

## On two approaches to the problem of instability of short-crested water waves

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The work is concerned with the problem of the linear instability of symmetric short-crested water waves, the simplest three-dimensional wave pattern. Two complementary basic approaches were used. The first, previously developed by Ioualalen & Kharif (1993, 1994), is based on the application of the Galerkin method to the set of Euler equations linearized around essentially nonlinear basic states calculated using the Stokes-like series for the short-crested waves with great precision. An alternative analytical approach starts with the so-called Zakharov equation, i.e. an integro-differential equation for potential water waves derived by means of an asymptotic procedure in powers of wave steepness. Both approaches lead to the analysis of an eigenvalue problem of the type

$$\det |\mathbf{A} - \lambda \mathbf{B}| = 0$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are infinite square matrices. The first approach should deal with matrices of quite general form although the problem is tractable numerically. The use of the proper canonical variables in our second approach turns the matrix  $\mathbf{B}$  into the unit one, while the matrix  $\mathbf{A}$  gets a very specific ‘nearly diagonal’ structure with some additional (Hamiltonian) properties of symmetry. This enables us to formulate simple necessary and sufficient *a priori* criteria of instability and to find instability characteristics analytically through an asymptotic procedure avoiding a number of additional assumptions that other authors were forced to accept.

A comparison of the two approaches is carried out. Surprisingly, the analytical results were found to hold their validity for rather steep waves (up to steepness 0.4) for a wide range of wave patterns. We have generalized the classical Phillips concept of weakly nonlinear wave instabilities by describing the interaction between the elementary classes of instabilities and have provided an understanding of when this interaction is essential. The mechanisms of the relatively high stability of short-crested waves are revealed and explained in terms of the interaction between different classes of instabilities. A helpful interpretation of the problem in terms of an infinite chain of interacting linear oscillators was developed.

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## 1. Introduction

The so-called short-crested waves† are three-dimensional progressive water waves composed of two progressive waves differing in the direction of propagation only. In the small-amplitude limit they tend to the simple superposition of two oblique plane waves. Short-crested waves appear naturally due to plane wave reflection from a sea-wall. Sometimes one can observe wave patterns closely resembling short-crested waves in unbounded natural basins. The nonlinear dynamics of short-crested waves is of evident interest for interpretation of some sea surface patterns; it is also of prime importance in the context of wave impact on structures.

Apart from the importance for the applications there is another and perhaps deeper motivation to study their properties: they are the simplest representatives of a wide completely uninvestigated class of three-dimensional waves. We believe that the study of short-crested waves, being of interest in itself, is the necessary first step for the understanding of more widespread and more complicated three-dimensional patterns.

A particular aspect we are interested in is the stability of these waves. The first studies of this problem were made by Roskes (1976) and by Mollo-Christensen (1981). These works were confined to the study of the one-dimensional (longitudinal) subharmonic instabilities only and were based on the analysis of two coupled Schrödinger equations (see e.g. Roskes 1976) or just on the formal application of the Lighthill criterion to the Chappellear (1961) Stokes-like expansion, truncated at the third order (Mollo-Christensen 1981).

The recent studies by Ioualalen & Kharif (1993, 1994, hereafter referred to as IK93, IK94) used a much more elaborated technique. The basic steady waves were described with great precision: the Stokes-like series for the short-crested waves were obtained employing a symbolic manipulator technique (Ioualalen 1993) and effective numerical algorithm. Although there is no rigorous proof of existence of the solutions, there are rather convincing numerical arguments along with the purely numerical solutions and extensive analysis by Roberts 1983 that give us enough confidence in the basic states under consideration for a wide range of wave steepness.

The stability study of essentially nonlinear waves with respect to infinitesimal perturbations reduces to an eigenvalue problem of the type

$$\det |\mathbf{A} - \lambda \mathbf{B}| = 0 \quad (1.1)$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are infinite square matrices of a rather general form. The analysis was carried out via the Galerkin method for truncated matrices  $\mathbf{A}$  and  $\mathbf{B}$  in order to calculate eigenvalues  $\lambda$  and to estimate the accuracy of these calculations. It was found that the strongest instabilities occur due to spatio-temporal resonance of the basic harmonics and two perturbation waves. In accordance with the concept of wave resonant interactions of Phillips (1960) (see also Zakharov 1968) the resonance zones were found to be close to the resonant curves prescribed by linear dispersion relations, while the instability rates were of order  $\varepsilon^n$ , where  $\varepsilon$  is a characteristic steepness of the basic waves and  $n$  is the number of interacting basic harmonics. The latter results give one an indication that perhaps instabilities of *essentially nonlinear waves* can nevertheless be effectively described within the framework of a *weakly nonlinear theory*.

Quite recently stability investigations of weakly nonlinear water waves based on the

† Following common practice we use this term throughout the paper for symmetric patterns only. The asymmetric patterns sometimes also called short-crested waves studied e.g. by Bryant (1985) lie beyond the scope of the present work.

Hamiltonian properties of water wave equations have got a new momentum through the work by Badulin & Shrira (1995, hereafter referred to as BS95), where a new approach has been developed. The essence of this approach can be summarized as follows. We start with the so-called Zakharov equation, i.e. an integro-differential equation for potential water waves derived by means of an asymptotic procedure in powers of  $\varepsilon$ ; the stability study as usual reduces to the analysis of an eigenvalue problem of the type (1.1) with an essential difference: the matrix  $\mathbf{B}$  is now the unit one, while the matrix  $\mathbf{A}$  has a very specific 'nearly diagonal' structure with some additional (Hamiltonian) properties of symmetry. This difference results in the following principal implications:

some general properties of the problem of instability can be easily obtained in explicit form without any calculations;

instability characteristics can be found analytically.

An approach similar in spirit was developed by Crawford *et al.* (1981), Stiassnie & Shemer (1984, 1987) and Tomita (1987) for the stability of plane gravity waves and by Okamura (1984) for standing waves. However, the version of the Zakharov equation they based their studies on does not utilize all the symmetries of the original system, which leads to the hardly tractable eigenvalue problem. Some additional assumptions were made to treat the eigenvalue problem numerically.

In the present work we apply the general approach of BS95 to analytical calculations of the instabilities of short-crested gravity waves and perform a detailed comparison with the numerical results by IK93, IK94. Combination of numerical and analytical approaches permits us to understand better physical mechanisms of water wave instabilities. The comparative advantages and disadvantages of the two approaches are discussed as well as ways of their possible synthesis. The specific questions we are aiming to clarify are the following:

Do numerical calculations provide the *full picture* of the instability domain in the parameter space? We can check the validity of numerical calculations in the small-amplitude range where the asymptotic analytical formulae are sure to be valid.

The asymptotic formulae often hold their validity well beyond the range of their formal applicability. Is it the case? We mentioned that there are some indications in the numerical computations by IK93, IK94. So, if yes, in what range and with what accuracy?

What are the physical mechanisms of instabilities belonging to different classes and the physics of interaction between the classes?

The paper is organized as follows. In § 2 we start with the governing equations both in standard Eulerian form and in terms of the Hamiltonian formulation for surface waves proposed by Zakharov (1966, 1968). Steady solutions for short-crested gravity waves in primitive and canonical variables are presented and their comparison is carried out. In the original variables the steady solutions are given by the Stokes-like series, whose coefficients are calculated by means of a successive asymptotic procedure (Roberts 1983; IK93) with, in principle, arbitrary desired accuracy. *Weakly nonlinear* solutions of the reduced Hamiltonian equations (see e.g. Zakharov 1968; Krasitskii 1994) are Stokes-like series truncated at low number of items which are presented in terms of some canonical variables in extremely simple form as a sum of two  $\delta$ -functions. Surprisingly good agreement between the two forms of solutions is found for the integral characteristics (e.g. wave energy, nonlinear frequency shift) in a wide range of waves parameters. Thus, the higher harmonics in the Stokes-like series at moderate wave steepness can affect significantly waves profiles but influence very slightly the integral characteristics of these waves.

In §3 the instability of the basic solutions of §2 is analysed. Within the framework of weakly nonlinear theory use of the properly chosen canonical variables results in an analytically tractable eigenvalue problem. The analysis of instability reduces to the question of the possibility and the type of coalescence of two or more partial frequencies of the perturbations, which is straightforwardly resolved for the generic situations. In its most general form this idea on the connection between an instability in the Hamiltonian system and coalescence of the roots of the eigenvalue equation goes back to work by Krein (1950) (for a review see e.g. Yakubovich & Starzhinskii 1975). In the context of water wave dynamics it was used by MacKay & Saffman (1986) but has not been utilized to its full extent. We suggest a very simple and natural classification of the instabilities as a classification of different cases of this coalescence.

A detailed comparison of the numerical and analytical results was carried out. We found good validity of analytical weakly nonlinear results for rather steep waves. The physical mechanisms of the relatively high stability of certain configurations of short-crested waves have been revealed. We suggest a qualitative explanation of this phenomenon in terms of interaction of the different classes of instabilities. The interaction of classes has already been considered by Zakharov & Rubenchik (1972) for a particular problem of instability of interacting high- and low-frequency waves.

In the Discussion we consider some possible ways of extension of the results and methods of this paper to more complex cases, including three-dimensional waves of arbitrary structure. Some possible ways of synthesis of numerical and analytical methods are discussed as well.

## 2. Statement of the problem

### 2.1. Governing equations

#### 2.1.1. Primitive equations

We consider potential gravity waves on the free surface of an inviscid, incompressible fluid of infinite depth. The governing equations in the Cartesian frame  $\{x, y, z\}$  having the  $z = 0$  plane on the undisturbed water surface and the  $z$ -axis oriented upward are standard:

$$\left. \begin{aligned} \nabla^2 \phi &= 0, \\ \lim_{z \rightarrow -\infty} \phi &= 0, \end{aligned} \right\} \quad z \leq \eta(x, y, t), \quad \left. \begin{aligned} \phi_t + \eta + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) &= 0, \\ \eta_t + \phi_x \eta_x + \phi_y \eta_y - \phi_z &= 0, \end{aligned} \right\} \quad z = \eta(x, y, t), \quad (2.1)$$

where  $\phi(x, y, z, t)$  is the velocity potential and  $z = \eta(x, y, t)$  is the equation of the free surface; the gravitational acceleration is taken to be unity.

#### 2.1.2. Hamiltonian formulation

We shall deal with the Hamiltonian formulation of the equations of motion (2.1) in Fourier space proposed by Zakharov (1968). In terms of Fourier amplitudes the complex variables  $b(\mathbf{k})$  are expressed by means of integral power series in Fourier amplitudes  $\eta(\mathbf{k})$  and  $\psi(\mathbf{k})$ , of the free surface elevation and the velocity potential at the free surface (see Appendix A). The basic equations (2.1) are presented in the form

$$i \frac{\partial b(\mathbf{k})}{\partial t} = \frac{\delta \mathcal{H}}{\delta b^*(\mathbf{k})} \quad (2.2)$$

where the Hamiltonian  $\mathcal{H}$  is expressed in terms of an integral power series in the complex amplitudes  $b(\mathbf{k})$ :

$$\mathcal{H} = \mathcal{H}_0 + \sum_{n=4}^{\infty} \mathcal{H}_n, \quad (2.3a)$$

$$\mathcal{H}_0 = \int \omega(\mathbf{k}) b(\mathbf{k}) b^*(\mathbf{k}) d\mathbf{k}. \quad (2.3b)$$

The first term in the expansion of the Hamiltonian in (2.3a,b),  $\mathcal{H}_0$ , is quadratic ( $\omega(\mathbf{k})$  is the linear dispersion law) and gives the linear terms in the equations for  $b(\mathbf{k})$ . The terms  $\mathcal{H}_n$  are of power  $n$  in amplitudes  $b(\mathbf{k})$  and thus they are responsible for nonlinear effects. The systematic derivation of the equation (2.2) is given in Zakharov (1968) and Krasitskii (1990, 1994). We shall use the notation of Krasitskii (1990, 1994) except for the notation of the arguments. The full form of these arguments will be used.

Confining ourselves to the first two terms in the series (2.3a) we arrive at the so-called four-wave reduced Zakharov's equation. Some key properties of this equation are important for our further consideration.

(a) The Hamilton function takes the form

$$\begin{aligned} \mathcal{H} = & \int \omega(\mathbf{k}) b(\mathbf{k}) b^*(\mathbf{k}) d\mathbf{k} \\ & + \frac{1}{2} \int \tilde{V}^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) b^*(\mathbf{k}) b^*(\mathbf{k}_1) b(\mathbf{k}_2) b(\mathbf{k}_3) \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \end{aligned} \quad (2.4a)$$

that is,  $\mathcal{H}$  is quartic in the amplitudes and the cubic term is absent in this truncated expansion.

(b) The Hamilton equation is cubic in the complex amplitudes  $b(\mathbf{k})$

$$\begin{aligned} i \frac{\partial b(\mathbf{k})}{\partial t} = & \omega(\mathbf{k}) b(\mathbf{k}) \\ & + \int \tilde{V}^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) b^*(\mathbf{k}_1) b(\mathbf{k}_2) b(\mathbf{k}_3) \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \end{aligned} \quad (2.4b)$$

All the formulae for the kernels  $\tilde{V}^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  and for  $b(\mathbf{k})$  as a function of  $\psi(\mathbf{k})$  and  $\eta(\mathbf{k})$  which will be often used in further calculations are given in Appendix A.

(c) Equation (2.4b) is the Hamiltonian that results in the following important symmetry properties of the kernels  $\tilde{V}^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  (see Krasitskii 1990, 1994):

$$\tilde{V}^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \tilde{V}^{(2)}(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2, \mathbf{k}_3) = \tilde{V}^{(2)}(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_3, \mathbf{k}_2) = \tilde{V}^{(2)}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}, \mathbf{k}_1). \quad (2.5)$$

The form of the kernels we use is derived by Krasitskii (1990, 1994). The previous presentations of the kernels by other authors (see Zakharov 1968; Stiassnie & Shemer 1984; Crawford *et al.* 1981) apart from having a number of misprints lacked these symmetry properties.

(d) There is a freedom in these kernels and canonical variables  $b(\mathbf{k})$  allowing one to add an arbitrary function to the kernels obeying the symmetries (2.5) and vanishing at the resonant curves

$$\left. \begin{aligned} \mathbf{k} + \mathbf{k}_1 &= \mathbf{k}_2 + \mathbf{k}_3, \\ \omega(\mathbf{k}) + \omega(\mathbf{k}_1) &= \omega(\mathbf{k}_2) + \omega(\mathbf{k}_3). \end{aligned} \right\} \quad (2.6)$$

It means that we can introduce a particular canonical transformation (see the formula

(A5)) to let some nonresonant kernels in formulae (2.4a), (2.4b) be plain zeros in order to simplify the form of the governing equations. This property is vital for getting exact solutions of the reduced equation (2.4b).

## 2.2. Basic states for short-crested waves

### 2.2.1. Stokes-like solutions in primitive variables

Being interested in the steady solutions to the basic equations (2.1) describing short-crested waves, which in the linear limit is the superposition of two plane waves with the wavevectors  $\kappa$  and  $\bar{\kappa}$  ( $|\kappa| = |\bar{\kappa}|$ ) and an angle  $(\pi - 2\Theta)$  between these wavevectors, we define a new coordinate frame as follows:

$$\left. \begin{aligned} X &= x |\kappa| \sin \Theta - \Omega_0 t, \\ Y &= y |\kappa| \cos \Theta, \\ Z &= z, \end{aligned} \right\} \quad (2.7)$$

where  $\Omega_0$  is the wave frequency. The values  $\Theta = 90^\circ$  and  $\Theta = 0^\circ$  correspond to the Stokes progressive waves and standing waves, respectively. Both are two-dimensional limits of short-crested waves. Using the scaling invariance of deep-water waves we assume further that  $|\kappa| = |\bar{\kappa}| = 1$ . Thus, the solution corresponding to the short-crested waves is given by the expansion in the formally small-amplitude parameter  $\varepsilon$  (Roberts 1983; IK93)

$$\left. \begin{aligned} \bar{\phi} &= \sum_{p=1}^N \sum_{m,n=1}^N d_{p,mn} \sin(mX) \cos(nY) \varepsilon^p \exp(\gamma_{mn} Z), \\ \bar{\eta} &= \sum_{p=1}^N \sum_{m,n=1}^N c_{p,mn} \cos(mX) \cos(nY) \varepsilon^p, \\ \Omega_0 &= \sum_{p=0}^N \omega_p \varepsilon^p, \\ \gamma_{mn} &= (m^2 \sin^2 \Theta + n^2 \cos^2 \Theta)^{1/2}. \end{aligned} \right\} \quad (2.8)$$

Symmetry and nonlinearity require the indices  $p, m, n$  to be of the same parity. Here  $\varepsilon$  is a parameter of nonlinearity, assumed to be small; we shall call it wave amplitude or wave steepness. Separating equations (2.1) for each power of  $\varepsilon$  we obtain an infinite set of algebraic equations for  $d_{p,mn}$ ,  $c_{p,mn}$ ,  $\omega_p$  which can be resolved successively.

It is important to emphasize that the nonlinear solutions are determined by a number of coefficients  $c_{p,mn}$ ,  $d_{p,mn}$ ,  $\omega_p$  which are functions of  $\Theta$  only (we set the wavenumber to be unity) and do not depend on the parameter  $\varepsilon$ , which is responsible for wave steepness. The method of calculation of  $c_{p,mn}$ ,  $d_{p,mn}$ ,  $\omega_p$  proposed by Roberts 1983 gives an error  $o(\varepsilon^{N/3})$  ( $N$  is an order of the truncated series in powers of amplitudes (2.8)). In IK93 all these coefficients were calculated up to the order 27 to obtain profiles of the short-crested waves, while a relatively small order of approximation in formulae (2.8) was often proved to be sufficient for the instability analysis.

### 2.2.2. Steady solutions in canonical variables

An alternative way to construct approximate solutions of (2.1) is to employ the asymptotic approach given by the Hamiltonian formulation of equations for surface gravity waves (Zakharov 1968). In our case the advantages of using this technique are especially palpable:

$$b(\mathbf{k}) = \sum_{i=1}^M b_i(t) \delta(\mathbf{k} - \mathbf{K}_i) \quad (2.9)$$

are exact solutions of the reduced equation (2.4b) (see BS95). In particular, the solution of (2.4b) corresponding to non-steady weakly nonlinear short-crested waves can be presented in the form (2.9) as well.

Let us seek steady solutions of (2.4b) in the form

$$b(\mathbf{k}) = c_0 [\delta(\mathbf{k} - \boldsymbol{\kappa}) + \delta(\mathbf{k} - \bar{\boldsymbol{\kappa}})] \exp(-i\Omega_0 t), \quad (2.10a)$$

$$\Omega_0 = \omega(\boldsymbol{\kappa}) + \tilde{V}^{(2)}(\boldsymbol{\kappa}, \boldsymbol{\kappa}, \boldsymbol{\kappa}) |c_0|^2 + 2\tilde{V}^{(2)}(\boldsymbol{\kappa}, \bar{\boldsymbol{\kappa}}, \boldsymbol{\kappa}, \bar{\boldsymbol{\kappa}}) |c_0|^2, \quad (2.10b)$$

$$|\boldsymbol{\kappa}| = |\bar{\boldsymbol{\kappa}}|. \quad (2.10c)$$

The solution (2.10) when inserted into the right-hand side of (2.4b) generates, generally speaking, higher harmonics. That is

$$\begin{aligned} \frac{\partial b(2\bar{\boldsymbol{\kappa}} - \boldsymbol{\kappa})}{\partial t} &= 2\tilde{V}^{(2)}(2\bar{\boldsymbol{\kappa}} - \boldsymbol{\kappa}, \boldsymbol{\kappa}, \bar{\boldsymbol{\kappa}}, \bar{\boldsymbol{\kappa}}) c_0^3 \exp(i\Omega_0 t), \\ \frac{\partial b(2\boldsymbol{\kappa} - \bar{\boldsymbol{\kappa}})}{\partial t} &= 2\tilde{V}^{(2)}(2\boldsymbol{\kappa} - \bar{\boldsymbol{\kappa}}, \bar{\boldsymbol{\kappa}}, \boldsymbol{\kappa}, \boldsymbol{\kappa}) c_0^3 \exp(i\Omega_0 t). \end{aligned}$$

However, these harmonics are not resonant, i.e. they do not satisfy resonant conditions (2.6) and, thus, they can be excluded by a particular canonical transformation that makes the kernels  $\tilde{V}^{(2)}(2\bar{\boldsymbol{\kappa}} - \boldsymbol{\kappa}, \boldsymbol{\kappa}, \bar{\boldsymbol{\kappa}}, \bar{\boldsymbol{\kappa}})$  and  $\tilde{V}^{(2)}(2\boldsymbol{\kappa} - \bar{\boldsymbol{\kappa}}, \bar{\boldsymbol{\kappa}}, \boldsymbol{\kappa}, \boldsymbol{\kappa})$  plain zeros. In our problem we can choose

$$\tilde{V}_{0,1,2,3}^{(2)} = \tilde{V}_{0,1,2,3}^{(2)Kras} \left[ 1 - \frac{(\omega_0 + \omega_1 - \omega_2 - \omega_3)^2}{(\omega(2\boldsymbol{\kappa} - \bar{\boldsymbol{\kappa}}) - \omega(\boldsymbol{\kappa}))^2} \right] \quad (2.11)$$

where  $\tilde{V}_{0,1,2,3}^{(2)Kras}$  is a particular expression for the kernel which is given by Krasitskii (1990, 1994). The expression (2.11) is not singular and obeys the same 'natural symmetry' conditions (2.5). We recall that in accordance with formula (A 5)  $b_i$  contains terms of orders  $\varepsilon$ ,  $\varepsilon^2$  and  $\varepsilon^3$ . Thus, our *exact solution of the reduced equation* (2.4b) has an accuracy  $O(\varepsilon^3)$ , i.e. its error is  $o(\varepsilon^3)$ . A similar trick of correction of kernels can be successfully used for different problems (say, for non-symmetric short-crested waves) in order to obtain exact solutions of the reduced equation of the type (2.9) (see BS95 for more details).

The solution (2.10) gives us the simplest but not trivial three-dimensional solutions for the waves of permanent form – the so-called short-crested waves. It is easy to present solution (2.10) in terms of original physical variables. The formulae for the vertical displacement of the surface are given in the Appendix B. The linear term in (B 1) gives the simple superposition of two plane oblique waves. Higher-order terms are forced harmonics. It is important that in our case none of the coefficients in these terms are singular. Thus in this order of approximation we have a very simple solution easy to operate with. The singularities and nonuniqueness of the solution are unavoidable if one attempts to get higher accuracy; however the principal point is that, as we shall see below, this solution is sufficient to study the dominating instabilities in the small- and moderate-amplitude range.

### 2.3. A comparison between weakly nonlinear and 'exact' numerical solutions

There is a question of true importance in itself and for our further analysis:

How good are the approximate formulae (B 1) for describing the basic states beyond the range of their asymptotic validity; what characteristics of the wave can be modelled that way?

First we again draw attention to the remarkable fact that the two- $\delta$ -functions ansatz of (2.10) or the formulae (B 1) provide an error  $o(\varepsilon^3)$ , but not  $o(\varepsilon^2)$  as it may be expected. The two-term ansatz gives the same accuracy as seven terms of Chappelear's (1961) solution or the series (2.8) truncated at the  $\varepsilon^3$  order.

Some conclusions can be deduced from figure 1(a,b) which shows results of our approximate (weakly nonlinear) and 'exact' (numerical) calculations for wave energy and wave frequency  $\Omega_0$ . Weakly nonlinear theory gives an accuracy better than 1% up to wave steepness 0.25 for all angles less than approximately  $60^\circ$ . For these angles the good agreement is retained for larger steepness as well: even at steepness 0.4 the discrepancies in energy and frequency do not exceed 5% and 0.2% (8% for the difference of the frequencies of linear and nonlinear waves), respectively. For angles exceeding  $75^\circ$  (like for all angles close to  $90^\circ$  - a plane wave) higher harmonics, which are responsible for higher-order corrections in Stokes-like series (2.7), play more important role. This leads to the growth of deviation in energy and frequency between the numerical and analytical results (see also Roberts & Schwartz 1983, figure 2.3)†.

Our comparison shows that our approximate solutions model fairly well the integral characteristics of the wave (energy, frequency) in a wide range of wave steepness. All the discrepancies between analytical and numerical approaches are due to higher harmonics and can affect significantly small-scale details of the wave profiles, but not the integral characteristics. It gives us hope that the wave instabilities could be also described fairly well within the weakly nonlinear approach. Although *a priori* there is no guarantee that the small discrepancies in the description of the basic states do not result in a qualitatively different outcome, in the next section we shall show that this is not the case using the simplest two- $\delta$ -function ansatz for the basic state in the analytical study and the maximal possible accuracy for the numerical computations.

## 3. The linearized stability problem

### 3.1. Instability of essentially nonlinear solutions

On linearizing the primitive equations (2.1) around the basic state discussed in §2 we obtain the infinite set of first-order perturbation equations, which we reduce via a normal modes decomposition to an eigenvalue problem for the frequency of a perturbation mode,  $\lambda$  (for more detail see IK94):

$$\mathbf{A}\mathbf{u} - \lambda\mathbf{B}\mathbf{u} = 0. \quad (3.1)$$

The complex matrices  $\mathbf{A}$  and  $\mathbf{B}$  depend on the basic solution and the real longitudinal and transversal wavenumbers of the perturbation. An eigenvector  $\mathbf{u}$  is a vector of

† It is important to note that both approaches (the analytical one and numerical schemes by Roberts 1983 and IK93) fail for nearly plane configurations. There are no continuous limits when three-dimensional patterns tend to become plane. This problem is far from being trivial but lies beyond the scope of the present consideration. The only attempt to address this problem has been done by Roberts & Peregrine (1983), where a particular continuous solution was derived. It should be noted however that the asymmetric patterns of the type studied by e.g. Bryant (1985) do have a continuous transition to plane waves.

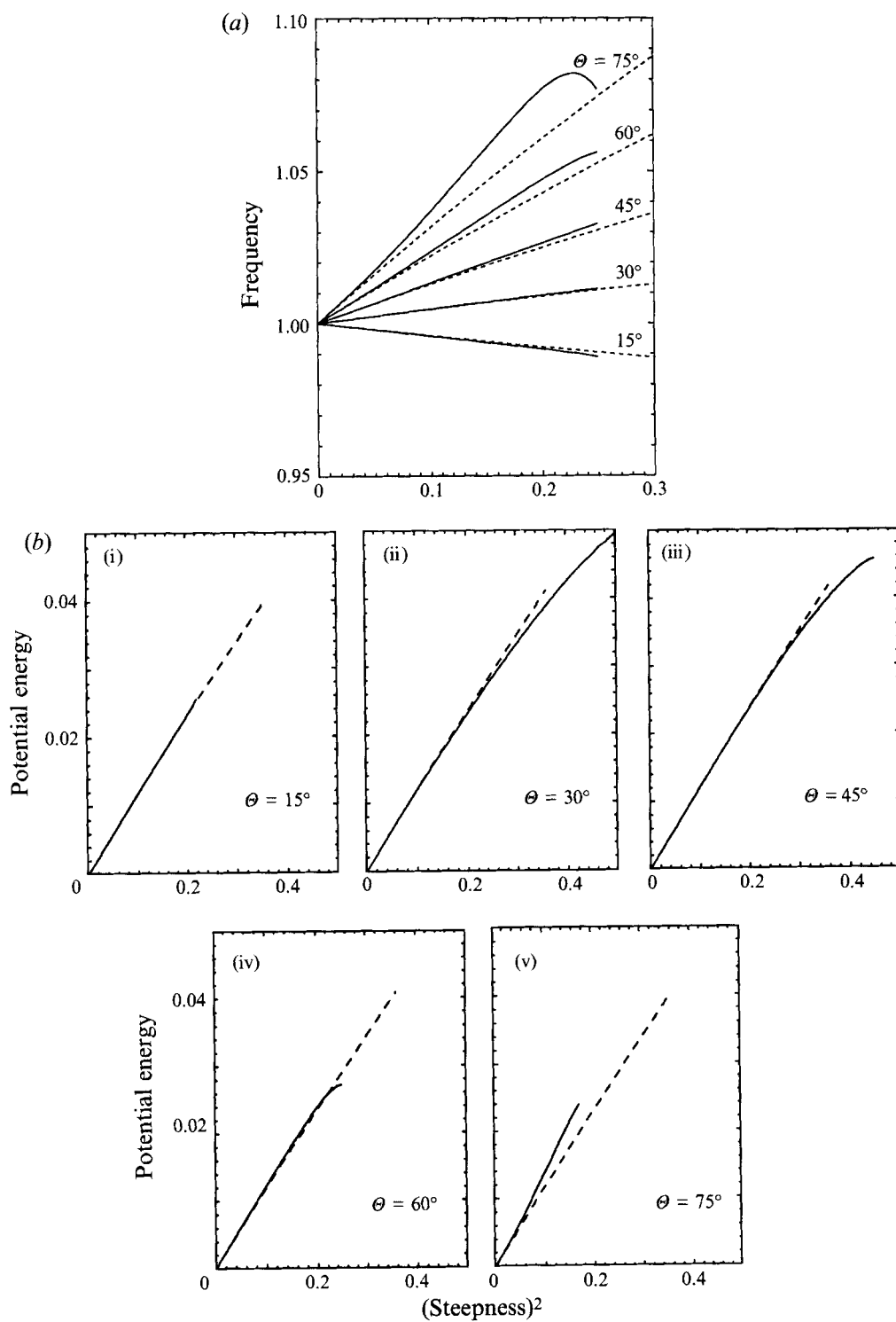


FIGURE 1. For caption see facing page.

Fourier amplitudes (both of surface elevation and velocity potential) of the wave of perturbation (see IK93). Instability occurs for  $\text{Im}(\lambda) \neq 0$ .

Numerical methods are commonly used to reduce the perturbation equations to the eigenvalue problem (3.1). We use Galerkin's method, which is as a rule rather efficient for these kinds of problems (e.g. Fletcher 1984). Generally, the method presupposes, first, a choice of a certain set of orthogonal basic functions, and second, casting of the primitive equations into a set of equations for the amplitudes of these modes. Accuracy of the method does not depend directly on the number of modes, but is sensitive to the proper choice of the basic functions, to their proximity to a certain set of 'natural' eigenfunctions. For our problem of short-crested waves in terms of primitive physical variables the most straightforward choice of the set of basic functions is a truncated series of Fourier harmonics, similar to that of a Stokes-like expansion. The effectiveness of this approach was demonstrated in IK93, IK94. All the numerical calculations throughout the paper were done using this version of the Galerkin method. However the problem of optimization of the basic set of functions has not been considered yet. It will be demonstrated below that the choice of the basic set could be optimized as well through casting the system in terms of the canonical variables.

### 3.2. General properties of the eigenvalue problem within the framework of the four-wave Zakharov equation

In terms of Hamiltonian variables  $b(\mathbf{k})$  the stability problem is formulated quite similarly to the Galerkin approach, as a set of equations for the amplitudes of the basic modes. These modes are related in a nontrivial way to the primitive physical variables (see Appendix A, (A 1) – (A 5)). The fact that the basic states (2.4b) take in terms of our canonical variables the exceptionally simple form of just two  $\delta$ -pulses allows us to expect that the stability problem will be optimized as well.

Consider a perturbed solution (2.10) in the form

$$\tilde{b}(\mathbf{k}) = b(\mathbf{k}) + \alpha(\mathbf{k})$$

where  $\alpha(\mathbf{k})$  ( $|\alpha(\mathbf{k})| \ll |b(\mathbf{k})|$ ) is a small continuous perturbation in  $\mathbf{k}$ -space. The linearization of the Zakharov equation picks out pairs of equations for perturbations with wavevectors  $(\mathbf{h} + n(\bar{\mathbf{k}} - \mathbf{\kappa}))$  and  $(\mathbf{\kappa} + \bar{\mathbf{k}} - \mathbf{h} - n(\bar{\mathbf{k}} - \mathbf{\kappa}))$  where  $\mathbf{h}$  is an arbitrary 'probing' wavevector, while  $n$  is an arbitrary integer number and  $(\bar{\mathbf{k}} \pm \mathbf{\kappa})$  are the wavevectors of the propagating component (+) and of the standing component (−) of the basic three-dimensional-pattern (their moduli give the inverse spatial scales of the basic pattern, longitudinal and transversal, respectively):

$$\begin{aligned} & i \frac{\partial \alpha(\mathbf{h} + n(\bar{\mathbf{k}} - \mathbf{\kappa}))}{\partial t} \\ &= D_n^+ \alpha(\mathbf{h} + n(\bar{\mathbf{k}} - \mathbf{\kappa})) + E(t) C_n \alpha^*(\bar{\mathbf{k}} + \mathbf{\kappa} - \mathbf{h} - n(\bar{\mathbf{k}} - \mathbf{\kappa})) \\ &+ T_{n+1} \alpha(\mathbf{h} + (n+1)(\bar{\mathbf{k}} - \mathbf{\kappa})) + E(t) Q_{n+1} \alpha^*(\bar{\mathbf{k}} + \mathbf{\kappa} - \mathbf{h} - (n+1)(\bar{\mathbf{k}} - \mathbf{\kappa})) \\ &+ T_n \alpha(\mathbf{h} + (n-1)(\bar{\mathbf{k}} - \mathbf{\kappa})) + E(t) S_n \alpha^*(\bar{\mathbf{k}} + \mathbf{\kappa} - \mathbf{h} - (n-1)(\bar{\mathbf{k}} - \mathbf{\kappa})), \end{aligned} \quad (3.2a)$$

FIGURE 1. Frequency  $\Omega$  (a) and potential energy (b) of short-crested waves vs. square of wave steepness,  $\epsilon$ , for different angles  $\Theta$ . Solid lines: 'exact' numerical calculations based on the algorithm by IK93; dashed lines: asymptotic weakly nonlinear theory. The plots demonstrate surprisingly good agreement between the leading-order terms of the asymptotic theory based on the Zakharov equation and 'exact' solutions for a wide range of angles  $\Theta$ . The weakly nonlinear description becomes less effective as  $\Theta$  tends to  $\pi/2$ , i.e. to the nearly plane wave. In particular, an essential qualitative feature is lost, namely the pronounced maximum of  $\Omega$ .

$$\begin{aligned}
 & i \frac{\partial \alpha^*(\bar{\mathbf{k}} + \boldsymbol{\kappa} - \mathbf{h} - n(\bar{\mathbf{k}} - \boldsymbol{\kappa}))}{\partial t} \\
 & = -E^*(t)C_n\alpha(\mathbf{h} + n(\bar{\mathbf{k}} - \boldsymbol{\kappa})) + D_n^-\alpha^*(\bar{\mathbf{k}} + \boldsymbol{\kappa} - \mathbf{h} - n(\bar{\mathbf{k}} - \boldsymbol{\kappa})) \\
 & \quad -E^*(t)S_{n+1}\alpha(\mathbf{h} + (n+1)(\bar{\mathbf{k}} - \boldsymbol{\kappa})) + N_{n+1}\alpha^*(\bar{\mathbf{k}} + \boldsymbol{\kappa} - \mathbf{h} - (n+1)(\bar{\mathbf{k}} - \boldsymbol{\kappa})) \\
 & \quad -E^*(t)Q_n\alpha(\mathbf{h} + (n-1)(\bar{\mathbf{k}} - \boldsymbol{\kappa})) + N_n\alpha^*(\bar{\mathbf{k}} + \boldsymbol{\kappa} - \mathbf{h} - (n-1)(\bar{\mathbf{k}} - \boldsymbol{\kappa})) \quad (3.2b)
 \end{aligned}$$

where  $E(t) = \exp(2i\Omega_0 t)$  and  $E^*(t)$  is its complex conjugate. The simple transformation

$$\alpha(\mathbf{k}) = \tilde{\alpha}(\mathbf{k})(\mathbf{k}) \exp(-i\Omega_0 t); \quad \alpha^*(\mathbf{k}) = \tilde{\alpha}^*(\mathbf{k}) \exp(i\Omega_0 t) \quad (3.3)$$

gives us an infinite countable set of coupled linear equations with real constant coefficients. We can cast equations (3.2) in matrix form to obtain an eigenvalue problem for the partial frequencies by setting

$$\left. \begin{aligned} & \alpha(\mathbf{k}) \sim \exp(-i\lambda t), \\ & \mathbf{A}\mathbf{u} - \lambda \mathbf{E}\mathbf{u} = 0; \quad \mathbf{u} = \{\alpha(\mathbf{k}), \alpha^*(\mathbf{k})\}^t. \end{aligned} \right\} \quad (3.4)$$

Here  $\mathbf{E}$  is the infinite unit matrix. The matrix  $\mathbf{A}$  has the form

$$\begin{bmatrix}
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 T_{n-1} & S_{n-1} & D_{n-1}^+ & C_{n-1} & T_n & Q_n & 0 & 0 \\
 -Q_{n-1} & -N_{n-1} & -C_{n-1} & D_{n-1}^- & -S_n & -N_n & 0 & 0 \\
 0 & 0 & T_n & S_n & D_n^+ & C_n & T_{n+1} & Q_{n+1} \\
 0 & 0 & -Q_n & -N_n & -C_n & D_n^- & -S_{n+1} & -N_{n+1} \\
 0 & 0 & 0 & 0 & T_{n+1} & S_{n+1} & D_{n+1}^+ & C_{n+1} \\
 0 & 0 & 0 & 0 & -Q_{n+1} & -N_{n+1} & -C_{n+1} & D_{n+1}^- \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
 \end{bmatrix}.$$

The expressions for the elements of matrix  $\mathbf{A}$  in terms of the interaction coefficients are given in Appendix C. Thus we have an eigenvalue problem looking similar to (3.1) but

$\mathbf{B}$  is the unit matrix now;

matrix  $\mathbf{A}$  belongs to the class of so-called Jacobian matrices, i.e. the elements sharply decrease with the distance from the main diagonal (in the general case as  $\varepsilon^{|n/2|}$  (BS95), where  $n$  is the number of a diagonal,  $[n/2]$  is the integer part of  $(n/2)$ ). In our particular case we have plane zeros for nondiagonal terms  $A_{k+p,p} = A_{p,p+k} = 0$  ( $k > 4$ ) and  $\mathbf{A}$  has the following essential properties:

- (a) diagonal terms in  $\mathbf{A}$  are differences of the frequencies of harmonics of perturbations and the basic frequency  $\Omega_0$ ;
- (b) nondiagonal terms, responsible for the interaction of these harmonics, have a order of amplitude squared ( $\varepsilon^2$ );
- (c) nondiagonal terms  $A_{ij}$  owing to symmetries (2.5) obey the equality

$$|A_{ij}| = |A_{ji}|; \quad (3.5)$$

(d) matrix  $\mathbf{A}$  is invariant relative to the renumbering of its terms, i.e. for the wavevectors of disturbances  $\mathbf{h}$  and  $(\mathbf{h} + n(\bar{\mathbf{k}} - \boldsymbol{\kappa}))$  we have the same set of eigenvalues. The latter is a general property of the periodic solutions (e.g. McLean *et al.* 1981). We draw attention to the fact that within the framework of exact equations (IK94) the eigenvalue problem possesses a double periodicity, both in along- and cross-directions. In the approximation we consider here there is a transversal periodicity in  $\mathbf{k}$ -space only. It can be shown that the longitudinal periodicity also appears when

the fifth-order terms in the Hamiltonian  $\mathcal{H}$  are taken into account (BS95). This translation invariance in the  $\mathbf{k}$ -space was ignored in the analytical approaches by Stiassnie & Shemer (1984), Okamura (1984). These authors assumed higher harmonics of perturbation to be unimportant. Within our approach we can easily avoid this assumption, which in particular will allow us to describe some new details of higher-order instabilities (see §3.5).

The smallness of nondiagonal terms of the matrix  $\mathbf{A}$  makes it natural to look for the solutions as an expansion in powers of these terms, i.e. in powers of  $\varepsilon^2$ . In the zeroth-order approximation  $\varepsilon \rightarrow 0$  we have an eigenvalue for an infinite diagonal matrix. The eigenvalues of this matrix can be easily found as a set of its diagonal terms  $D_n^+$ ,  $D_n^-$ . Thus we have a set of frequencies of free surface waves relative to the frequency of the basic wave pattern. All these frequencies are real (no instability of perturbations) and, in general, as can easily be seen, are different. Such a method of analysis breaks down when two or more of these frequencies are close to each other. In this case small nondiagonal terms appear to be important and we can expect the emergence of instabilities; moreover just a few nondiagonal terms proved to be important (Krein 1950; MacKay & Saffman; Yakubovich & Starzhinskii 1975). This understanding allows us to reduce sharply the number of modes of perturbation which should be taken into account for an instability analysis and thus the dimension of the corresponding matrix as well.

This qualitative picture indicates us how to proceed with the further analysis.

### 3.3. Four-wave (class 1) instabilities; classification of instabilities

As the first step of our analysis we consider the simplest case of the emergence of instability when only two eigenvalues (or just two diagonal terms in the Hamiltonian formulation) of the matrix  $\mathbf{A}$  are close to each other. Let us fix these diagonal terms and denote them, say,  $d_0$  and  $d_k$ . We can present the eigenvalue problem (3.4) in terms of power series in  $\varepsilon^2$ . Assume  $A_{0k} \neq 0$ , which means that  $d_0$  and  $d_k$  are in the same block  $4 \times 4$  of matrix  $\mathbf{A}$ . In this case we have the following expansion for the determinant in (3.4):

$$[(d_0 - \lambda)(d_k - \lambda) - A_{0k}A_{k0}] \prod_{-\infty (p \neq k, 0)}^{+\infty} (d_p - \lambda) + O(|\varepsilon|^6) = 0 \quad (3.6)$$

and hence the leading-order approximation for the eigenvalues is

$$\lambda_{1,2} = \frac{1}{2} \{ (d_0 + d_k) \pm [(d_0 - d_k)^2 + 4A_{0k}A_{k0}]^{1/2} \}. \quad (3.7)$$

In our formulation the matrix  $\mathbf{A}$  possesses, as already mentioned, the important specific property  $|A_{ij}| = |A_{ji}|$  due to the Hamiltonian symmetries (2.5).

Thus we have got the necessary and sufficient *a priori* criterion of instability: *instabilities in order  $\varepsilon^2$  (complex  $\lambda$  in (3.7)) occur when:*

(a) any two of diagonal elements, designated, say  $d_0$  and  $d_k$ , are close to each other, that is  $|d_0 - d_k|$  is of order of  $\varepsilon^2$  or smaller. This criterion corresponds to the well-known necessary criterion of instability (see Krein 1950; MacKay & Saffman 1986; Yakubovich & Starzhinskii 1975);

(b) the corresponding nondiagonal elements,  $A_{0k}$  and  $A_{k0}$ , have opposite signs. This is the sufficient criterion of crossing of colliding eigenvalues (Krein 1950; MacKay & Saffman; Yakubovich & Starzhinskii 1975).

There are four cases of possible instabilities of this type for the 'basic' block ( $4 \times 4$ )

of the matrix  $\mathbf{A}$  that satisfy both to the necessary (a) and sufficient (b) criteria:

$$D_n^+ \approx D_n^- \quad \text{or} \quad D_{n+1}^+ \approx D_{n+1}^-, \quad (3.8a)$$

$$D_n^+ \approx D_{n+1}^- \quad \text{or} \quad D_{n+1}^+ \approx D_n^-. \quad (3.8b)$$

The upper indices of  $D_n^\pm$  in our notation correspond to the signatures of the colliding eigenvalues. Thus the qualitative result by MacKay & Saffman (1986, see pp. 119–120) is expressed in the following explicit form (see formulae for  $D_n^\pm$  in Appendix C): two eigenvalues have opposite signatures if one of them corresponds to a wave of perturbation propagating faster and the other to a wave propagating slower than the basic wave. In this case the crossing of eigenvalues and, hence, instability occurs. In the opposite case of collision of two eigenvalues of the same signature ‘the cross avoiding’ of these eigenvalues occurs and there is no any instability.

Our eigenvalue problem (3.4) is formulated for an arbitrary wavevector of perturbation  $\mathbf{h}$ . These cases of matching of the diagonal terms (3.8a,b) in the limit of small wave amplitude correspond to the following resonance conditions prescribing two-dimensional curves in wavevector space:

$$\left. \begin{aligned} 2\Omega_0 &= \omega(\mathbf{h} + n(\bar{\mathbf{\kappa}} - \mathbf{\kappa})) + \omega(\bar{\mathbf{\kappa}} + \mathbf{\kappa} - \mathbf{h} - n(\bar{\mathbf{\kappa}} - \mathbf{\kappa})), \\ 2\Omega_0 &= \omega(\mathbf{h} + (n+1)(\bar{\mathbf{\kappa}} - \mathbf{\kappa})) + \omega(\bar{\mathbf{\kappa}} + \mathbf{\kappa} - \mathbf{h} - (n+1)(\bar{\mathbf{\kappa}} - \mathbf{\kappa})), \end{aligned} \right\} \quad (3.9a)$$

$$\left. \begin{aligned} 2\Omega_0 &= \omega(\mathbf{h} + n(\bar{\mathbf{\kappa}} - \mathbf{\kappa})) + \omega(2\bar{\mathbf{\kappa}} - \mathbf{h} - n(\bar{\mathbf{\kappa}} - \mathbf{\kappa})), \\ 2\Omega_0 &= \omega(\mathbf{h} + n(\bar{\mathbf{\kappa}} - \mathbf{\kappa})) + \omega(2\mathbf{\kappa} - \mathbf{h} - n(\bar{\mathbf{\kappa}} - \mathbf{\kappa})). \end{aligned} \right\} \quad (3.9b)$$

The cases (3.9b) give us the well-known resonance conditions for the instability of plane waves with vectors  $\bar{\mathbf{\kappa}}$  or  $\mathbf{\kappa}$  (Phillips 1960). The conditions (3.9a) give us another case of the resonance of two different basic waves  $\bar{\mathbf{\kappa}}$  and  $\mathbf{\kappa}$  and two perturbation waves. All these instabilities are four-wave ones or the ‘class 1’ ones in the classification by McLean *et al.* (1981) and McLean (1982) and we can interpret (3.9b) as 1b, and (3.9a) as 1a subclasses of this type of instability. This classification of instabilities was proposed in IK94.

The results of this section are based on the formal asymptotic expansion in  $\varepsilon^2$  and the assumption that just two diagonal elements are nearly equal. Are these results sufficient to describe the instabilities of short-crested waves? Does the interaction of subclasses of instabilities (IK93, IK94) play any important role? As a first step we shall try to discuss this question by considering numerical (IK93, IK94) and analytical results of calculations of instability rates in accordance with the approximate formulae (3.7).

### 3.3.1. ‘First’ comparison of the analytical and numerical results

Some results of the calculations based on the simplest formulae (3.7), i.e. calculated for the  $(2 \times 2)$  matrix, are presented in the graphic form. The dependence of the maximal growth rate on the angle  $\Theta$  is plotted in figure 2(a,b) for two subclasses of instability given by (3.8a,b). Analytical results given by formula (3.7) are compared with the numerical computations. The maximal growth rate is the most representative instability characteristic for a comparison, as our asymptotic formulae are expected to work best at the resonance curves, where exact matching of the diagonal terms occurs. The boundaries of the instability regions are described less accurately.

The growth rates given by the weakly nonlinear formulae (3.7) for relatively small values of angles (up to  $40^\circ$ ) are in excellent agreement with the numerical results. Discrepancies in analytical and numerical results are relatively larger for greater

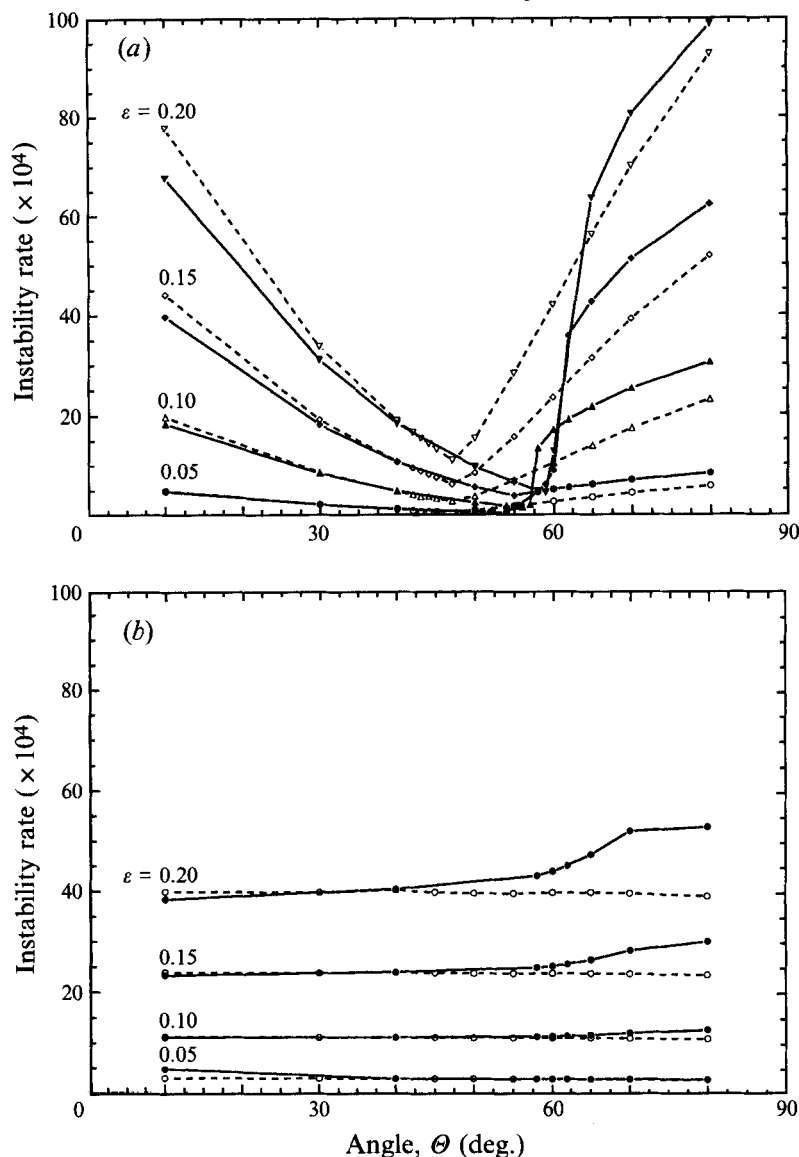


FIGURE 2. Maximal instability rates of the '1a' (a) and the '1b' (b) instabilities *vs.*  $\theta$  for different values of the wave steepness  $\epsilon$ . Solid lines: numerical calculations based on the algorithm by IK94; dashed lines: asymptotic weakly nonlinear theory (the leading-order approximation, formula (3.7)). The plot illustrates good agreement between the numerical and asymptotic results for almost the whole range of  $\theta$  except nearly plane wave patterns. It also indicates the very low sensitivity of the class 1b instability to the angle. Deviation of the asymptotic curve for large  $\theta$  can be attributed to higher-order nonlinearity.

angles  $\theta$ . There are two possible reasons for these discrepancies. The first one is the contribution of the higher-order harmonics to the basic solutions. We have seen from the earlier analysis in §2 that this contribution becomes more important for configurations close to the plane wave (see figure 1a,b).

The second reason for discrepancies is an interaction between two subclasses of instabilities which has not been taken into account by the simplest formula (3.7). It corresponds to the proximity of more than two diagonal terms of matrix  $\mathbf{A}$  to each

other. The analysis of the eigenvalue problem for the matrix  $\mathbf{A}$  block greater than  $(2 \times 2)$  is needed in this case. This is the subject of the next subsection.

### 3.4. Interaction between the subclasses

#### 3.4.1. Movements of the roots in the eigenvalue problem

Matching of just two diagonal elements is the most common, we may say generic, situation and, as we have shown in the previous subsection, being described in the main order by a simplest formula (3.7), it still provides the good quantitative predictions of the instability rates for a large range of basic wave configurations and parameters. Nevertheless, matching of more than two diagonal terms in the matrix  $\mathbf{A}$  inevitably occurs and as we show below its analysis is of importance for understanding of the physics of instabilities.

Formally, the multiple coincidence of diagonal terms can be treated by means of an asymptotic procedure similar to that of the previous subsection. The eigenvalue problem of infinite dimension reduces in a similar way to that of dimension equal to the number of close diagonal elements. We note however that within the adopted approximation of the Zakharov equation this number is effectively limited by the maximal order of the block of the matrix  $\mathbf{A}$  with all non-zero terms, that is by the order 4. Thus we consider a truncated matrix  $\mathbf{A}$  of the order  $(4 \times 4)$ , which leads to a quartic equation for the eigenvalues:

$$\lambda^4 + L_3\lambda^3 + L_2\lambda^2 + L_1\lambda + L_0 = 0. \quad (3.10)$$

The coefficients of (3.10) are real and can be easily expressed in terms of interaction coefficients and of the basic solution parameters (see Appendix C). This equation is solvable analytically using the well-known Descartes–Euler formulae (e.g. Abramovitz & Stegun 1964) and the problem of its analysis can be considered closed (despite rather cumbersome expressions for the roots). However, being interested in a qualitative understanding of what essentially new element brings an increase of the order of the eigenvalue problem, instead of the analysis of the cumbersome exact solutions of (3.10) we shall attempt to treat this problem as an *interaction between the classes of instabilities* we have already identified. In other words, knowing the asymptotics in parameter space where solutions are given by the simplest  $(2 \times 2)$  equations, we try to figure out what happens to eigenvalues in the ‘interaction zone’, i.e. we are interested in the qualitative understanding of movement of the roots of (3.10) in parameter space.

An asymptotic approach, based on the ‘pure’ instabilities of the previous subsection as a zero approximation, seems to be most illuminating. At least we can reveal the tendencies of transformation of these ‘pure’ instabilities. Obviously this procedure inevitably breaks down when the roots come too close or in other words interaction becomes too strong. However at the moment we are more interested in qualitative analysis and this asymptotic approach suits our purposes well.

‘Weak’ interaction of the instabilities can be described by an eigenvalue equation of the form

$$[\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2][\lambda^2 - (\lambda_3 + \lambda_4)\lambda + \lambda_3\lambda_4] = \mathcal{P}. \quad (3.11)$$

Here  $\mathcal{P}$  is a formally small perturbation. In this form the problem can be easily analysed qualitatively as a movement of  $\lambda$  relative to its unperturbed values  $\lambda_i$  ( $i = 1, 2, 3, 4$ ) (all the expressions in brackets in the left-hand side and  $\mathcal{P}$  are real). Figure 3 illustrates this movement.

The left-hand side of (3.11) is a fourth-order parabola. Its intersection with

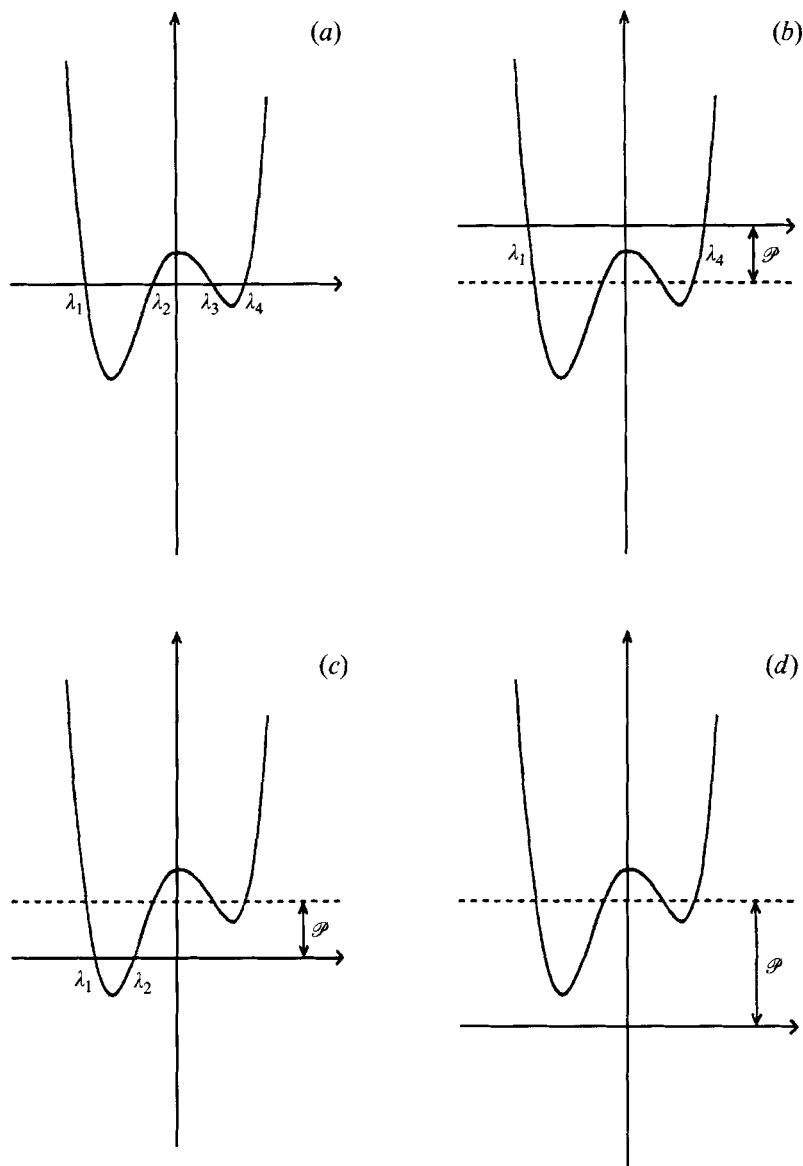


FIGURE 3. Interaction of 'pure' instabilities in terms of the simplest asymptotic approach. (a) Unperturbed eigenvalues. (b) Positive perturbation enlarges the range of frequencies or destabilizes stable states. This perturbation, can result in partial (b) destabilization of stable (a) 'pure' instabilities. (c,d) Negative perturbation diminishes the range of real frequencies  $\lambda$  and destabilizes partially (c) or fully (d) stable 'pure' states (a).

the abscissa corresponds to real values of  $\lambda$ , i.e. to stability (figure 3a). A small perturbation (the right-hand side of (3.11)) corresponds to a shift of the abscissa in the vertical direction. We emphasize that in our problem this perturbation may be both positive or negative. This means that there is a possibility of partial ( $\mathcal{P}$  is positive or negative – figure 3b,c) or full (when all four eigenvalues  $\lambda$  are imaginary) destabilization ( $\mathcal{P}$  is negative – figure 3d) of the original stable 'pure' states. Evidently, stabilization of the original unstable 'pure' states is possible as well.

Coming back to the general situation described by equation (3.10) we can conclude that interaction can either increase ‘pure’ instability or stabilize it. The point we would like to stress is that instabilities *can both appear or disappear owing to this interaction*. We note that the employed perturbation technique does not describe this phenomenon quantitatively in the general case, i.e. for ‘strong’ interaction, and to get quantitative results in the general case one should solve (3.10).

### 3.4.2. When is this interaction of instabilities important?

Having got a qualitative understanding of the possible behaviour of the eigenvalues we can pose the question ‘When is this interaction important?’ or more precisely, ‘Where in parameter space is it most essential?’ as the eigenvalues  $\lambda_i$  depend parametrically on the perturbation wavevector  $\mathbf{h}$  and the parameters of the basic state. While resonant curves (3.9a,b) of the instability classes 1a, 1b correspond to matching of two diagonal terms of a block of the matrix  $\mathbf{A}$ , the domains of their *intersection or proximity* in the  $\mathbf{h}$ -plane are the zones of relatively strong interaction of the ‘pure’ instabilities. Thus the interaction between subclasses occurs in the narrow zones in  $\mathbf{k}$ -space lying in the immediate vicinity of two resonance curves belonging to different subclasses. It is easy to see that these resonance zones are very narrow indeed,  $\sim O(\varepsilon^2)$ . One may expect their influence on the *global* characteristics of instability in the generic situations is unlikely to be crucial. However situations where this influence is important are possible as well and merit special consideration.

These specific situations occur when these zones contain the maxima of a subclass instability, provided the maximum is well localized, and the overlapping zones themselves are relatively large. To distinguish these exceptional situations we need:

simple analysis of the resonance curves (3.9a,b);

a basic knowledge of the properties of the instabilities belonging to each of the interacting subclasses derived in §3.3.1.

Consider the properties which are necessary in this context in more detail. We recall that there are two basic classes of instabilities. The instability of class 1b has maximal rates for small wavenumbers of perturbation,  $\mathbf{h}_{\max}$  (see Benjamin & Feir 1967; Zakharov 1966, 1968; Longuet-Higgins 1978)

$$|\mathbf{h}_{\max}^b| \approx \eta/4, \quad (3.12a)$$

$$\text{Im}(\lambda_{\max}^b) \approx \tilde{V}^{(2)}(\mathbf{\kappa}, \mathbf{\kappa}, \mathbf{\kappa}, \mathbf{\kappa})|c_0|^2 = \eta^2/8 \quad (3.12b)$$

where  $\eta$  is nondimensional wave height. The instability of class 1b is the Benjamin–Feir instability, which means that the dispersive terms are balanced by the nonlinear frequency shift. The wavenumber of the most unstable perturbation is of the order of the wave steepness  $\varepsilon$  (see (3.12a)).

Formula (3.7) and numerical computations by IK93, IK94 show that long-wave instability dominates among the resonant interactions of class 1a as well. In the corresponding limit formula (3.7) yields

$$|\mathbf{h}_{\max}^a| \approx \eta^2/(4 \cos \Theta), \quad (3.13a)$$

$$\text{Im}(\lambda_{\max}^a) \approx 2\tilde{V}^{(2)}(\bar{\mathbf{\kappa}}, \mathbf{\kappa}, \bar{\mathbf{\kappa}}, \mathbf{\kappa})|c_0|^2. \quad (3.13b)$$

For the class 1a instability the amplitude dispersion term for the most unstable perturbation is of order of the convective term (the difference of group velocities of two perturbation waves). In accordance with formula (3.13a) the most unstable perturbations in this case are much longer than those for the classical Benjamin–Feir instability ( $|\mathbf{h}_{\max}^a| \sim \varepsilon^2$ , while  $|\mathbf{h}_{\max}^b| \sim \varepsilon$ ).

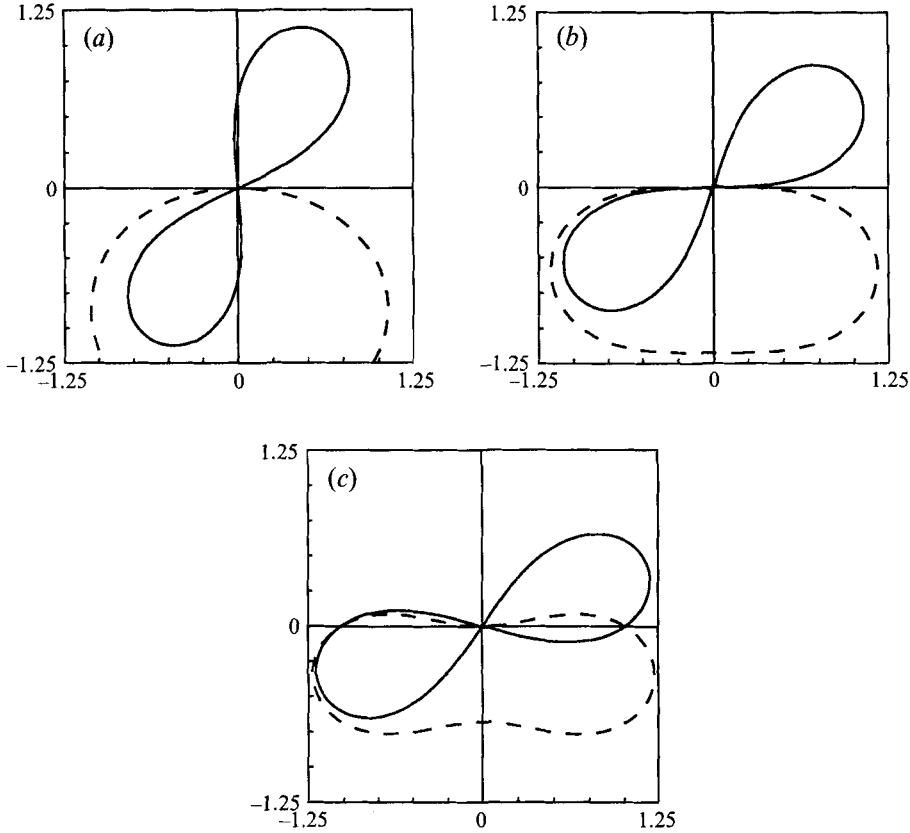


FIGURE 4. Samples of resonant curves in the  $h(p, q)$ -plane for class 1a (dashed lines) and 1b (solid lines) for different short-crested patterns. (a) At  $\Theta = 30^\circ$  a simple intersection of resonance curves for relatively long-wave perturbations occurs. Strong interaction is localized in this zone. (b)  $\Theta = 54^\circ$ . At the point  $h = 0$  two resonant curves are tangential to each other and strong interaction takes place in relatively large parts of the resonance zones. (c)  $\Theta = 70^\circ$ . The strong interaction between subclasses is localized in the zone  $O(\epsilon)$ .

We see that instabilities of these two subclasses are caused by different physical mechanisms. We note that 1a instability dominates in a wide range of angles  $\Theta$  excluding cases when the interaction coefficient  $\tilde{V}^{(2)}(\bar{\kappa}, \kappa, \bar{\kappa}, \kappa)$  is anomalously small (near  $\Theta \approx 43.9^\circ$ ). In terms of figure 3 this means that in the vicinity of this angle the corresponding minimum of the curve (3.11) for a 'pure' instability of class 1a is close to the abscissa. A small perturbation  $\mathcal{P}$  can strongly affect this slightly unstable state. The effect manifests itself in global instability characteristics in a relatively narrow range of angles  $45^\circ < \Theta < 60^\circ$ . In this case the instability of class 1a is weak and zones of class 1a and class 1b instabilities are overlapping or lie very close to each other. This purely geometric fact is illustrated by figure 4.

When  $\Theta$  differs essentially from  $\Theta^* = (\pi/2 - \Theta_{\text{Kelvin}})$ , two resonance zones intersect each other (see figure 4a,c). When resonance curves become tangential to each other at  $\Theta = \Theta^*$  (figure 4b), relatively large parts of resonance zones can interact strongly, which results in a sharp reduction of the 1a instability rate near  $\Theta^*$  (see figure 2).

The speculations given above are supported by a quantitative illustration: figure 5 demonstrates the effect of interaction between instabilities. The results of the direct numerical analysis (IK94) are compared with the solutions of the eigenvalue problem

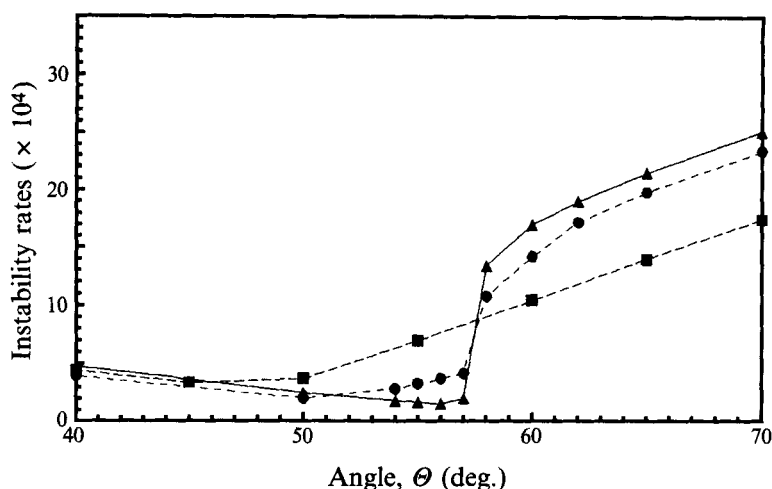


FIGURE 5. Maximum rate of class 1b instability for the wave steepness 0.1 vs. the angle  $\Theta$  calculated by different methods.  $\blacktriangle$ , Numerical approach (see IK94).  $\blacksquare$ , Analytical results: interaction of instabilities is neglected (approximate formula (3.7)).  $\bullet$ , Analytical approach: interaction of instabilities is taken into account (solution to the quartic equation (3.10)).

for the block  $(4 \times 4)$  (see (3.10)) which takes the interaction into account and the analytical results provided by the formula (3.7) where interaction was neglected. The comparison is carried out for the most interesting range of angles, where in accordance with the speculations above the interaction is essential. One can see that the interaction influence upon the global instability characteristics indeed becomes negligible outside this zone.

We see that the Hamiltonian approach gives an impressive accuracy for the global characteristics of instabilities even in the lowest order of the matrix  $\mathbf{A}$  truncation. We have got a confirmation of our qualitative conclusions about when the interaction between the instabilities is important and when it can be *a priori* neglected.

### 3.5. A qualitative analysis of higher-order resonances within the four-wave equation

Throughout the previous subsection we analysed cases when the matching of the diagonal terms occurs within the 'basic' block  $(4 \times 4)$  of matrix  $\mathbf{A}$ . The instability rate in this case has order  $\varepsilon^2$ . However the matching of, say, two diagonal terms located in two different basic blocks is also possible. An approximate expansion like (3.6) cannot be applied in the latter case because  $A_{0k} = 0$  within the accepted accuracy of the Zakharov equation and higher-order terms should be taken into account to describe the instability. It can be shown that the Hamiltonian expansion being extended to higher orders in  $\varepsilon$  gives  $A_{0k}$  to be of order  $\varepsilon^{2[k/4+1]}$  (see BS94) ( $[x]$  means integer fraction of  $x$ ).

An asymptotic procedure similar to (3.6) leads to similar resonant conditions

(a) for diagonal terms of the same

$$D_0^+ \approx D_n^+ \quad \text{or} \quad (D_0^- \approx D_n^-) \quad (3.14a)$$

(b) and opposite signatures

$$D_0^+ \approx D_n^- \quad \text{or} \quad (D_0^- \approx D_n^+). \quad (3.14b)$$

Here  $n = [k/2]$  is the number of a  $(4 \times 4)$  block. If  $k < 3$  ( $n < 2$ ) (3.14a,b) give harmonic resonance conditions which have been analysed in the previous section.

In the small-amplitude limit the resonance curves for an arbitrary  $k$  can be easily obtained. The case (3.14a) gives us two straight lines which are parallel to the axis of symmetry of our short-crested pattern

$$\left. \begin{aligned} |\mathbf{h}| &= |\mathbf{h} + n(\bar{\mathbf{\kappa}} - \mathbf{\kappa})|, \\ |\bar{\mathbf{\kappa}} + \mathbf{\kappa} - \mathbf{h}| &= |\bar{\mathbf{\kappa}} + \mathbf{\kappa} - \mathbf{h} - n(\bar{\mathbf{\kappa}} - \mathbf{\kappa})|. \end{aligned} \right\} \quad |n| > 1. \quad (3.15)$$

This case corresponds to the resonance of two probing waves with the same frequencies and  $[n/2 + 1]$  pairs of basic harmonics  $\mathbf{\kappa}$  and  $\bar{\mathbf{\kappa}}$ . Generally, this resonance does not lead to the instability, because of the same signature of colliding eigenvalues. In other words, corresponding nondiagonal terms  $A_{0k}$  and  $A_{k0}$  have the same signs (see BS94) and the simplest asymptotic procedure of the type (3.6) gives real eigenvalues – avoiding crossing of the eigenvalues. This is true as long as  $A_{0k} \neq 0$  and our asymptotic procedure (3.6) is not degenerated. For discrete sets of resonant points

$$\begin{aligned} (a) \quad \mathbf{h} &= n(\bar{\mathbf{\kappa}} - \mathbf{\kappa}), \\ (b) \quad \mathbf{h} &= \bar{\mathbf{\kappa}} + \mathbf{\kappa} - n(\bar{\mathbf{\kappa}} - \mathbf{\kappa}) \end{aligned}$$

it follows from a general asymptotic procedure (see Zakharov 1966, 1968; Krasitskii 1994) being extended to higher-order terms in wave amplitude, that corresponding nondiagonal  $A_{0k}$  are plain zeros. Thus, the answer to the question on instability can be given in this degenerate case neither in terms of the asymptotic procedure (3.6) nor within the general consideration (see Krein 1950; Yakubovich & Starzhinskii 1975; MacKay & Saffman 1986). The problem has to be studied by other methods, say numerically. Resonances of this kind – ‘bubbles’ of superharmonic instability for plane (Longuet-Higgins 1978) and short-crested gravity waves (IK94) – have been investigated by this method. It was found that these instabilities are strongly localized in the space of wave parameters as ‘bubbles’. The instability rates corresponding to these ‘bubbles’ are of order  $\varepsilon^{2[n/2+1]}$  and the widths of the zones of instability are of the same order in amplitude. Being weak and capriciously dispersed over parameter space they represent a quite difficult task for a numerical analysis.

The case (3.14b) gives us the resonance curves

$$2\Omega_0 = \omega(\mathbf{h}) + \omega(\bar{\mathbf{\kappa}} + \mathbf{\kappa} - \mathbf{h} - n(\bar{\mathbf{\kappa}} - \mathbf{\kappa})). \quad (3.16)$$

Consider the family of instabilities specified in (3.16) in more detail. The resonances (3.16) are the fourth-order curves for gravity deep-water waves. These curves exist if the following condition is satisfied:

$$n^2 < \frac{3}{\cos^2 \Theta} + 1. \quad (3.17)$$

Inequality (3.17) gives us the number of resonance curves of the type (3.16) for any given angle  $\Theta$ . From (3.16) and the condition (3.17) an important conclusion follows. *There are higher-order instabilities of class 1, specific for three-dimensional waves. These instabilities do not disappear as the patterns tend to long-crested ones ( $\Theta \rightarrow 90^\circ$ ).*

Moreover, the number of possible resonances for the case (3.14b) tends to infinity as the wave pattern tends towards a two-dimensional one. The existence of these resonances was demonstrated in IK93 as well.

Both resonance conditions (3.15) and (3.16) correspond, as can be easily seen, to the Phillips resonances of two perturbation waves and  $2[n/2 + 1]$  basic harmonics. Within the adopted order of accuracy of the Zakharov equation we can predict the existence of these instabilities and their location in  $\mathbf{k}$ -space, but cannot describe quantitatively their rates. To conclude whether the instability takes place we check

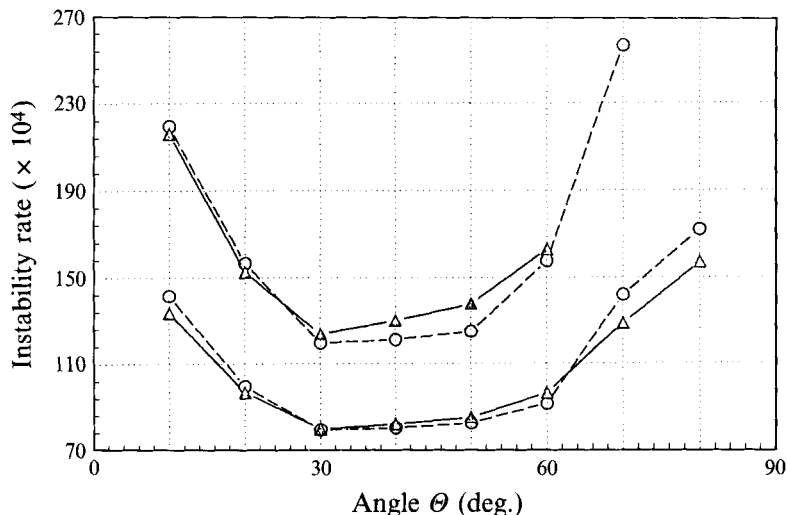


FIGURE 6. Maximal instability rates for short-crested waves of high steepness. Solid lines: numerical calculations based on the algorithm by IK94; dashed lines: the asymptotic weakly nonlinear approach for the block  $(4 \times 4)$  of the matrix  $\mathbf{A}$ .

the signatures of diagonal terms or the signatures of the nondiagonal terms  $A_{0k}$  and  $A_{k0}$  for the case  $k > 3$  ( $n > 1$ ), which are equivalent to the Krein signatures or energy signs of the corresponding perturbations. Thus we come to a criterion similar to that obtained by MacKay & Saffman (1986) (Theorem 1, p. 118, see also § 3.3): *instability occurs if corresponding diagonal terms have opposite signatures (3.14b) or nondiagonal terms  $A_{0k}$  are equal to zero (3.14a).*

The results of this subsection confirm the completeness of the picture of these resonances constructed by numerical methods in IK93. They explain the underlying physics and allow one to pinpoint the locations of possible resonances in more complex situations.

### 3.6. Waves of high steepness

The analytical part of our analysis was based on the asymptotic expansion in wave steepness  $\varepsilon$ ; the leading-order terms were retained only. Strictly speaking, we could not claim their validity beyond the range of very small steepness. The only way to investigate the range of their true applicability is through comparison with numerical computations using the exact equations. The plots of growth rates of instabilities of different classes (see figure 2) demonstrates that the analytical approach works well at least up to moderate steepness  $\varepsilon = 0.2$ . What happens at even higher steepness? Does the validity of our analytical results hold? Does it depend on the wave parameters (the angle  $\theta$ )? To answer these questions computations of the maximal growth rate (the most representative characteristic) were performed for short-crested waves of high steepness, namely for  $\varepsilon = 0.3$  and  $0.4$ . The results of these computations are presented in figure 6.

This figure illustrates very good agreement between the numerical and asymptotic results for the range  $10^\circ \leq \theta \leq 70^\circ$ . The largest discrepancies occur for nearly plane waves:  $\theta = 70^\circ, 80^\circ$  at  $\varepsilon = 0.3, 0.4$ , where the theory *a priori* was not supposed to work well. The smaller gap between the numerical and analytical growth rates for  $\theta = 40^\circ, 50^\circ$  at  $\varepsilon = 0.4$  is certainly due to the interaction of instabilities. The good agreement for nearly standing waves is surprising. For the point  $\theta = 70^\circ$ ,  $\varepsilon = 0.4$  the

convergence of the numerical procedure was not good enough to ensure confidence in the result. Therefore the corresponding growth rate is not given here. The poor accuracy might be due to the occurrence of a strong singularity in the vicinity of this point. For this point we can say now only that the asymptotic values seems to be too high.

We conclude that the Zakharov approach works very well for the range of angles  $\Theta = 10^\circ, 60^\circ$  up to  $\varepsilon = 0.4$ , the error being less than 10%. Moreover if we exclude some particular configurations (say,  $\Theta = 40^\circ - 50^\circ$  at  $\varepsilon = 0.3, 0.4$ ) we gain almost an order in accuracy.

Further analysis of very steep short-crested waves, especially those close to the highest ones, requires special consideration and will be reported elsewhere.

### 3.7. Wave instabilities and mechanical analogies

In the previous subsection we have accomplished our analysis of the linear instabilities of the short-crested waves by two complementary approaches, which was the immediate goal of the present work. However having in mind our more distant goal to investigate more complex wave patterns it is highly desirable to acquire from this simplest example as much insight as possible into the physics of wave instabilities. We note that a characteristic feature of the weakly nonlinear problems cast into normal variables is their universality, i.e. there is a very limited number of different Hamiltonians, as almost all the specifics of the particular system is hidden in the corresponding kernels. This prompts us to look for simpler or better studied physical models governed by the same set of equations as, say, three-wave interaction might be modelled by gyroscope dynamics (e.g. Rabinovich & Trubetskov 1989).

#### 3.7.1. Quantum oscillators

The most straightforward analogy lies in the domain of quantum mechanics. It could be noted that our basic system of linear equations (3.2), which describes instabilities of symmetric short-crested waves, also describes the motion of an infinite chain of interacting quantum oscillators. Indeed, a single quantum oscillator is an object governed by a linear equation of the type (e.g. Feynman, Leighton & Sands 1963)

$$i \frac{da}{dt} = \omega^{quant} a \quad (3.18)$$

where  $a$  is its complex amplitude and  $\omega^{quant}$  is called its partial frequency. If we take an infinite number of oscillators having different partial frequencies, put them in line and connect them through a set of linear 'springs' such that each oscillator interacts with the five nearest neighbours, we arrive at our basic system. The partial frequencies of the oscillators are equal to the diagonal elements

$$\omega_n^{quant} = D_n^\pm$$

while the 'strength of the springs' providing the interaction should be taken equal to corresponding nondiagonal elements of the matrix  $\mathbf{A}$ . In this model the signs of the diagonal elements or the signs of the partial frequencies correspond literally to the signs of the energies of the oscillators. An instability within this model is naturally interpreted as matching of partial frequencies of any two (in the simplest case) oscillators having direct bonds and characterized by different energy signs. Matching of the frequencies means resonant interaction between the two oscillators, while the requirement of opposite energy signs permits the energy of each oscillator to grow, preserving the total energy. Multiple matching corresponds to multiple resonances.

However an interpretation of multiple resonances is not straightforward within this model.

Little help can be found in the literature, as the dynamics of our chain of interacting linear oscillators differs qualitatively from the typical models studied in condensed matter physics. The latter deals with ensembles of quantum oscillators where all or almost all oscillators are identical, or at least there are few sorts of identical oscillators. As a rule 'far' coupling is neglected as well, i.e. only the interaction with the two nearest neighbours is taken into account in these models. A particular feature of these systems is that the states of identical oscillators are infinitely degenerated. A weak coupling in this system produces specific modes of group motions in this chain – phonons.

In the context under consideration we have a quite different problem. All oscillators are different and propagation of phonon-like disturbances as in crystals is prohibited. The only qualitative conclusion we can make on the multiple resonances, which are not collective motion yet but differ strongly from the two-oscillator resonance, is that an increase of the number of identical oscillators leads to suppressing of 'individual' properties of oscillators and, in particular, might result in suppressing of 'intrinsic' instabilities of the unstable resonant couples involved.

### 3.7.2. A chain of classical oscillators

For obvious reasons it seems preferable to have an interpretation of the physics of the wave instabilities in terms of a simple model from the domain of classical mechanics. To elaborate it let us make some additional transformation of variables  $\alpha_n(\mathbf{k})$ . Consider a diagonal block ( $2 \times 2$ ) of the system of equations (3.2)

$$\left. \begin{aligned} i \frac{\partial \alpha_n^+}{\partial t} &= D_n^+ \alpha_n^+ + C_n \alpha_n^-, \\ i \frac{\partial \alpha_n^-}{\partial t} &= -C_n \alpha_n^+ + D_n^- \alpha_n^-. \end{aligned} \right\} \quad (3.19)$$

A linear transformation of dependent variables

$$\begin{aligned} W_n^+ &= p \alpha_n^+ + q \alpha_n^-, \\ W_n^- &= q \alpha_n^+ - p \alpha_n^-, \end{aligned}$$

where

$$p = [D_n^+ / (D_n^+ - D_n^-)]^{1/2}, \quad q = [-D_n^- / (D_n^+ - D_n^-)]^{1/2}$$

results in the single second-order equation for, say,  $W^+$

$$\frac{\partial^2 W_n^+}{\partial t^2} + i(D_n^+ + D_n^-) \frac{\partial W_n^+}{\partial t} - (D_n^+ D_n^- + 4C_n^2) W_n^+ = 0. \quad (3.20)$$

Equation (3.20) is the well-known equation for the Foucault pendulum (see e.g. Arnol'd 1978, § 27), which describes the oscillation of a pendulum in a rotating frame of reference. Physical coordinates  $x$  and  $y$  for this pendulum correspond to real and imaginary parts of the complex variable  $W_n^+$ . The angular velocity of the reference system is

$$\Omega_n^z = (D_n^+ + D_n^-)/2.$$

and the frequency of the classical linear oscillator (the pendulum) in the nonrotating reference system is

$$\omega_n^{\text{classic}} = \pm [(D_n^+ - D_n^-)^2 / 4 - C_n^2]^{1/2}. \quad (3.21)$$

The latter value is responsible for the stability of the oscillator. The rotation splits two frequencies of the oscillator  $\pm\omega_n^{\text{classic}}$  that are equal in modulae into two different ones, which are close to  $D_n^+$  and  $D_n^-$  (if we let  $C_n^2$  be small). It also changes whimsically coordinates but does not influence the condition of the oscillator stability. The oscillator is stable if  $\omega_n^{\text{classic}}$  is real, which corresponds to the motion of the oscillator in a bowl or to oscillations of a pendulum. The instability occurs when the bowl transforms into a hill or the pendulum is reversed. This is a case when  $D_n^+$  and  $D_n^-$  – oscillator frequencies are close to each other. Thus, we can interpret the instability occurrence as ‘internal resonance’ of the classical oscillator or ‘degeneration’ of the oscillator, when its potential surface becomes nearly flat. Thus, for the case of the simplest resonance we have found a very simple model in classical mechanics, that of an oscillator in a rotating frame of reference. A similar model can be suggested where instead of rotation of the frame the magnetic field effect upon a charged oscillator is taken into account.

The coordinate transformation can consequently be executed for all pairs of perturbations  $\alpha_n^+$ ,  $\alpha_n^-$  in (3.2) and thus our problem can be interpreted as an analysis of an infinite chain of interacting Foucault pendulae, where each oscillator interacts with the two nearest neighbours only. The interaction of two different oscillators might be essential when these oscillators are close to their intrinsic instabilities. As an example we can imagine two coupled oscillators in very shallow bowls or at very low hills: weak interaction could affect qualitatively their intrinsic dynamics. Bearing in mind that each of these oscillators has two different (split by rotation) frequencies and their bonds contain a ‘nonpotential component’ (see below), the possible outcome of this interaction is richer than in the simpler problem of two non-rotating oscillators coupled by potential bonds (e.g. Andronov, Witt & Khaikin 1966; Rabinovich & Trubetskov 1989). We have

$$\left. \begin{aligned} \frac{\partial^2 W_1}{\partial t^2} + i(D_1^+ + D_1^-) \frac{\partial W_1}{\partial t} - (D_1^+ D_1^- + 4C_1^2) W_1 &= F_{2p} W_2 + iF_{2g} \frac{\partial W_2}{\partial t}, \\ \frac{\partial^2 W_2}{\partial t^2} + i(D_2^+ + D_2^-) \frac{\partial W_2}{\partial t} - (D_2^+ D_2^- + 4C_2^2) W_2 &= F_{1p} W_1 + iF_{1g} \frac{\partial W_1}{\partial t}. \end{aligned} \right\} \quad (3.22)$$

A feature which makes the interaction of our oscillators different from that of genuine pendulae is that symmetry restrictions for the coefficients in the right-hand side of (3.22) are much milder in our case. The only requirement is that the eigenvalue equation should have real coefficients. That results in richer dynamics of the system.

## 4. Conclusions and discussion

### 4.1. Conclusions

The main goal of this work, to make a comparison of the two approaches to the wave stability problems for the simplest but still representative example of three-dimensional waves, has been achieved:

the ability of the methods employed in IK93, IK94 to provide the full picture of instabilities, including the finest details of weak ‘bubble’ instabilities, has been confirmed by comparison with the asymptotic results in the small-amplitude limit;

the hypothesis that the asymptotic results hold their validity for rather steep waves has been confirmed as well, for a wide range of wave patterns, although only the leading-order terms were used for the analysis. Some new physical mechanisms responsible for the occurrence of the wave instabilities have been identified.

## 4.2. Discussion

In this paper we have considered a particular problem – that of the instability of short-crested surface gravity waves, which we believe to be the best, simple but still representative, example for understanding the general situation of water wave instabilities. Two complementary basic approaches IK93, IK94 and BS95 were used. The Hamiltonian approach by means of the development of an asymptotic procedure allowed us to get from an infinite system of equations for perturbations some very simple analytical expressions for the instability growth rates, avoiding a number of additional approximations that other authors were forced to accept. The results of the direct (numerical) analysis enabled us to claim surprisingly good validity of these simple formulae for waves of moderate and high steepness within a wide range of wave parameters as well.

The detailed comparison of the results of the two approaches also allowed us to specify the contributions due to the particular classical Phillips instabilities. We generalized this concept by describing the interaction between the elementary classes. The remarkable fact we found is that *most of the deviations from the classical theory predictions can be quantitatively described with a good accuracy as a result of this interaction*. The effect of strong interaction of subclasses is localized in a certain range of angles  $\Theta$ . That enables us to quantify the effects due to higher-order harmonics as well.

Summarizing we would like to emphasize that the applicability of the concepts developed, which obviously is not confined to the particular problem of short-crested waves, is in effect much wider than the range of validity of weakly nonlinear theory. Indeed, some well-known effects typical of strongly nonlinear waves, such as, say, restabilization of a plane wave of large steepness with respect to four-wave instability (e.g. Lighthill 1965), or to be more precise and using McLean terminology, disappearance of class 1 ( $m = 1$ ) instability, are better interpreted in terms of the interaction of classes of instabilities, like the inhibition of the four-wave instability by the five-wave processes.

The results of the work give a good overview of directions of further investigations in the field of wave stability problem.

The analytical asymptotic approach presented here has two main directions of development. First, there are numerous opportunities to extend straightforwardly these results, as they are, to such problems as the stability of some more complex patterns consisting of more than two basic waves, and to take into account finite depth, capillarity (remaining in the range where triad interactions are prohibited)<sup>†</sup>. The corresponding expressions for the kernels in the Hamiltonian series have been already derived by Krasitskii (1990, 1994). Second, it seems realistic to improve the present analytical results given now by the leading-order terms in the asymptotic expansion by taking into account the higher-order corrections employing a symbolic manipulator technique.

The numerical approach seems to have large reserves in efficiency. The problem being formulated in Hamiltonian variables (not confined to the leading order as in our present consideration) contains ordering in  $\varepsilon$  and a rather ‘good’ structure of matrix **A**. This structure could be improved considerably through transformations

<sup>†</sup> Some additional specific questions concerning higher-order resonances arise in this problem, which we believe in some cases could be avoided. In any case the problem of instability of gravity-capillary waves is the subject of further studies.

leading to the 'normal' variables, which should give an enormous gain in efficiency of computations.

It should be noted that our study has also outlined a range of wave patterns, namely nearly plane waves, where neither the developed asymptotic approach nor the numerical treatment work well. This specific and very important area requires a quite new approach, which is still to be invented. Here we confine ourselves to just drawing attention to this challenging problem.

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## Appendix A

Canonical variables  $a(\mathbf{k})$ ,  $a^*(-\mathbf{k})$  are expressed in terms of Fourier amplitudes of surface velocity potential and surface displacement

$$\eta(\mathbf{k}) = M(\mathbf{k})[a(\mathbf{k}) + a^*(-\mathbf{k})], \quad \psi(\mathbf{k}) = -iN(\mathbf{k})[a(\mathbf{k}) - a^*(-\mathbf{k})] \quad (\text{A } 1)$$

with

$$M(\mathbf{k}) = \left[ \frac{|\mathbf{k}|}{2\omega(\mathbf{k})} \right]^{1/2}, \quad N(\mathbf{k}) = \left[ \frac{\omega(\mathbf{k})}{2|\mathbf{k}|} \right]^{1/2} \quad (\text{A } 2)$$

where  $\omega(\mathbf{k})$  is the dispersion relation of linear waves defined by

$$\omega(\mathbf{k}) = [g|\mathbf{k}|]^{1/2}$$

and

$$\eta(\mathbf{x}) = \frac{1}{2\pi} \int \eta(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k}, \quad \eta(\mathbf{k}) = \eta^*(-\mathbf{k}), \quad (\text{A } 3)$$

$$\psi(\mathbf{x}) = \frac{1}{2\pi} \int \psi(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k}, \quad \psi(\mathbf{k}) = \psi^*(-\mathbf{k}). \quad (\text{A } 4)$$

Here  $\mathbf{k} = (k_x, k_y)$  is the horizontal wavevector, integration with respect to  $\mathbf{k}$  is extended over the entire  $\mathbf{k}$ -plane, the asterisk denotes complex conjugate, and explicit dependence of  $\eta$  and  $\psi$  on  $t$  is suppressed for simplicity of notation.

A canonical transformation to the new canonical variables  $b(\mathbf{k})$ ,  $b^*(\mathbf{k})$  is

$$\begin{aligned} a_0 &= b_0 + \int A_{0,1,2}^{(1)} b_1 b_2 \delta_{0-1-2} d\mathbf{k}_{12} \\ &+ \int A_{0,1,2}^{(2)} b_1^* b_2 \delta_{0+1-2} d\mathbf{k}_{12} + \int A_{0,1,2}^{(3)} b_1^* b_2^* \delta_{0+1+2} d\mathbf{k}_{12} \\ &+ \int B_{0,1,2,3}^{(1)} b_1 b_2 b_3 \delta_{0-1-2-3} d\mathbf{k}_{123} + \int B_{0,1,2,3}^{(2)} b_1^* b_2 b_3 \delta_{0+1-2-3} d\mathbf{k}_{123} \\ &+ \int B_{0,1,2,3}^{(3)} b_1^* b_2^* b_3 \delta_{0+1+2-3} d\mathbf{k}_{123} + \int B_{0,1,2,3}^{(4)} b_1^* b_2^* b_3^* \delta_{0+1+2+3} d\mathbf{k}_{123}. \end{aligned} \quad (\text{A } 5)$$

The arguments  $k_j$  in  $a$ ,  $\omega$ ,  $U^{(n)}$ ,  $V^{(n)}$  and  $\delta$ -functions are replaced by subscripts  $j$ , with the subscript zero assigned to  $k$ . Thus, for example,  $a_j = a(k_j, t)$ ,  $\omega_j = \omega(k_j)$ ,  $U_{0,1,2}^{(n)} = U^{(n)}(k, k_1, k_2)$ ,  $\delta_{0-1-2} = \delta(k - k_1 - k_2)$  etc. For differentials the notation  $dk_0 = dk$ ,  $dk_{012} = dk dk_1 dk_2$  etc. is used, and the integral signs denote corresponding multiple integrals with limits from  $-\infty$  to  $+\infty$ .

The kernels in (A 5) are

$$\begin{aligned} A_{0,1,2}^{(1)} &= -\Delta_{0-1-2}^{-1} U_{0,1,2}^{(1)}, \\ A_{0,1,2}^{(2)} &= -2\Delta_{0+1-2}^{-1} U_{2,1,0}^{(1)} = -2A_{2,1,0}^{(1)}, \\ A_{0,1,2}^{(3)} &= -\Delta_{0+1+2}^{-1} U_{0,1,2}^{(3)}, \end{aligned}$$

$$\begin{aligned} B_{0,1,2,3}^{(2)} &= A_{0,1,-0-1}^{(3)} A_{2,3,-2-3}^{(3)} + A_{1,2,1-2}^{(1)} A_{3,0,3-0}^{(1)} + A_{1,3,1-3}^{(1)} A_{2,0,2-0}^{(1)} \\ &\quad - A_{0+1,0,1}^{(1)} A_{2+3,2,3}^{(1)} - A_{0,2,0-2}^{(1)} A_{3,1,3-1}^{(1)} - A_{0,3,0-3}^{(1)} A_{2,1,2-1}^{(1)}, \end{aligned}$$

$$\begin{aligned} B_{0,1,2,3}^{(1)} &= -\Delta_{0-1-2-3}^{-1} [Z_{0,1,2,3}^{(1)} + V_{0,1,2,3}^{(1)}], \\ B_{0,1,2,3}^{(3)} &= -\Delta_{0+1+2-3}^{-1} [Z_{0,1,2,3}^{(3)} + 3V_{3,2,1,0}^{(1)}], \\ B_{0,1,2,3}^{(4)} &= -\Delta_{0+1+2+3}^{-1} [Z_{0,1,2,3}^{(4)} + V_{0,1,2,3}^{(4)}], \end{aligned}$$

where

$$\begin{aligned} \Delta_{0-1-2} &= \omega(k_0) - \omega(k_1) - \omega(k_2), \\ \Delta_{0+1+2} &= \omega(k_0) + \omega(k_1) + \omega(k_2), \end{aligned}$$

$$\begin{aligned} U_{0,1,2}^{(1)} &= -U_{-0,1,2} - U_{-0,2,1} + U_{1,2,-0}, \\ U_{0,1,2}^{(3)} &= U_{0,1,2} + U_{0,2,1} + U_{1,2,0}, \end{aligned}$$

$$\begin{aligned} V_{0,1,2,3}^{(1)} &= \frac{1}{3} (-V_{-0,1,2,3} - V_{-0,2,1,3} - V_{-0,3,1,2} + V_{1,2,-0,3} + V_{1,3,-0,2} + V_{2,3,-0,1}), \\ V_{0,1,2,3}^{(2)} &= (V_{-0,-1,2,3} + V_{2,3,-0,-1} - V_{-0,2,-1,3} - V_{-1,2,-0,3} - V_{-0,3,-1,2} - V_{-1,3,-0,2}), \\ V_{0,1,2,3}^{(4)} &= \frac{1}{3} (V_{0,1,2,3} + V_{0,2,1,3} + V_{0,3,1,2} + V_{1,2,0,3} + V_{1,3,0,2} + V_{2,3,0,1}), \end{aligned}$$

$$U_{0,1,2} = -N_0 N_1 M_2 E_{0,1,2}^{(3)},$$

$$V_{0,1,2,3} = -2N_0 N_1 M_2 M_3 E_{0,1,2,3}^{(4)},$$

$$E_{0,1,2}^{(3)} = -\frac{1}{2(2\pi)} [(k_0 k_1 + |k_0| |k_1|)],$$

$$\begin{aligned} E_{0,1,2,3}^{(4)} &= -\frac{1}{8(2\pi)^2} [2|k_0|^2 |k_1| + 2|k_1|^2 |k_0| \\ &\quad - |k_0| |k_1| (|k_0 + k_2| + |k_1 + k_2| + |k_0 + k_3| + |k_1 + k_3|)], \end{aligned}$$

and

$$\begin{aligned}
 Z_{0,1,2,3}^{(1)} &= \frac{2}{3} [U_{0,1,0-1}^{(1)} A_{2+3,2,3}^{(1)} + U_{0,2,0-2}^{(1)} A_{1+3,1,3}^{(1)} + U_{0,3,0-3}^{(1)} A_{1+2,1,2}^{(1)} \\
 &\quad + U_{1,0,1-0}^{(1)} A_{2,3,-2-3}^{(3)} + U_{2,0,2-0}^{(1)} A_{1,3,-1-3}^{(3)} + U_{3,0,3-0}^{(1)} A_{1,2,-1-2}^{(3)}], \\
 Z_{0,1,2,3}^{(2)} &= -2 [U_{0,2,0-2}^{(1)} A_{3,1,3-1}^{(1)} + U_{2,0,2-0}^{(1)} A_{1,3,1-3}^{(1)} + U_{0,3,0-3}^{(1)} A_{2,1,2-1}^{(1)} \\
 &\quad + U_{3,0,3-0}^{(1)} A_{1,2,1-2}^{(1)} - U_{0+1,0,1}^{(1)} A_{2+3,2,3}^{(1)} - U_{0,1,-0-1}^{(3)} A_{2,3,-2-3}^{(3)}], \\
 Z_{0,1,2,3}^{(3)} &= 2 [U_{3,0,3-0}^{(1)} A_{1+2,1,2}^{(1)} - U_{0+1,0,1}^{(1)} A_{3,2,3-2}^{(1)} - U_{0+2,0,2}^{(1)} A_{3,1,3-1}^{(1)} \\
 &\quad - U_{0,1,-0-1}^{(3)} A_{2,3,2-3}^{(3)} - U_{0,2,-0-2}^{(3)} A_{1,3,1-3}^{(3)} + U_{0,3,0-3}^{(3)} A_{1,2,-1-2}^{(3)}], \\
 Z_{0,1,2,3}^{(4)} &= \frac{2}{3} [U_{0,1,-0-1}^{(3)} A_{2+3,2,3}^{(1)} + U_{0,2,-0-2}^{(3)} A_{1+3,1,3}^{(1)} + U_{0,3,-0-3}^{(3)} A_{1+2,1,2}^{(1)} \\
 &\quad + U_{0+1,0,1}^{(1)} A_{2,3,-2-3}^{(3)} + U_{0+2,0,2}^{(1)} A_{1,3,-1-3}^{(3)} + U_{0+3,0,3}^{(1)} A_{1,2,-1-2}^{(3)}].
 \end{aligned}$$

The reduced ‘four-wave’ Hamiltonian is

$$\mathcal{H} = \int \omega_0 b_0^* b_0 dk_0 + \frac{1}{2} \int \tilde{V}_{0,1,2,3}^{(2)} b_0^* b_1^* b_2 b_3 \delta_{0+1-2-3} d\mathbf{k}_{0123} \quad (\text{A } 6)$$

and the corresponding *four-wave reduced equation* reads

$$i \frac{\partial b_0}{\partial t} = \frac{\delta \mathcal{H}}{\delta b_0^*} = \omega_0 b_0 + \int \tilde{V}_{0,1,2,3}^{(2)} b_1^* b_2 b_3 \delta_{0+1-2-3} d\mathbf{k}_{123}. \quad (\text{A } 7)$$

The coefficients  $\tilde{V}^{(2)}$  can be expressed in explicit form as

$$\tilde{V}_{0,1,2,3}^{(2)} = \frac{1}{4} [Z_{0,1,2,3}^{(2)} + Z_{1,0,2,3}^{(2)} + Z_{2,3,0,1}^{(2)} + Z_{3,2,0,1}^{(2)}] + V_{0,1,2,3}^{(2)} + \mathcal{A}_{0,1,2,3} \quad (\text{A } 8)$$

and contain ‘natural’ symmetry conditions in the explicit form

$$\tilde{V}_{0,1,2,3}^{(2)} = \tilde{V}_{1,0,2,3}^{(2)} = \tilde{V}_{0,1,3,2}^{(2)} = \tilde{V}_{2,3,0,1}^{(2)}.$$

The function  $\mathcal{A}_{0,1,2,3}$  in (A 8) is an arbitrary one, satisfies the ‘natural symmetry’ conditions and is equal to zero at the resonant curve

$$\omega_0 + \omega_1 - \omega_2 - \omega_3 = 0.$$

## Appendix B

The elevation of the free surface for the short-crested waves (2.10) is

$$\begin{aligned}
 \eta &= \frac{2M(\kappa)c_0}{\pi} \cos(X) \cos(Y) + \frac{2c_0^2}{\pi} \{ M(\bar{\kappa} - \kappa) A^{(2)}(\kappa - \bar{\kappa}, \bar{\kappa}, \kappa) \cos(2Y) \\
 &\quad + M(2\kappa) [A^{(1)}(2\kappa, \kappa, \kappa) + A^{(3)}(-2\kappa, \kappa, \kappa)] \cos(2X) \cos(2Y) \} \\
 &\quad + \frac{2c_0^3}{\pi} \{ M(3\kappa) [B^{(1)}(3\kappa, \kappa, \kappa, \kappa) + B^{(4)}(-3\kappa, \kappa, \kappa, \kappa)] \cos(3X) \cos(3Y) \\
 &\quad + 3M(2\kappa + \bar{\kappa}) [B^{(1)}(2\kappa + \bar{\kappa}, \kappa, \kappa, \bar{\kappa}) + B^{(4)}(-2\kappa - \bar{\kappa}, \kappa, \kappa, \bar{\kappa})] \cos(3X) \cos(Y) \\
 &\quad + M(2\kappa - \bar{\kappa}) [B^{(2)}(2\bar{\kappa} - \kappa, \kappa, \bar{\kappa}, \bar{\kappa}) + B^{(3)}(\kappa - 2\bar{\kappa}, \bar{\kappa}, \bar{\kappa}, \kappa)] \cos(X) \cos(3Y) \\
 &\quad + M(\kappa) [B^{(2)}(\kappa, \kappa, \kappa, \kappa) + 2B^{(2)}(\kappa, \bar{\kappa}, \kappa, \bar{\kappa}) + B^{(3)}(-\kappa, \kappa, \kappa, \kappa) \\
 &\quad + 2B^{(3)}(-\kappa, \kappa, \bar{\kappa}, \bar{\kappa})] \cos(X) \cos(Y) \} + O(\varepsilon^4).
 \end{aligned} \quad (\text{B } 1)$$

Here  $X, Y$  are determined by (2.7) and expressions for functions  $B^{(1)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ ,  $B^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ ,  $B^{(3)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ ,  $B^{(4)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  and  $M(\mathbf{k})$  are given in Appendix A (see also Krasitskii 1994 for finite-depth gravity-capillary waves).

## Appendix C

Nondiagonal terms in  $\mathbf{A}$  are proportional to  $\varepsilon^2$ :

$$\begin{aligned} N_n &= 2\tilde{V}^{(2)}(\bar{\kappa} + \kappa - \mathbf{h} - n(\bar{\kappa} - \kappa), \bar{\kappa}, \kappa, \bar{\kappa} + \kappa - \mathbf{h} - (n-1)(\bar{\kappa} - \kappa)) |c_0|^2, \\ Q_n &= \tilde{V}^{(2)}(\bar{\kappa} + \kappa - \mathbf{h} - n(\bar{\kappa} - \kappa), \mathbf{h} + (n-1)(\bar{\kappa} - \kappa), \kappa, \kappa) |c_0|^2, \\ T_n &= 2\tilde{V}^{(2)}(\mathbf{h} + n(\bar{\kappa} - \kappa), \kappa, \bar{\kappa}, \mathbf{h} + (n-1)(\bar{\kappa} - \kappa)) |c_0|^2, \\ S_n &= \tilde{V}^{(2)}(\bar{\kappa} + \kappa - \mathbf{h} - (n-1)(\bar{\kappa} - \kappa), \mathbf{h} + n(\bar{\kappa} - \kappa), \bar{\kappa}, \bar{\kappa}) |c_0|^2, \\ C_n &= 2\tilde{V}^{(2)}(\mathbf{h} + n(\bar{\kappa} - \kappa), \bar{\kappa} + \kappa - \mathbf{h} - n(\bar{\kappa} - \kappa), \bar{\kappa}, \kappa) |c_0|^2. \end{aligned}$$

Diagonal terms :

$$\begin{aligned} D_n^+ &= -\Omega_0 + \omega(\mathbf{h} + n(\bar{\kappa} - \kappa)) \\ &\quad + 2[\tilde{V}^{(2)}(\mathbf{h} + n(\bar{\kappa} - \kappa), \bar{\kappa}, \mathbf{h} + n(\bar{\kappa} - \kappa), \bar{\kappa}) \\ &\quad + \tilde{V}^{(2)}(\mathbf{h} + n(\bar{\kappa} - \kappa), \kappa, \mathbf{h} + n(\bar{\kappa} - \kappa), \kappa)] |c_0|^2, \end{aligned}$$

$$\begin{aligned} D_n^- &= \Omega_0 - \omega(\bar{\kappa} + \kappa - \mathbf{h} - n(\bar{\kappa} - \kappa)) \\ &\quad - 2[\tilde{V}^{(2)}(\bar{\kappa} + \kappa - \mathbf{h} - n(\bar{\kappa} - \kappa), \bar{\kappa}, \bar{\kappa} + \kappa - \mathbf{h} - n(\bar{\kappa} - \kappa), \bar{\kappa}) \\ &\quad + \tilde{V}^{(2)}(\bar{\kappa} + \kappa - \mathbf{h} - n(\bar{\kappa} - \kappa), \kappa, \bar{\kappa} + \kappa - \mathbf{h} - n(\bar{\kappa} - \kappa), \kappa)] |c_0|^2 \end{aligned}$$

have the order of differences of the frequencies of the basic wave and perturbation waves.

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