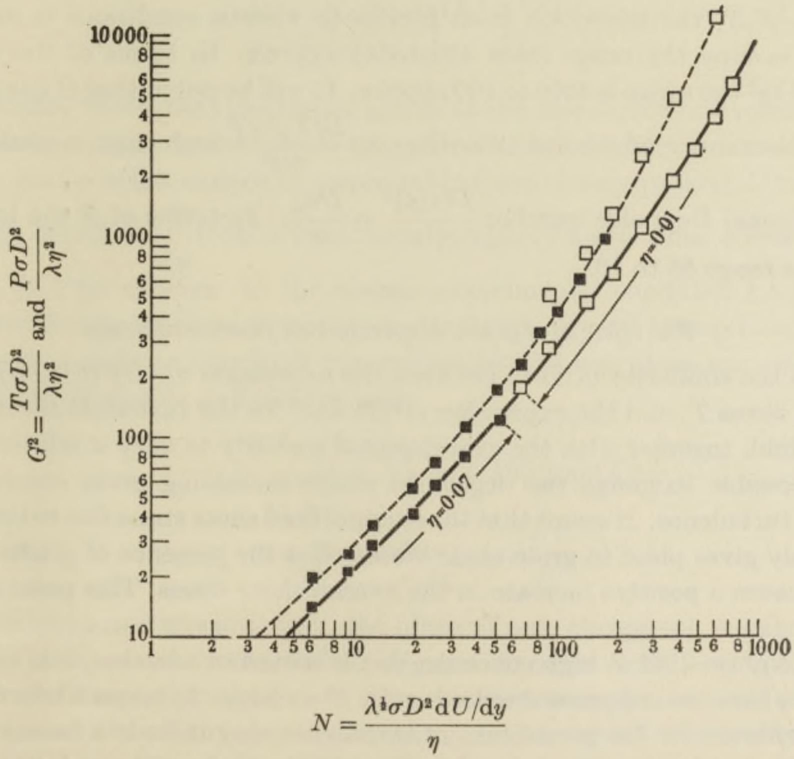

Figure 7. Conformity of experimental results at different viscosities. Complete line shear stress; broken line pressure. $\lambda = 11$. □, $\eta = 0{\cdot}01$ (water); ■, $\eta = 0{\cdot}07$.

Now suppose that $f'(\lambda)$ in (5) were the same function as $f(\lambda)$ in (3) which was found experimentally to be proportional to λ. It so happens that if we put $f'(\lambda) = \lambda$ in (5) and evaluate $\dfrac{\mathscr{T}}{\eta\, \mathrm{d}U/\mathrm{d}y} = (1+\lambda)(1+\frac{1}{2}\lambda)$ over the experimental range of λ between 2 and 10 we find that $\dfrac{\mathscr{T}}{\eta\, \mathrm{d}U/\mathrm{d}y} \sim \dfrac{T}{\eta\, \mathrm{d}U/\mathrm{d}y} \sim 2{\cdot}0\lambda^{\frac{3}{2}}$, within the limits of experimental error. This close correspondence is of course fortuitous.

The ratio $\dfrac{2\lambda^{\frac{3}{2}}}{(1+\lambda)(1+\frac{1}{2}\lambda)}$ only approximates to unity for $\lambda > 2{\cdot}5$. It decreases rapidly as λ is made smaller. It seems likely therefore that until more precise measurements are made the semi-empirical expressions $(1+\lambda)(1+\frac{1}{2}\lambda)$ may be the more reliable. If so we have for the viscous region

$$\mathscr{T} = (1+\lambda)(1+\tfrac{1}{2}\lambda)\,\eta\, \mathrm{d}U/\mathrm{d}y,$$
$$P = 1{\cdot}3\mathscr{T}. \tag{7a}$$

The point at once arises that whereas $\mathscr{T}$ in (7a) properly reduces to $\eta\, \mathrm{d}u/\mathrm{d}y$ when $\lambda = 0$ for a grainless fluid, i.e. $\mathscr{T}$ remains finite, the experiments show that P is proportional to $\mathscr{T}$ and so should also remain finite. If, however, the pressure P is in fact isotropic its existence in a grainless fluid could only be detected as a volume expansion. Since the possibility of such detection seems very remote, the point need raise no real difficulty.

(iv) *The transition region*

In terms of N, the transition from inertia to viscous conditions is seen from figure 7 to occupy the range from 450 to 40 approx. In terms of the ordinate $G^2 = \sigma D^2 T/\lambda\eta^2$ the range is 3000 to 100 approx. It will be noted that G has the form of a Reynolds number which can be written as $\dfrac{D(T/\lambda\sigma)^{\frac{1}{2}}}{\eta/\sigma}$ and which is analogous to the conventional Reynolds number $\dfrac{D(\tau/\rho)^{\frac{1}{2}}}{\eta/\rho} = \dfrac{Du_*}{\nu}$. In terms of R the transition occupies the range 55 to 10.

(v) *The effect of a grain dispersion on fluid turbulence*

Owing to the similarity of form between the expression $\sigma(\lambda D)^2 (\mathrm{d}U/\mathrm{d}y)^2$ for the grain shear stress T, and the expression $\rho l^2(\mathrm{d}u/\mathrm{d}y)^2$ for the turbulent shear stress τ in a plain fluid, together with the experimental inability to vary σ relatively to ρ, it was impossible to gauge the degree to which increasing grain concentration suppressed turbulence. It seems that the residual fluid shear stress due to turbulence progressively gives place to grain shear stress. But the presence of grains appears always to cause a positive increase in the overall shear stress. This point seems of some importance.

When the grain fluid at high concentration is stirred in a beaker, it is noticeable that small eddies are suppressed more readily than large. It seems likely therefore that any criterion for the persistence of turbulence may include a function of the ratio, size of whole system/size of grains. The additional shear stress due to residual turbulence must, however, be negligible at high grain concentrations.

5. Various applications

(a) *Flow of dry sand under gravity*

When dry sand of fairly uniform size is supplied at the top of a slope of the same sand inclined at an angle very near that of collapse, the supplied sand moves down the slope as a flow, i.e. every grain moves relatively to its neighbours and the shearing is general. The moving mass can be distorted round obstacles placed in its path. The flow has a well-defined front and advances at a constant speed depending only on the grain size and the depth of the flow. This phenomenon has hitherto never been explained quantitatively, though it can be seen on any steep dune slope.

If the results of the rotating drum experiments are generally applicable, beyond the very special experimental conditions, they should enable the above speed to be calculated.

The applied shear stress $\mathscr{T}$ on any plane y below the upper, free, surface of the sand is $\sigma g \sin\beta \int_y^0 C\,\mathrm{d}y$, where β is the slope angle. Anticipating that the very low viscosity of the fluid, air, will put the internal resisting grain shear stress T in the inertia region, conditions for steady flow require that, from (6),

$$a_i \sigma \lambda^2 D^2 (\mathrm{d}U/\mathrm{d}y)^2 \sin\alpha = \sigma g \sin\beta \int_y^0 C\,\mathrm{d}y,$$

whence

$$\frac{\mathrm{d}U}{\mathrm{d}y} = \left(\frac{g\sin\beta}{a_i \sin\alpha}\right)^{\frac{1}{2}} \frac{\left(\int C\,\mathrm{d}y\right)^{\frac{1}{2}}}{\lambda D}. \tag{8}$$

Observation shows that the visible grains at the free surface are about as closely packed as they are in random static piling for which $C = 0{\cdot}6$ for most natural sands. Since the grains below cannot be appreciably more closely packed, C must be constant with depth, so $\int_y^0 C\,\mathrm{d}y = 0{\cdot}6y$, and λ probably has a value corresponding to the limit 22·5 for spheres. In the nearest experimental condition $\lambda = 17$, and the experimental value of $a_i \sin\alpha$ from figure 5 is 0·076. Since T is seen to increase very rapidly with λ at these very high concentrations, the real figure may well be twice this. But taking $a_i \sin\alpha$ as 0·076 we have

$$\frac{\mathrm{d}U}{\mathrm{d}y} = \left(\frac{0{\cdot}6\sin\beta}{0{\cdot}076}\right)^{\frac{1}{2}} \frac{g^{\frac{1}{2}} y^{\frac{1}{2}}}{\lambda D} = 0{\cdot}165\,(g\sin\beta)^{\frac{1}{2}} \frac{y^{\frac{1}{2}}}{D}$$

and

$$U = \tfrac{2}{3} \times 0{\cdot}165 (g\sin\beta)^{\frac{1}{2}} \frac{y'^{\frac{3}{2}}}{D}, \tag{9}$$

where y' is measured upwards from the plane of zero movement.

A fairly uniform quartz sand of mean grain diameter 0·035 mm was allowed to flow down the simple flume shown in figure 8. By raising the reservoir gate a given distance a sand flow of any required height could be started, and could be stopped by closing the gate. The passage of repeated flows was found to leave the 'level' of the sand bed constant to within one grain diameter. The height Y of the flow above this level was measured together with the speed. The angle β was constant at 33°. The results are given in table 1. The calculated values are 50 % too big, which was rather to be expected; but the figures are consistent and are of the right order.

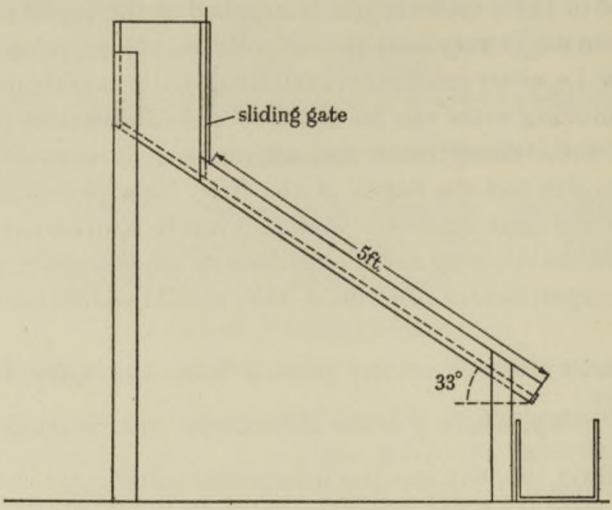

FIGURE 8

TABLE 1

flow height Y (cm)	measured speed (cm/sec)	speed, from (9) (cm/sec)	ratio
0·5	17·2	26·4	1·53
0·65	27·5	38·8	1·41
0·75	30·0	48·0	1·6
0·9	39·0	63·0	1·61

(b) *Sorting of grains of mixed size*

In the inertia region of grain flow the dispersive pressure P should vary as D^2, for a given shear stress dU/dy. This suggests that when grains of mixed size are sheared together, the larger grains should tend to drift towards the zone of least shear strain, e.g. towards the free surface of a gravity flow, and the smaller grains towards that of greatest shear strain, e.g. the bed. This certainly conforms with experience. The effect seems very analogous to that of 'partial pressures'. If this explanation is sound the sorting effect should disappear when N or G_g is small enough, e.g. when $D_{max.}$ on the bed of a stream of water < say 0·2 mm. This again appears to conform with experience.

(c) '*Flowing gravel*'

Professor K. Terzaghi has kindly given me the following first-hand description of a phenomenon which has was able to observe from a distance of a few yards.

'A river of gravel, estimated as up to the size of hens' eggs at the visible surface, was watched in movement along the floor of a mountain valley. The gradient did not exceed a few degrees. The speed of flow at the surface was estimated as that of walking. No supporting fluid was visible between the surface pebbles.'

This curious phenomenon, evidently somewhat akin to that of the creep of a supersaturated soil, now seems capable of a simple explanation.

Let the density of the stones be 2·6, and suppose that beneath the surface the voids were filled by mud of density 2·0 and of a consistency equivalent to that of a Newtonian fluid having a viscosity 120 poise (about twice that of castor oil). Let the stones' concentration be constant with depth at $C = 0{\cdot}55$ (for which $\lambda = 11$), and let the internal friction coefficient $\tan\alpha_v$ have the experimental value of 0·75.

The gravity components parallel to and normal to the flow are related by

$$g\sin\beta\{\rho+(\sigma-\rho)\,C\}\,y = \tan\alpha_v g\cos\beta(\sigma-\rho)\,Cy,$$

whence $$\tan\beta = \tan\alpha_v\frac{(\sigma-\rho)\,C}{\rho+(\sigma-\rho)\,C} = 0{\cdot}106 \quad\text{and}\quad \beta = 6^\circ.$$

Equating the applied and internal shear stresses by (7)

$$2{\cdot}25\lambda^{\frac{3}{2}}\eta\,\mathrm{d}U/\mathrm{d}y = g\sin\beta\{\rho+(\sigma-\rho)\,C\}\,y,$$

$$\mathrm{d}U/\mathrm{d}y = 0{\cdot}0029gy/\eta,$$

$$U = 0{\cdot}00145gy'^2/\eta^2.$$

If the whole depth Y' of the flow was 1 m

$$U_{\text{surface}} = 1{\cdot}2\,\text{m/sec}.$$

But in order that (7) may be applicable, the condition $N < 40$ must be satisfied at all depths. $\mathrm{d}U/\mathrm{d}y$ has a maximum value at the bed of $0{\cdot}29g/\eta$, so at the bed $N = \lambda^{\frac{1}{2}}\sigma D^2\frac{0{\cdot}29g}{\eta^2}$, and the condition is that

$$D < \eta\sqrt{\frac{40}{0{\cdot}29g\lambda^{\frac{1}{2}}\sigma}},$$

$$D < 15\,\text{cm}.$$

If the stones had been 60 cm in diameter, the mud remaining of the same viscosity, the application of equation (6) for the inertia conditions give a surface speed of 1·35 m/sec, and a bed value for N of 550, and the inertia condition would persist to a considerable height above the bed.

The above applications (*a*) and (*c*) are capable of simple treatment because the grain concentration can be assumed approximately constant with height. It is hoped in a further paper to discuss application to the important general problem of the transport of grains over the grain bed of a fluid current, when the grain concentration is not constant with depth and when the applied shear stress is transferred downwards from layers of grainless fluid.

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References

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