A new unified theory for nonlinear steady travelling waves in constant, but arbitrary, depth

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ABSTRACT

A non-linear coupled-mode system of equations on the horizontal plane is derived with the aid of Luke's (1967) variational principle, which models the evolution of nonlinear water waves in intermediate depth and over a general bathymetry. The vertical structure of the wave field is exactly represented by means of a local-mode series expansion of the wave potential, Athanassoulis & Belibassakis (2002). This series contains the usual propagating and evanescent modes, plus two additional modes, the free-surface mode and the sloping-bottom mode, enabling the consistent treatment of the non-vertical end-conditions at the free-surface and the bottom boundaries. The coupled-mode system fully accounts for the effects of non-linearity and dispersion. The main features of the present approach are the following: (i) various standard models of water-wave propagation are recovered by appropriate simplifications of the coupled-mode system, and (ii) a small number of modes are enough for a precise numerical solution, provided that the two new modes (the free-surface and the sloping-bottom ones) are included in the local-mode series. In the present work, the coupled-mode system is applied to the numerical derivation and investigation of families of steady travelling wave solutions in constant depth regions, corresponding to various water depths, ranging from intermediate to shallow wave conditions.

1 INTRODUCTION

The nonlinear water-wave problem is a difficult and interesting free-boundary problem, for which a broad class of mathematical models and approximation techniques have been developed. An important feature of this problem is that propagation phenomena take place in horizontal directions, and non-local couplings (wave-wave and seabed-wave) exist through the vertical structure of the flow field. Various equivalent reformulations of the fully nonlinear-nonlocal water-wave problem have been obtained, as e.g., by means of Hamilton's principle and the Dirichlet to Neumann (DtN) map, Craig & Sulem (1993), and by means of Lagrange equations of fluid dynamics and analyticity of the wave potential in the liquid domain, Craig (1985), Wu (1999) and others; see also Groves & Toland (1997).

In the present work, we consider the problem of non-linear gravity waves propagating over a general bathymetry. An essential feature of this problem is that the wave field is not spatially periodic. Extra difficulties are introduced by the fact that no asymptotic assumptions concerning the free-surface and bottom slope are made. Using Luke's (1967) variational principle, in conjunction with an enhanced local-mode series expansion of the wave potential, Athanassoulis & Belibassakis (2002), we first present a new non-linear coupled-mode system of equations on the horizontal plane modelling the evolution of nonlinear water waves in intermediate depth and over a general bathymetry. Then, the above coupled-mode system is applied to the numerical investigation of families of steady travelling wave solutions in constant depth regions, corresponding to a wide range of water depths, ranging from intermediate to shallow wave conditions. The derivation of the latter solutions is important for comparison with known theories and validation of the present model, as well as for the consistent initialization of the coupled-mode system in the case of time evolution problems in non-homogeneous environments.

2 VARIATIONAL FORMULATION

We restrict ourselves to the two-dimensional problem corresponding to normally incident waves. However, all the analysis presented in this work can be generalised to three spatial dimensions, i.e. the two horizontal dimensions associated with the propagation space and the vertical (cross space) dimension. The liquid domain is a generally-shaped (non-uniform) strip D, bounded below by the seabed $z=-h(x)$, and above by the free surface $z=\eta(x,t)$. The function $h(x)$ represents the local depth, measured from the mean water level. The functions $h(x)$ and $\eta(x,t)$ are assumed bounded and smooth functions of x. Moreover, the function $\eta(x,t)$ is continuously dependent on time t, ranging over the half-line $(t\geq 0)$. These functions satisfy the inequality $-h(x)<\eta(x,t)$, ensuring the connectedness of D.

A main feature of the water-wave problem is that the propagation space does not coincide with the physical space. While the latter is the whole liquid domain (an irregularly shaped horizontal strip), the former is only the horizontal direction(s). This fact, leads to the reformulation of the propagation problem as a non-local wave equation in the propagation (horizontal) space. Under the assumptions of incompressibility and irrotationality, the problem of evolution of water waves, propagating over a variable bathymetry region, can be reformulated as a variational equation by means of Luke's (1967) variational principle. According to this formulation, the admissible fields are free of essential conditions, except, for smoothness and completeness (compatibility) prerequisites. Luke's functional, modelling the homogeneous, nonlinear water-wave problem, is

$$F[\Phi,\eta]=\int_{t_1}^{t_2}\int_{x_1}^{x_2}dxdt\int_{z=-h(x)}^{z=\eta(x,t)}\left\{\frac{\partial\Phi}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right\}+gz\right\}dz \quad , \tag{1}$$

where x is the horizontal and z is the vertical (positive upwards) co-ordinates, $\Phi=\Phi(x,z,t)$ is the velocity potential, and $\eta=\eta(x,t)$ is the free surface elevation. In this case, the Lagrangian density is just the pressure. The nonlinear water-wave problem is then expressed by the variational equation

$$\delta F[\Phi,\eta]=0\,, \tag{2}$$

where the first variation of $F[\Phi,\eta]$ can be obtained as the sum of its partial variations with respect to the fields $\Phi=\Phi(x,z,t)$ and $\eta=\eta(x,t)$, i.e. $\delta F[\Phi,\eta]=\delta_\Phi F[\Phi,\eta]+\delta_\eta F[\Phi,\eta]$. On the basis of the above it is directly seen that the condition of stationarity of functional $F[\Phi,\eta]$ is equivalent to the non-homogeneous, nonlinear water-wave problem; see, e.g., Witham (1974). More precisely, the variational equation $\delta_\Phi F=0$ models the *water-wave kinematics*,

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}\;=\;0,\quad x_1<x<x_2,\quad -h(x)<z<\eta(x;t) \quad , \tag{3a}$$

$$\frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \Phi}{\partial z} + \frac{\partial \eta}{\partial t} = 0, \quad z = \eta(x;t), \tag{3b}$$

$$\frac{\partial \Phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial z} = 0, \qquad z = -h(x), \tag{3c}$$

while the variational equation $\delta_\eta F = 0$ models the *water-wave dynamics* (Bernoulli's integral)

$$\frac{1}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \Phi}{\partial z}\right)^2 + \frac{\partial \Phi}{\partial t} + g\eta = 0, \quad x_1 < x < x_2,\ z = \eta(x;t). \tag{3d}$$

3 LOCAL-MODE SERIES EXPANSION

In this section a new *local-mode series expansion* of the wave potential $\Phi(x,z,t)$ in variable bathymetry regions, derived by Athanasoulis & Belibassakis (2002), is presented. This expansion has the general form

$$\Phi(x,z,t) = \sum_{n=-2}^{\infty} \varphi_n(x,t)\, Z_n\big(z, h(x), \eta(x,t)\big) \ , \tag{4}$$

where

$$Z_{-2}(z,h,\eta) = \frac{\mu_0 h_0 + 1}{2(\eta+h)h_0}(z+h)^2 - \frac{\mu_0 h_0 + 1}{2h_0}(\eta + h) + 1, \tag{5}$$

represents the vertical structure of the term $\varphi_{-2}Z_{-2}$, which is called the *free-surface mode*,

$$Z_{-1}(z,h,\eta) = \frac{\mu_0 h_0 - 1}{2h_0(\eta+h)}(z+h)^2 + \frac{1}{h_0}(z+h) + \frac{2h_0 - (\eta+h)(\mu_0 h_0 + 1)}{2h_0}, \tag{6}$$

represents the vertical structure of the term $\varphi_{-1}Z_{-1}$, called the *sloping-bottom mode,* and

$$Z_0(z,h,\eta) = \frac{\cosh\left[k_0(z+h)\right]}{\cosh\left[k_0(\eta+h)\right]}, \quad Z_n(z,h,\eta) = \frac{\cos\left[k_n(z+h)\right]}{\cos\left[k_n(\eta+h)\right]}, \quad n = 1,2,3,... \tag{7}$$

are corresponding functions associated with the rest of the terms, which are called the *propagating* ($\varphi_0 Z_0$) and the *evanescent* ($\varphi_n Z_n$, $n = 1,2,...$) modes. The (numerical) parameters $\mu_0,\ h_0 > 0$ are positive constants, not subjected to any a-priori restrictions. Moreover, the z-independent quantities $k_n = k_n(h,\eta),\ n = 0,1,2...,$ appearing in Eqs. (7), are defined as the positive roots of the transcendental equations,

$$\mu_0 - k_0\, tanh\left[k_0(h+\eta)\right] = 0, \qquad \mu_0 + k_n \tan\left[k_n(h+\eta)\right] = 0. \tag{8}$$

A detailed proof about the above expansion can be found in Belibassakis & Athanassoulis (2005). The usefulness of the above local-mode representation is that, substituted in the variational equation (2), it leads to a non-linear coupled-mode system of differential equations on the horizontal plane, with respect to unknown modal amplitudes $\varphi_n(x,t)$ and the unknown elevation $\eta(x,t)$. The coupled-mode system greatly facilitates the numerical solution of the present problem and will be presented in the next section.

A similar modal-type series expansion has been earlier introduced by Nadaoka *et al* (1997) for the development of a fully dispersive, weakly nonlinear, multiterm-coupling model for water waves, with application to slowly varying bottom topography. In that work, the vertical modes have been selected to have the form: $\cosh\left(k_n\left(z+h\right)\right)\cosh^{-1}\left(k_n h\right)$, where the parameters $k_n > 0$ are independent from the upper surface elevation $\eta\left(x,t\right)$. The major part of the present set of vertical modes $\left\{Z_n\left(z,h,\eta\right),\, n=0,1,2,...\right\}$ is obtained by solving a Sturm-Liouville problem, formulated at the local vertical interval $-h\left(x\right)<z<\eta\left(x,t\right)$, ensuring L_2 – completeness. This set contains both hyperbolic and trigonometric functions, dependent both on the local depth $h\left(x\right)$ and the (instantaneous) free surface elevation $\eta\left(x,t\right)$. However, the boundary conditions satisfied by these local vertical eigenfunctions are not compatible with the boundary conditions of the problem at the bottom surface, if the bottom is not horizontal or mildly sloping, and at the upper (free) surface. In order to overcome the mild-slope bottom approximation and to consistently satisfy the free-surface boundary conditions, the present set has been enhanced by including the two additional modes $\left\{Z_{-2}\left(z,h,\eta\right),\, Z_{-1}\left(z,h,\eta\right)\right\}$ with unknown amplitudes $\left\{\varphi_{-2}\left(x,t\right),\, \varphi_{-1}\left(x,t\right)\right\}$. The latter are the additional degrees of freedom required for the consistent satisfaction of the free-surface and the sloping-bottom boundary conditions, respectively. The idea of the sloping-bottom mode has been presented by Athanassoulis & Belibassakis (1999) for the propagation of linearised waves in general bathymetry regions. The latter work has been extended to second-order Stokes waves (in the frequency domain) by Belibassakis & Athanassoulis (2002), where also the necessity of a free-surface additional mode has been discussed for the satisfaction of the (second-order) free-surface boundary condition.

3 THE COUPLED MODE SYSTEM OF EQUATIONS (CMS)

The series expansion (4) permits us to obtain corresponding expansion of the variation $\delta\Phi$ of the wave potential, in terms of the variations of the modal amplitudes $\delta\varphi_n$ and the free surface elevation $\delta\eta$. Then, it can be shown that the examined hydrodynamic problem in the variable bathymetry region reduces to the following *nonlinear Coupled-Mode System* (see, e.g., Athanassoulis & Belibassakis, 2002, Belibassakis & Athanassoulis 2005) with respect to the mode amplitudes $\varphi_n\left(x,t\right)$ and the free-surface elevation $\eta\left(x,t\right)$:

$$\frac{\partial\eta}{\partial t} + \sum_{n=-2}^{\infty}\left(A_{mn}\left(\eta\right)\frac{\partial^2\varphi_n}{\partial x^2}+B_{mn}\left(\eta\right)\frac{\partial\varphi_n}{\partial x}+C_{mn}\left(\eta\right)\varphi_n\right) = 0, \quad m=-2,-1,0,1,2...\,, \tag{9a}$$

$$g\eta + \sum_{n=-2}^{\infty}\left(\frac{\partial\varphi_n}{\partial t}+\left[W_n\right]_{z=\eta}\varphi_n\frac{\partial\eta}{\partial t}\right) - \sum_{\ell=-2}^{\infty}\sum_{n=-2}^{\infty}\left(a^{(0,2)}_{\ell n}\left(\eta\right)\varphi_\ell\frac{\partial^2\varphi_n}{\partial x^2}+a^{(1,1)}_{\ell n}\left(\eta\right)\frac{\partial\varphi_\ell}{\partial x}\frac{\partial\varphi_n}{\partial x}+b_{\ell n}\left(\eta\right)\varphi_\ell\frac{\partial\varphi_n}{\partial x}+c_{\ell n}\left(\eta\right)\varphi_\ell\varphi_n\right)=0\,. \tag{9b}$$

where $W_n\left(z,h,\eta\right)=\dfrac{\partial Z_n\left(z,h,\eta\right)}{\partial\eta}$. The matrix-coefficients $A_{mn}(\eta),\, B_{mn}(\eta), C_{mn}(\eta)$, and $a^{(0,2)}_{mn}\left(\eta\right),\ a^{(1,1)}_{mn}\left(\eta\right)$, $b_{mn}\left(\eta\right),\, c_{mn}\left(\eta\right)$, appearing in the above equations are expressed in terms of the local vertical modes $\left\{Z_n\right\}_{n=-2,-1,0,1,..}$ and their derivatives, and can be found in Athanasoulis & Belibassakis (2002) and in Belibassakis Athanassoulis (2005). Also, in the latter works numerical applications of the CMS (9) to various shoaling environments are presented, clearly demonstrating the rapid decay of the modal amplitudes and the fast convergence of the modal series (4).

The non-linear CMS, Eqs. (9), has been obtained without any assumptions concerning the vertical structure of the wave potential. Thus, this system, being equivalent with the complete formulation, fully accounts for wave non-linearity and dispersion. Detailed results concerning the dispersion characteristics of the linearised system are presented in Belibassakis &

Athanassoulis (2005, Sec.7), where it is shown that retaining the first few terms in the series (up to 5 modes) is sufficient for numerical convergence to the exact result, for an extended range of wave frequencies, ranging from shallow to deep water-wave conditions.

Moreover, a distinctive feature of the present CMS is that no simplifications have been introduced for its derivation. Thus, in principle, various simplified models can be recovered as appropriate limiting forms of Eqs. (9). For example, keeping only the propagating mode $Z_0(z)$ in the expansion (4) and linearising the coupled-mode equations, the classical *mild-slope* model is obtained, see, e.g., Dingemans (1997). If the evanescent modes $Z_n(z)$, $n=1,2,\ldots$, are also retained, an *extended mild-slope* model is obtained, see, e.g., Massel (1993). If we keep only the quadratic vertical mode $Z_{-2}(z)$, defined by Eq. (5), in the vertical expansion of the wave potential and retain up to second-order terms in the present CMS, a *Boussinesq-type* model is obtained, see, e.g., Liu (1995). On the other hand, if we keep in the local-mode series only the propagating mode $Z_0(z)=\cosh\left(k_0(z+h)\right)\cosh^{-1}\left(k_0(\eta+h)\right)$ and again retain up to second-order terms, a two-equation, nonlinear, mild-slope model is derived, quite similar as the time-dependent, nonlinear, mild-slope equation by Beji & Nadaoka (1997).

A convenient *reformulation of the CMS* can be obtained by subtracting by parts Eqs. (10a, for $m=-1,0,1,\ldots$) from Eq. (10a, for $m=-2$). Thus, it is possible to eliminate the $\partial\eta/\partial t$ term from the left-hand side of the former equations. Moreover, by introducing the function $\varphi(x,t)=\sum_{n=-2}\varphi_n(x,t)$, which equals to the wave potential on the free-surface (since all vertical functions are normalized, i.e. $Z_n(x,z=\eta)=1$)

$$\Phi(x,z=\eta;t)=\sum_{n=-2}\varphi_n(x;t)Z_n(x,z=\eta)=\sum_{n=-2}\varphi_n(x;t)=\varphi(x;t),$$

the CMS (10) is equivalently reformulated as a set of two evolution equations on $\{\varphi,\eta\}$:

$$-\frac{\partial\eta}{\partial t}=\frac{\partial}{\partial x}\left((\eta+h)\frac{\partial\varphi}{\partial x}\right)+\sum_{n=-2}^{\infty}\left(\hat{A}_{-2n}(\eta)\frac{\partial^2\varphi_n}{\partial x^2}+\hat{B}_{-2n}(\eta)\frac{\partial\varphi_n}{\partial x}+\hat{C}_{-2n}(\eta)\varphi_n\right)\quad,\tag{10a}$$

$$-\frac{\partial\varphi}{\partial t}-g\eta=N\quad,\tag{10b}$$

where N is defined as follows:

$$N=\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2+\sum_{n=-2}^{\infty}\left(\left[W_n\right]_{z=\eta}\varphi_n\frac{\partial\eta}{\partial t}\right)-\sum_{\ell=-2}^{\infty}\sum_{n=-2}^{\infty}\left(a_{\ell\mathrm{n}}^{(0,2)}(\eta)\varphi_\ell\frac{\partial^2\varphi_n}{\partial x^2}+b_{\ell n}(\eta)\varphi_\ell\frac{\partial\varphi_n}{\partial x}+c_{\ell n}(\eta)\varphi_\ell\varphi_n\right).\tag{10c}$$

Furthermore, the two-equation system (10) is subjected to the constraints imposed by the equations:

$$\sum_{n=-2}^{\infty}\left(A_{-2n}(\eta)-A_{mn}(\eta)\right)\frac{\partial^2\varphi_n}{\partial x^2}+\left(B_{-2n}(\eta)-B_{mn}(\eta)\right)\frac{\partial\varphi_n}{\partial x}+\left(C_{-2n}(\eta)-C_{mn}(\eta)\right)\varphi_n=0,\quad m=-1,0,1,2..,\tag{11a}$$

$$\sum_{n=-2}\varphi_n(x;t)=\varphi(x;t),\tag{11b}$$

which are shown to be equivalent to the kinematical subproblem materializing the DtN map, associated with the calculation of the wave potential in the whole domain *D*, given the instantaneous values of the free-surface potential and the free-surface elevation, and satisfying the bottom boundary condition. In Eq. (10a), the coefficients $\hat{A}_{mn}(\eta)$, $\hat{B}_{mn}(\eta)$, $\hat{C}_{mn}(\eta)$ are defined as follows

$$\hat{A}_{mn}(\eta)=\int_{z=-h(x)}^{z=\eta(x;t)}\left(-1+Z_n(z;h,\eta)Z_m(z;h,\eta)\right)dz\,,\quad \hat{B}_{mn}(\eta)=2\left\langle\frac{\partial Z_n}{\partial x},Z_m\right\rangle+\frac{\partial h}{\partial x}\left[-1+Z_nZ_m\right]_{z=-h}\,,\quad (12a,b)$$

$$\hat{C}_{mn}(\eta)=\langle\Delta Z_n,Z_m\rangle+\left[\left(\frac{\partial h}{\partial x}\frac{\partial Z_n}{\partial x}+\frac{\partial Z_n}{\partial z}\right)Z_m\right]_{z=-h}+\left[\left(\frac{\partial \eta}{\partial x}\frac{\partial Z_n}{\partial x}-\frac{\partial Z_n}{\partial z}\right)Z_m\right]_{z=\eta}\,,\quad (12c)$$

where $\Delta Z_n=\partial^2 Z_n/\partial x^2+\partial^2 Z_n/\partial z^2$. Moreover, the matrix-coefficients $a_{mn}^{(0,2)}(\eta),\ b_{mn}(\eta),\ c_{mn}(\eta)$, in Eq. (10c), are also dependent on the free-surface elevation, and are defined as follows:

$$a_{\ell n}^{(0,2)}(\eta)=\langle Z_n,W_\ell\rangle=\int_{z=-h(x)}^{z=\eta(x;t)}Z_n(z;h,\eta)W_\ell(z;h,\eta)dz\,,\quad (13a)$$

$$b_{\ell n}(\eta)=2\left\langle\frac{\partial Z_n}{\partial x},W_\ell\right\rangle+\frac{\partial h}{\partial x}\left[Z_nW_\ell\right]_{z=-h}-\left[\frac{\partial Z_n}{\partial x}\right]_{z=\eta}\,,\quad c_{\ell n}(\eta)=\langle\Delta Z_n,W_\ell\rangle+\left[\left(\frac{\partial h}{\partial x}\frac{\partial Z_n}{\partial x}+\frac{\partial Z_n}{\partial z}\right)W_\ell\right]_{z=-h}\,.\quad (13c,d)$$

4 DERIVATION OF STEADY TRAVELLING SOLUTIONS IN CONSTANT DEPTH

Looking for steady travelling wave solutions in constant but arbitrary depth h, characterised by the (unknown) wave celerity c, we use the transformation

$$\varphi(x;t)=\varphi(\xi)\,,\quad \varphi_n(x;t)=\varphi_n(\xi)\,,\quad \eta(x,t)=\eta(\xi),\quad \xi=x-ct \quad \text{(and thus, } \partial/\partial t=-c\partial/\partial x\text{)}.$$

Consequently, the CMS (10) is put in the following equivalent (time independent) form:

$$c\,L_1(\mathbf{u})+L_0(\mathbf{u})=N(\mathbf{u})\,,\quad (14)$$

where $\mathbf{u}=\begin{cases}\eta(x)\\ \varphi(x)\end{cases}$, $L_1(\mathbf{u})=\begin{cases}\partial\eta/\partial x\\ \partial\varphi/\partial x\end{cases}$, $L_0(\mathbf{u})=\begin{cases}0\\ -g\eta\end{cases}$, and the components of the nonlinear operator $N(\mathbf{u})$ are defined by the right-hand side of Eqs. (10a) and (10b), with simplified coefficients due to the fact that in the present (flat bottom) case $dh/dx=0$, and thus, the sloping-bottom mode vanishes, $\varphi_{-1}=0$.

Given the length of the periodic cell (the wavelength) λ and the depth h (or the flow rate Q under the waves), steady travelling wave solutions of Eq. (14) are numerically constructed by calculating the free-surface elevation $\eta(x)$, the modes $\varphi_n(x)$, $n=-2,0,1,2...$, the wave potential $\Phi(x,z)$, and the wave speed c. We remark here that a nontrivial solution should contain (at least) one crest. In order to obtain solution of mode type I (one crest and one trough per periodic cell) we need to specify the location of the crest. Without loss of generality, we suppose that

$$\frac{d\eta(x=x_*)}{dx}=0,\quad \text{at a given point } x_* \text{ within the periodic cell.}\quad (15)$$

The above condition introduces an additional algebraic constraint, that can be considered to be equivalent to a "nonlinear dispersion" relation. The system (12) is iteratively solved by: (i) guessing initial $c_0,\mathbf{u}_0$ (e.g., from a linear model), (ii) calculating $\mathbf{u}_{k+1}=\left(c_kL_1+L_0\right)^{-1}N(\mathbf{u}_k)$ from Eq. (14), and c_{k+1} from Eq. (15), and (iii) iterating until convergence: $\|\mathbf{u}_{k+1}-\mathbf{u}_k\|+|c_{k+1}-c_k|<$ tolerance.

Numerical results are presented in Figs. 1, 2, for two different cases corresponding to shallow water (*λ/h*=20, Fig.1), and intermediate (*λ/h*=5, Fig.2) water-depth conditions, respectively, and relatively strong wave nonlinearity. The initial guess, based on the linearised (time-harmonic) solution, is plotted in the left part of these figures, as well as the consecutive iterations. We observe that in all cases the rate of convergence is very fast and 8-10 iterations suffice for convergence. In the right part of Fig. 1 the final (convergent) solution from the present model is compared with results from nonlinear cnoidal theory (Fenton, 1990), and in Fig. 2 with 5th-order Stokes theory (Fenton, 1985), and the agreement is found to be excellent. Moreover, in Fig. 3 corresponding results are presented for a case characterised by balanced nonlinearity and dispersion, in the regime of Ursell number $U = (H/h)(\lambda/h)^2 \approx 8\pi^2$ (where H is the waveheight), where Boussinesq equation(s) are applicable. In this case, the present CMS provides reasonable results, in agreement with cnoidal theory, while the Stokes expansion fails.

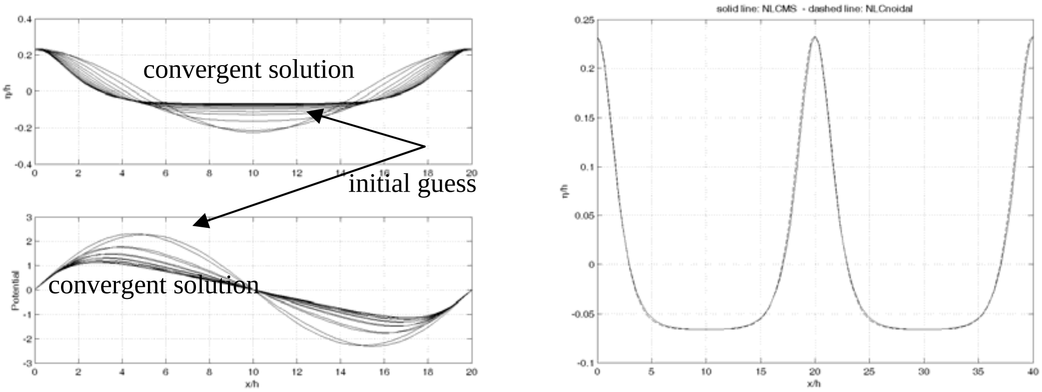

Fig 1. Steady travelling solution in the long-wave regime, characterized by moderate nonlinearity (λ/h=20, H/h=0.3, c/c$_0$=1.054). Right: Comparison of present CMS (solid line) with nonlinear cnoidal theory (Fenton 1990), shown by dashed line.

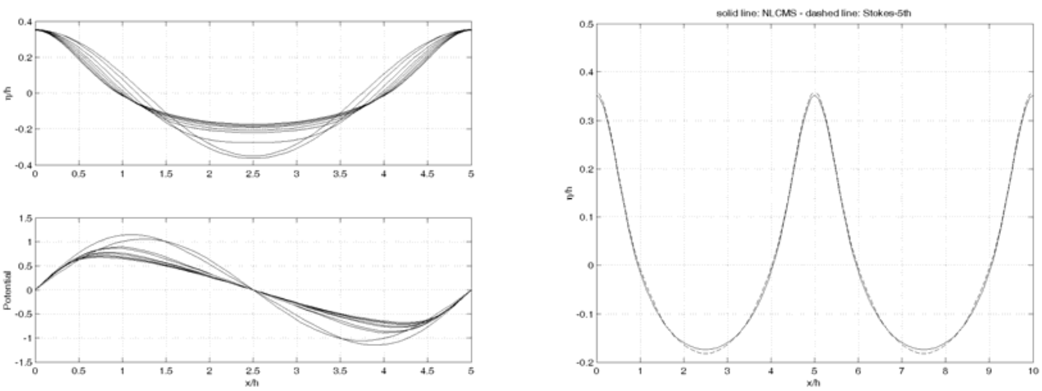

Fig 2. Steady travelling solution in the intermediate-wave regime, characterized by strong nonlinearity (λ/h=5, H/h=0.52, c/c$_0$=1.089). Right: Comparison of present CMS (solid line) with 5th-order Stokes theory (Fenton, 1985), shown by dashed line.

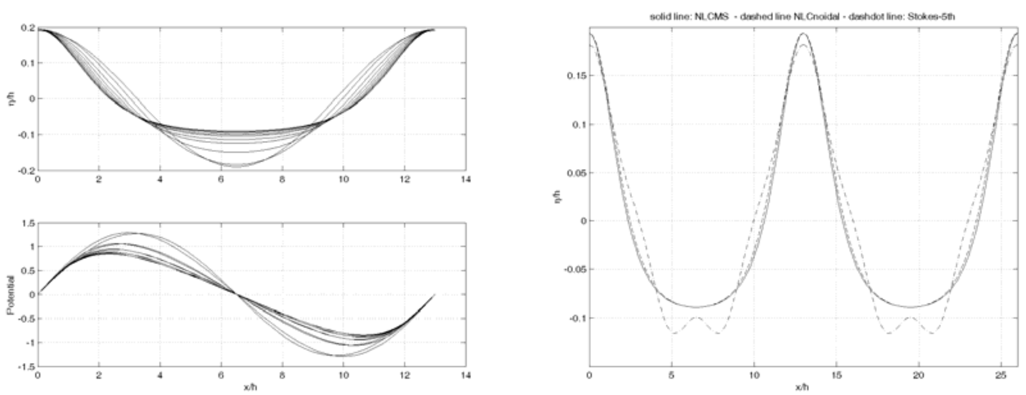

Fig 3. Steady travelling solution in the Boussinesq regime (U=75), characterized by balanced non-linearity and dispersion (λ/h=13, H/h=0.3, c/c$_0$=1.033). Right: Comparison of present CMS (solid line) with 5th-order Stokes theory (Fenton, 1985) shown by dash-dotted line, and nonlinear cnoidal theory (Fenton 1990) shown by dashed line.

5 CONCLUSIONS

The present non-linear CMS has been obtained without any assumptions concerning the nonlineariry and the vertical structure of the wave potential, being thus, equivalent with the complete water-wave formulation. The theoretical value and practical effectiveness of the present model, except of its universal character, is that a small number of modes is found to be enough for numerical convergence, even in cases of very steep free-surface elevation and for arbitrary depth. Extensive numerical evidence suggests that the rate of decay of the mode amplitudes is very fast and thus, truncation of the modal series to its first few terms (up to 5 modes) is found to be sufficient for an accurate numerical solution. The present system is applied to the numerical investigation of steady travelling waves over horizontal bottom, corresponding to various water depths, and its results at intermediate and shallow water depth, respectively, are found to be consistent with Stokes 5^{th}-order and nonlinear cnoidal wave theory.

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