GEOPHYSICAL RESEARCH LETTERS, VOL. 24, NO. 7, PAGES 823-826, APRIL 1, 1997

Similarity of waves in dispersive media

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Abstract. In a previous letter, *D.A.Gurnett* [1995] discussed the remarkable fact that atmospheric whistlers propagate under the same emission cone relative to the earth's magnetic field as bow waves do with respect to the ship's velocity – namely under $19^o 28'$. The present contribution is a comment on Gurnett's letter and considers a prerequisite for the occurence of the similarity of both phenomena: the independence of the wave features of parameters such as the ship's speed or the earth's magnetic field. It is found that media with dispersion relations of the form $\omega \sim k^\alpha$ show such independence; a more general solution is also given.

1. Introduction

D.A. Gurnett discussed, in a previous letter, a nice example of two phenomena which, at first glance, seem to be quite different but are commonly described in the framework of linear waves propagating through a dispersive medium. They are atmospheric whistlers and ship waves which both propagate under an emisson cone of $arcsin\,\frac{1}{3}$ with respect to the earth's magnetic field respectively to the ship's velocity. More remarkably, their emission cones do not depend on the special parameters such as the magnetic field strength or the ship speed.

Both phenomena are linear waves and their characteristic features originate from the dispersion relations. We shall therefore concentrate on dispersion and assume the medium to be excited by a (moving) point source.

The lack of dependence of the emission cone on the medium parameters restricts the form of the dispersion relation; but this does not lead to simple statements if only the aperture angle is prescribed. Nevertheless, simple statements become possible if we demand that the wavepatterns keep spatially similar under a change of parameters. Spatial similarity is a sufficient (although not necessary) condition for equal emission cones and it leads in a natural way to the consideration of scaling properties of waves in dispersive media.

Historically, a famous example of scaling laws goes back to O. Reynolds and provides a similarity condition for the streaming of viscous fluids. Related concepts were used in magnetohydrodynamics or in mechanical similarity. We shall come back to general scaling later and start with an example remarkable in itself: the independence of the Kelvin wedge on the ship's speed.

Paper number 97GL00674.
0094-8534/97/97GL-00674$05.00

2. Independence of the source speed

The independence of the Kelvin wedge on the ship's speed is astonishing and contradictory to intuition which usually refers to a Mach cone whose aperture angle depends on the source speed. Is water just a very particular medium or is independence of the source- respective streaming speed quite common? To answer this, we must look closer at the general mathematical description of waves.

We assume that the wave can be represented as a superposition of plane waves satisfying the medium's dispersion:

$$\Phi(\mathbf{x},t) = \int d\mathbf{k}\, \tilde{\Phi}(\mathbf{k})\, e^{i\mathbf{k}\mathbf{x} - i\omega(\mathbf{k})t}\,.$$

If a norm pulse $\Phi(\mathbf{x},0) = (2\pi)^{dim}\,\delta(\mathbf{x})$ is applied, $\tilde{\Phi} \equiv 1$ and the response to the pulse is for $t > 0$:

$$\mathcal{G}(\mathbf{x},t) = \int d\mathbf{k}\, e^{i\mathbf{k}\mathbf{x} - i\omega(\mathbf{k})t}\,. \tag{1}$$

By linearity, the wave pattern of a moving source can be written as superposition of responses to pulses emitted along the trajectory $\mathbf{x_0}(t_0)$ of the source:

$$\begin{aligned} F(\mathbf{x},t) &= \int_{-\infty}^{t} dt_0\, \mathcal{G}\Big(\mathbf{x} - \mathbf{x}_0, t - t_0\Big) \\ &= \int_{-\infty}^{t} dt_0 \int d\mathbf{k}\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}_0) - i\omega(t-t_0)}\,. \end{aligned}$$

Let us assume that the source moves with constant velocity, $\mathbf{x}_0 = \mathbf{v}t_0$. For an observer fixed to the source frame of reference ($\mathbf{x}' = \mathbf{x} - \mathbf{v}t$), the wave pattern takes the form

$$\begin{aligned} F'(\mathbf{x}',t) &= \int d\mathbf{k} \int_{-\infty}^{t} dt_0\, e^{i\mathbf{k}(\mathbf{x}'+\mathbf{v}t-\mathbf{v}t_0) - i\omega(t-t_0)} \\ &= \int d\mathbf{k} \int_0^{\infty} d\tau\, e^{i\mathbf{k}\mathbf{x}' + i(\mathbf{k}\mathbf{v}-\omega)\tau}\,. \end{aligned} \tag{2}$$

We recognize that the expression (2) does not contain t, indicating that the wave pattern is *stationary*. Note that this is a consequence of the uniform source motion.

We shall now come to the question under which circumstances the wave field remains similar under a change of the source speed. More precisely, we ask if a change in $v = |\mathbf{v}|$ can be compensated by scaling all spatial and time coordinates. Let us denote the scaling by

$$\mathbf{v} \to \lambda\mathbf{v} \quad \text{and} \quad \mathbf{x} \to \mathbf{x}/\xi \quad \text{resp.} \quad \mathbf{k} \to \xi\mathbf{k}$$

and look what happens with equation (2). The first phase term $i\mathbf{k}\mathbf{x}'$ remains invariable due to the reciprocal scaling of $\mathbf{x}$- and $\mathbf{k}$-spaces. But the second phase term $(\mathbf{vk} - \omega)$ changes. However, if this change just multiplies the whole term by a positive constant β, this can be compensated by a new integration variable $\tau' = \tau/\beta$, contributing only a factor β to the wave amplitude and thereby conserving its similarity. So the condition for similar wavepatterns at different source speeds demands that the second phase term keeps proportional to itself when the source speed changes, implying that

$$\lambda\xi\,\omega(\mathbf{k}) = \omega(\xi\mathbf{k}) \tag{3}$$

for a given value of λ. This functional equation has the solution

$$\omega = k^{\alpha} f(\hat{\mathbf{k}}) \qquad \xi = \lambda^{1/(\alpha-1)} \tag{4}$$

where $k = |\mathbf{k}|$, $\hat{\mathbf{k}}$ is a unit vector in the direction of $\mathbf{k}$, α is an (almost) arbitrary power law index and $f(\hat{\mathbf{k}})$ is an arbitrary function of the direction of the wave vector. Thus, a scaling of the velocity by a factor λ can be compensated by the scaling of the space coordinates by a factor $\lambda^{1/(\alpha-1)}$. Note the singularity at $\alpha = 1$. Dispersionless wave patterns must depend on the source speed as the Mach cone does.

3. The group speed approach

We shall now build the connection to the group speed approach used by *Gurnett* [1995] and see that this concept leads to a simple geometrical classification of the wave behaviour. Let us return to the wave field created by a moving source, i.e. to Equation (2). For most values of $\mathbf{k}$ and τ, the integrand exhibits strong oscillations. Contributions to the integral come only from those parts where the oscillation vanishes, i.e. where the exponent's derivatives with respect to $\mathbf{k}$ and τ vanish:

$$\mathbf{x}' = \left(\frac{\partial\omega}{\partial\mathbf{k}} - \mathbf{v}\right)\tau \quad \text{and} \quad \omega = \mathbf{vk}\,. \tag{5}$$

This is the concept of *stationary phase approximation* introduced by Kelvin (*Thomson* [1910]). It demonstrates the

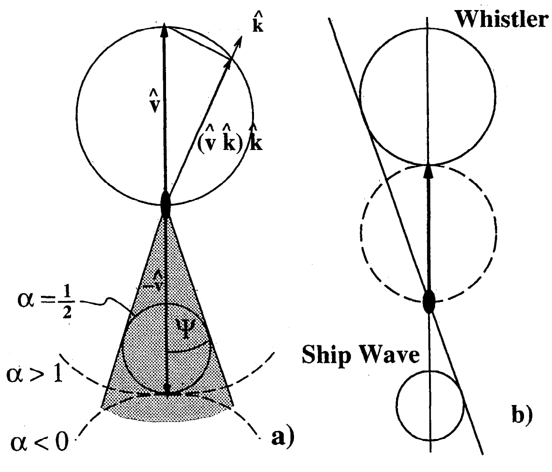

Figure 1. Construction of the emission cone in the stationary phase approximation (see text). Source frame of reference. a) General power law $\omega = k^{\alpha}$. The solid circle, drawn for $\alpha = \frac{1}{2}$, corresponds to the square bracket in Eq. 7. b) The equal emission cones of whistlers and ship waves.

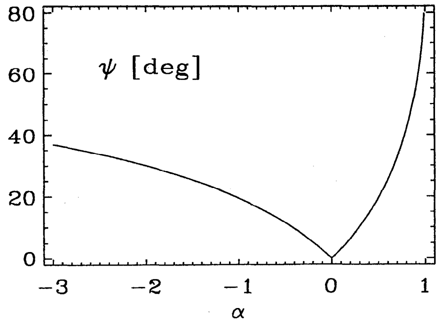

Figure 2. The aperture angle of the emission cone in dependence of the power law index.

physical meaning of the group speed as seen from an observer moving with the source: the vector

$$\mathbf{v}'_g = \frac{\partial\omega}{\partial\mathbf{k}} - \mathbf{v} \tag{6}$$

points to those positions for which a stationary phase contribution to the integral (2) exists. The range of $\mathbf{v}'_g$ is obtained from (6) by inserting all values of $\mathbf{k}$ which satisfy the second stationarity condition in Eq. (5). This is Gurnett's approach.

If a power law medium is isotropic, $\omega = k^{\alpha}$, and the pattern anisotropy is due to the movement of the source, Eq. (6) gives a simple geometrical construction of the allowed directions of $\mathbf{v}'_g$. In this case, we have

$$\mathbf{v}'_g = \frac{\partial\omega}{\partial k}\hat{\mathbf{k}}\Big|_{\omega=\mathbf{vk}} - \mathbf{v} = v[\alpha(\hat{\mathbf{v}}\cdot\hat{\mathbf{k}})\,\hat{\mathbf{k}} - \hat{\mathbf{v}}]\,. \tag{7}$$

The square brackets describe a circle of radius $|\alpha|/2$ with center at $-\hat{\mathbf{v}}(1-\alpha/2)$ (see Fig. 1a). For any $\mathbf{v}'_g$ (and hence $\mathbf{x}'$) within the shaded cone (Fig. 1a), contributions of the stationary phase approximation to the integral in Eq. (2) exist; outside they vanish. For $\alpha < 1$, the cone half angle is (see Fig. 2)

$$\Psi = \arcsin\left|\frac{\alpha}{2-\alpha}\right|\,.$$

For bow waves, we have $\alpha = \frac{1}{2}$ and $\Psi = 19.5^{o}$.

The graphical picture may also be used to explain the equal emission cones of ship waves and whistlers. Whistlers are excited by a lightning at a fixed position in the earth's atmosphere. Consequently, we have to consider the pulse response and the second condition in Eq. (5) does not apply. The whistler dispersion can be approximated in the form $\omega \propto k(\mathbf{k}\cdot\mathbf{B})$ (*Gurnett* [1995]) and the group speed is

$$\mathbf{v}_g = \frac{\partial\omega}{\partial\mathbf{k}} \propto kB\left\{(\hat{\mathbf{B}}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}} + \hat{\mathbf{B}}\right\}\,.$$

Due to the special dependence of the azimuthal coordinate, the group speed lies again on a circle, but shifted by $+\hat{\mathbf{B}}$. Fig. 1b shows how the different shifts of whistlers and ship waves lead to the same emission cone.

4. Simulated Examples

After having established the power law dispersions, Fig.3 gives some examples. The plots were obtained by numeri-

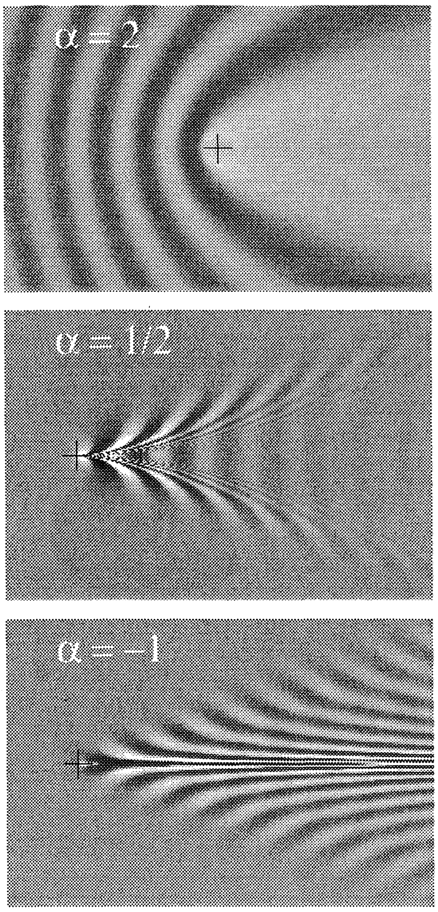

Figure 3. Examples of power law wave patterns. Sources (crosshair) moving to the left.

cal superposition of two-dimensional pulse responses (Eq.1) computed within the stationary phase approximation. A damping term $-\epsilon k$ was added to the exponent of the integral (Eq.1) which supports numerical convergence and filters out high spatial frequencies which could not be resolved by the printing device.

For power law media ($\omega \sim k^\alpha$) of arbitrary dimension, the stationary phase approximation contains terms of the form $\exp\left\{iCr(r/t)^{1/(\alpha-1)}\right\}$ with r the radius and t the time. The stationary points travel with $r \sim t^{1/\alpha}$.

Fig.3 shows the wavepatterns created by a moving source in power law media with $\alpha = 2, \frac{1}{2}$ and -1. These cases were chosen since they demonstrate the qualitative behaviour in the different regimes and since the integrals in the two-dimensional pulse responses (Eq. 1) are analytically solvable (see Appendix). This allows to control the validity of the stationary phase approximation used in Fig. 3. It turns out that the stationary phase approximation fits the analytical curves astonishingly well.

The case $\alpha = 2$ gives no emission cone and the medium dams up in front of the moving source. In nature, the dispersion $\omega \sim k^2$ is realized e.g. in the Schrödinger Equation or in bending waves in elastic rods and plates. The case $\alpha = \frac{1}{2}$ is the well-known water wave; an emission cone exists. The case $\alpha = -1$, although provided by an emission cone, shows a somewhat different behaviour since $r \sim t^{-1}$ (see above) and the group speed is opposite to the phase speed. As a consequence, the wave crests of the pulse response travel *inwards*; and so do the crests in vertical direction in Fig. 3, $\alpha = -1$. Note that by coincidence, the aperture angle of the emission cone is again $\Psi = 19^o 28'$.

5. General Formulation

In the previous sections we made a clear distinction between moving and resting sources. This is physically meaningful but somewhat artificial from the mathematical point of view. Let us assume now that the dispersion contains a set $\mathbf{p}$ of general parameters which may be inherent to the medium or due to the transformation to a moving frame of reference (as for the medium 'streaming water')

$$\omega = \omega(\mathbf{k}, \mathbf{p})\,.$$

The 'scaling property' requires that the wave pattern undergoes only a scaling transformation if the parameters are changed. Therefore, the dispersion relation must satisfy

$$\omega(\mathbf{k}, \mathbf{p}) = \eta\, \omega(\xi \mathbf{k}, \mathbf{p}') \tag{8}$$

for suitable, positive ξ and η. In other words: it must always be possible to compensate the effect of a parameter change by re-scaling the spatial and time coordinates as described in Sect. 2. A physically interesting solution of the scaling Equation (8) is

$$\omega^\gamma = A(\mathbf{p})\Psi_A\, k^\alpha + B(\mathbf{p})\Psi_B k^\beta \tag{9}$$

where Ψ_A and Ψ_B are arbitrary functions of the direction $\hat{\mathbf{k}}$, $\alpha \neq \beta$ and

$$\xi = \left[\frac{A(\mathbf{p})B(\mathbf{p}')}{A(\mathbf{p}')B(\mathbf{p})}\right]^{\frac{1}{\alpha-\beta}} \tag{10}$$

$$\eta = \left[\left(\frac{A(\mathbf{p})}{A(\mathbf{p}')}\right)^\beta \left(\frac{B(\mathbf{p}')}{B(\mathbf{p})}\right)^\alpha\right]^{\frac{1}{\gamma(\alpha-\beta)}} \tag{11}$$

with $A \not\equiv B$, $A \not\equiv 0$ and $B \not\equiv 0$. The presence of two independent terms in the solution (9) is due to the fact that in general, there are two variables to be adjusted, namely ξ and η. Note that in general, the singularity occurs if $\alpha = \beta$ and not if $\alpha = 1$ as it was the case for a moving source.

6. Physical Realizations

The physical interest of the scaling solution (9) comes from the possibility to describe composed effects of the form $\omega^2 = \omega_1^2 + \omega_2^2$. This situation arises when *two* backacting forces are present. Among the numerous examples, we mention the following:

- Capillary-gravity waves: $\omega^2 = gk + Tk^3/\rho$
- Light waves in a plasma: $\omega^2 = c^2k^2 + \omega_p^2$
- Waves within a fluid, emitted from a moving source: $\omega = \omega_0 \sin\theta - \mathbf{v}\cdot\mathbf{k}$.

For the first example, see *Lighthill* [1978]. As a consequence of the scaling solution (9) the capillary-gravity wavepattern remains similar when, for instance, only the surface tension T is changed. In the last example (see *Landau-Lifshitz* [1986]), θ is the angle between the gravity acceleration and the wave vector. These waves are the oscillating counterpart of convective instability; they occur when an adiabatically rising bubble cools, contracts and becomes restored by the less dense ambient fluid.

Appendix: Analytical pulse responses

Here we summarize the analytical, two-dimensional pulse responses (Eq. 1) of power law media:

$$\omega = k^2: \quad \mathcal{G}(r,t) = \frac{\pi}{it}\, e^{i\frac{r^2}{4t}}$$

$$\omega = k^{1/2}: \quad \mathrm{Re}[\mathcal{G}(r,t)] = \frac{\sqrt{2}\pi^2 u}{r^2}\Big\{J_{\frac{1}{4}}(u)J_{-\frac{1}{4}}(u) - 2uJ_{\frac{1}{4}}(u)J_{\frac{3}{4}}(u) + 2uJ_{-\frac{1}{4}}(u)J_{-\frac{3}{4}}(u)\Big\}$$

$$\omega = k^{-1}: \quad \mathcal{G}(r,t) = \frac{4\pi t}{ir}\, J_1(\sqrt{2itr})\, K_1(\sqrt{2itr})$$

where $u = \frac{t^2}{8r}$, J_ν and K_ν are Bessel functions (notation according to *Gradshteyn and Ryzhik* [1980]). The case $\omega = k^2$ is elementary; for the case $\omega = k^{1/2}$ see *Wehausen and Laitone* [1960].

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(Received February 4, 1997; accepted February 11, 1997.)