CORRESPONDENCE

Comments on "A Combined Derivation of the Integrated and Vertically Resolved, Coupled Wave–Current Equations"

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ABSTRACT

Several equivalent equations for the evolution of the wave-averaged current momentum have been proposed, implemented, and used. In contrast, the equation for the total momentum, which is the sum of the current and wave momenta, has not been widely used because it requires a less practical wave forcing. In an update on previous derivations, Mellor proposed a new formulation of the wave forcing for the total momentum equation. Here, the authors show that this derivation misses a leading-order term that has a zero depth-integrated value. Corrected for this omission, the wave forcing is equivalent to that in the first paper by Mellor. When this wave forcing effect on the currents is approximated it leads to an inconsistency. This study finally repeats and clarifies that the vertical integration of several various forms of the current-only momentum equations are consistent with the known depth-integrated equations for the mean flow momentum obtained by subtracting the wave momentum equation from the total momentum equation. Several other claims in prior Mellor manuscripts are discussed.

1. Introduction

The mass and momentum conservation equations for the ocean circulation involve the effects of ocean surface gravity wave properties. An exact formulation of this problem is provided by Andrews and McIntyre (1978a). For practical applications, the wave-induced forcing can be obtained from an asymptotic expansion of the wave effects to some order in wave steepness $\varepsilon_1 = ka$, where k and a are a typical wavenumber and amplitude of surface elevation, normalized amplitude gradient $\varepsilon_2 = (ka)^{-1} \times \partial a/\partial x$, and current vorticity.

One family of these equations is for the current momentum (see Table 1). Members of this family have been derived by different methods, with different reference frames for the wave averaging and for different asymptotic regimes (e.g., Craik and Leibovich 1976; Leibovich 1980; McWilliams et al. 2004; Ardhuin et al. 2008b; Aiki and Greatbatch 2014). Such current momentum equations have been implemented (e.g., Rascle 2007; Uchiyama et al. 2009; Bennis et al. 2011) and used for various applications (e.g., Uchiyama et al. 2010; Weir et al. 2011; Michaud et al. 2012; Delpey et al. 2014). Some members of this family express the wave effects on the current momentum in the form of the vortex force introduced by Craik and Leibovich (1976). The same wave effects can be cast in a different form [e.g., Holm 1996; Andrews and McIntyre 1978a, their (3.8)], allowing a different physical interpretation and analysis of energy fluxes (Suzuki and Fox-Kemper 2016).

For dynamical and material completeness, these wave-averaged current momentum equations need to be augmented by concentration advection and internal energy equations that contain additional Stokes drift advection as well as by incompressible mass balance and the equation of state.

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TABLE 1. List of previous studies for the effect of surface waves on mean flows.

Horizontal coordinate	Vertical coordinate	Prognostic quantity of momentum equations	
		Lagrangian mean velocity	Eulerian mean velocity
Euler	Integrated	Longuet-Higgins and Stewart (1962) Phillips (1977) Weber et al. [2006, their (15)]	Garrett (1976) Smith (2006) Weber et al. [2006, their (19)]
Euler	Euler	Craik and Leibovich (1976) Skyllingstad and Denbo (1995)	Rivero and Arcilla (1995) Zou et al. (2006) McWilliams et al. (2004)
		Alternative momentum equations	Standard momentum equations
Lagrange	Lagrange	Lagrange [1788, his (C), p. 445]	Lagrange [1788, his (D), p. 447] Lamb [1932, his (2), p. 13]
		Pierson [1962, his (5), (9)]	Pierson [1962, his (4), (10)]
Generalized Lagrange	Lagrange	Andrews and McIntyre [1978a, their (8.7a)]	Andrews and McIntyre [1978a, their (3.8)] Leibovich (1980) Ardhuin et al. (2008b)
		Direct momentum equations	Transformed momentum equations
Euler	Lagrange	Mellor (2003, 2005), Ardhuin et al. (2008a)	
		Aiki and Greatbatch (2012, 2013) Mellor (2015, main text) Mellor (2016)	Aiki and Greatbatch (2013, 2014) Mellor (2015, his appendix B)

A second family of wave-averaged momentum equations, following (8.7a) in Andrews and McIntyre (1978a), is for the total momentum: the sum of mean current and wave momenta (see Table 1). This family involves a wave forcing in the form of the three-dimensional radiation stress (3DRS) term. In a vertically Lagrangian and horizontally Eulerian coordinate system, the 3DRS term is written as the sum of the horizontal Reynolds stress term and the negative of the form stress term (Mellor 2003; Ardhuin et al. 2008a; Aiki and Greatbatch 2012). Recently Aiki and Greatbatch (2013, 2014) have shown that, if terms at higher order in an asymptotic expansion are retained, the wave-averaged momentum equations with the 3DRS term may be transformed to the wave-averaged momentum equations with the vortex force term. However, Mellor (2015) claimed to have found a practical 3DRS expression by considering only leading-order wave quantities in terms of an asymptotic expansion. In a follow-up paper, Mellor (2016) discussed the consistency/inconsistency of the two families of equations, concluding that one must be incorrect if not consistent with the other. Here, in section 2, we show that it is Mellor's (2015) 3DRS that is incorrect because of a derivation error. When inferring his (30) from his (28), one can add any term that has a depth-integrated value of zero but can be very large locally. In fact, a vertical flux is missing that makes his new 3DRS equivalent to the form given in Mellor (2003). This omitted vertical flux involves the vertical profile of the perturbation pressure $\tilde{p}$. If taken proportional to $\cosh(kz + kh)$, as appropriate for a flat bottom with h the mean water depth, $\tilde{p}$ is not accurate enough for 3DRS at the leading order, as shown by Ardhuin et al. (2008a). This is recalled and clarified in section 3.

Since Mellor (2015, 2016) also claimed that the current momentum equations with the vortex force were inconsistent with classical depth-integrated equations with the radiation stress term, we take this opportunity to reaffirm their consistencies in section 4. Conclusions and recommendations on future work on wave–current theory follow in section 5.

Finally, it is important to note that there are two types of wave effects: one depends only on wave properties, and the other depends both on wave and current properties. A typical example of the former is the wave setup/setdown effect, and a typical example of the latter are Langmuir circulations. As the former is independent of current properties, it is possible to compute such an effect with a lower-order wave solution that does not consider modifications of the wave solution because of an underlying current. In contrast, computing the latter effect requires knowledge of a higher-order wave solution that does reflect the influences of the underlying current. Therefore, it is impossible for a theory based on the lower-order wave solution such as Mellor (2003, 2015, 2016) to find the latter effect. On the other hand, a theory that includes the wave modifications by the current can find both the former and latter effects. Indeed, the setup/setdown effect is detailed in section 9.2 of McWilliams et al. (2004) and section 4.1 of Ardhuin et al. (2008b).

2. Near equivalence of Mellor (2015) and Mellor (2003)

a. The correct part in Mellor (2003)

In a seminal paper, Mellor (2003) proposed a very elegant and insightful derivation of the 3DRS based on the momentum equation averaged in a control volume that moves up and down with the wave motion. The resulting wave-averaged equation for the total horizontal momentum contains the divergence of the 3DRS tensor S,

$$\frac{\partial}{\partial x^*_\beta}(DS^{2003}_{\alpha\beta}) - \frac{\partial}{\partial \varsigma}(S^{2003}_{\alpha z}), \tag{1}$$

where D is the wave-averaged water depth, α, β are dummy indices for the horizontal directions, and ς is the vertical coordinate equal to -1 at the bottom and 0 at the wave-averaged free surface. The horizontal and vertical components of the 3DRS in (1) have been defined in (34c) and (34f), respectively, of Mellor (2003) as

$$S^{2003}_{\alpha\beta} \equiv \overline{\tilde{u}_\alpha \tilde{u}_\beta} + \frac{\delta_{\alpha\beta}}{D}\overline{\frac{\partial \tilde{s}}{\partial \varsigma}\tilde{p}} = \overline{\tilde{u}_\alpha \tilde{u}_\beta} + \frac{\delta_{\alpha\beta}}{D}\overline{\tilde{s}_\varsigma \tilde{p}}, \quad \text{and} \tag{2}$$

$$S^{2003}_{\alpha z} \equiv \overline{\frac{\partial \tilde{s}}{\partial x^*_\alpha}\tilde{p}} = \overline{\tilde{s}_\alpha \tilde{p}}, \tag{3}$$

where $\tilde{s} = \tilde{s}(x^*_\alpha, \varsigma, t^*)$ is the vertical displacement of the surfaces of constant ς. The symbol p represents the combined nonhydrostatic and hydrostatic pressure (hereinafter dynamic pressure) for which normalization by mean density is understood. The symbol $\tilde{p} \equiv p - \hat{p} = p + g(z - \hat{\eta})$ is the Eulerian perturbation of p where $\hat{p} = g(\hat{\eta} - z)$. Namely, $\tilde{p}$ is the perturbation measured at a given depth z. To be useful later in the manuscript, the vertically Lagrangian (VL) perturbation of p may be written using a Taylor expansion as $\tilde{p} + \tilde{s}\,\partial\hat{p}/\partial z = \tilde{p} - g\tilde{s}$. Namely, $\tilde{p} - g\tilde{s}$ is the perturbation measured along the surfaces of constant ς. In the absence of wind forcing, the Eulerian perturbation of dynamic pressure becomes $\tilde{p} = g\tilde{\eta}$ at the sea surface, where $z = \eta$ and $p = 0$ (relative to a constant atmospheric surface pressure). On the other hand, the VL perturbation of dynamic pressure becomes $(\tilde{p} - g\tilde{s}) = 0$ at the sea surface where $z = \eta$ and $p = 0$.

The expressions of 3DRS in Mellor (2003) are correct up to his (34a) and (34c) if the tilde variables in (2)–(3) above contain all fluctuations. Difficulties arise when $\tilde{p}$ and $\tilde{s}$ are approximated.

b. The inconsistent part in Mellor (2003)

In evaluating the right-hand sides of (2)–(3) of the present manuscript, Mellor (2003) used vertical profiles of $\tilde{p}$ and $\tilde{s}$ given by Airy theory; namely, for a monochromatic wave train of radian frequency σ and phase $\psi = k_a x_a - \sigma t$ and amplitude a, the Eulerian perturbation of dynamic pressure is

$$\tilde{p}(x_\alpha, z, t) = ga\frac{\cosh k(z+h)}{\cosh(kD)}\cos\psi, \tag{4}$$

and the vertical velocity field is

$$\tilde{w}(x_\alpha, z, t) = \sigma a\frac{\sinh k(z+h)}{\sinh(kD)}\cos\psi, \tag{5}$$

giving the vertical displacement of water parcels

$$\tilde{s}(x^*_\alpha, \varsigma, t^*) = a\frac{\sinh kD(1+\varsigma)}{\sinh(kD)}\cos\psi, \tag{6}$$

where $z + h = z + D - \hat{\eta} = D(1+\varsigma) + \tilde{s}$.

It is crucial to note that (2)–(3) are correct only when $\tilde{p}$ represents the exact perturbation pressure (including higher-order terms in terms of an asymptotic expansion) induced by both the wave motion and the wave–current nonlinear advection. However, (4)–(6) are approximations, strictly valid only for a flat bottom, a constant amplitude, and a uniform current, that is, all the assumptions of Airy wave theory. Most importantly, the influence of the wave–current interactions is not included in these approximations. As a result, this neglect of the wave–current interactions unavoidably results in inconsistency with the investigation of wave forcing effects on the wave-averaged current. This point will be discussed more in section 3.

c. The missing term in Mellor (2015)

One way to understand the result of Mellor (2015) is to rewrite (1) in the present manuscript as

$$\begin{aligned}
&\frac{\partial}{\partial x^*_\beta}(DS^{2003}_{\alpha\beta}) - \frac{\partial}{\partial \varsigma}(S^{2003}_{\alpha z}), \\
&= \frac{\partial}{\partial x^*_\beta}(DS^{2003}_{\alpha\beta} - \delta_{\alpha\beta} g\overline{\tilde{s}_\varsigma \tilde{s}}) - \frac{\partial}{\partial \varsigma}(S^{2003}_{\alpha z} - g\overline{\tilde{s}_\alpha \tilde{s}}), \\
&= \frac{\partial}{\partial x^*_\beta}\underbrace{[D\overline{\tilde{u}_\alpha \tilde{u}_\beta} + \delta_{\alpha\beta}\overline{\tilde{s}_\varsigma(\tilde{p} - g\tilde{s})}]}_{\equiv DS^{2015}_{\alpha\beta}} - \frac{\partial}{\partial \varsigma}\underbrace{[\overline{\tilde{s}_\alpha(\tilde{p} - g\tilde{s})}]}_{\text{missing in M15}}.
\end{aligned} \tag{7}$$

Mellor (2015) continues to evaluate $\tilde{p}$, $\tilde{w}$, and $\tilde{s}$ in the same way as Mellor (2003). Namely, they are taken as the leading-order approximations of (4)–(6) in the ς-coordinate system [i.e., (20a), (20b), and (20c) of Mellor (2015)]. When these equations together with the linear dispersion relation $\sigma^2 = gk\tanh(kD)$ are substituted for $\tilde{p}$, $\tilde{w}$, and $\tilde{s}$, we see that $\overline{(\partial\tilde{p}/\partial\varsigma)\tilde{s}}/D = \overline{\tilde{w}^2}$. Substitution of the Airy wave solution equations (4)–(6) to $S^{2015}_{\alpha\beta}$ in (7) yields

$$S_{\alpha\beta}^{2015} = \overline{\tilde{u}_\alpha \tilde{u}_\beta} - \delta_{\alpha\beta}\overline{\tilde{w}^2} + \frac{\delta_{\alpha\beta}}{D}\frac{\partial}{\partial \varsigma}(\overline{\tilde{p}\tilde{s}} - g\overline{\tilde{s}^2}/2). \quad (8)$$

This is the 3DRS given by (29) of Mellor (2015), removing the D subscripts for clarity and correcting for the missing $1/D$ factor in the last term. This expression corresponds to the $S_{\alpha\beta}^{2003}$ of Mellor [2003, his (34)] divided by D, with this D due to a different definition of the radiation stress between Mellor (2003, 2015).

So what has become of the vertical flux term $\overline{\tilde{s}_\alpha \tilde{p}}$ in (34a) of Mellor (2003)? This term is $\overline{\tilde{s}_\alpha(\tilde{p} - g\tilde{s})}$ with the Mellor (2015) notations used here. The vertical flux in the second term on the last line of (7) may be interpreted as the VL perturbation of dynamic pressure $\tilde{p} - g\tilde{s}$, acting on the top- or bottom-tilted material surface with its vertical displacement $\tilde{s}$ induced by the waves (Fig. 1 of Ardhuin et al. 2008a).[1] Without relying on the wave average (denoted by the overbar), the vertical flux $\tilde{s}_\alpha(\tilde{p} - g\tilde{s})$ is zero at the surface, where $\tilde{p} = g\tilde{s}$ is at the lowest order, and bottom, where $\tilde{s}$ is zero. Thus, the depth integral of $\partial\tilde{s}_\alpha(\tilde{p} - g\tilde{s})/\partial\varsigma$ vanishes, but this term can be very large in the water column, and it is indeed a leading term in the example shown in Fig. 1. If evaluated using the Airy wave solution (4)–(6), the wave average of the vertical flux $\overline{\tilde{s}_\alpha(\tilde{p} - g\tilde{s})}$ vanishes, owing to the phase relationship $(\overline{\cos\psi \sin\psi} = 0)$ at all surfaces of constant ς, but it is not zero when using a proper wave solution. For these reasons, the last term on the last line of (7) has been forgotten by Mellor (2015).

3. Necessary accuracy of the vertical flux in 3DRS

Although radiation stresses in (34a) and (34c) in Mellor (2003) are correct, the approximation of his (34e), using Airy wave theory [e.g., our (4) and (6)], is not consistent. For simplicity of the argument we consider the case of waves propagating in the x direction, with all parameters uniform along the y direction. This is easily generalized to a full three-dimensional setting. The horizontal momentum balance contains the body forces F_{xx} coming from the divergences of the horizontal radiation stress tensor

$$F_{xx} = \frac{\partial S_{xx}}{\partial x}, \quad (9)$$

and F_{xz} from the vertical radiation stress tensor in (3), the missing term mentioned above:

$$F_{xz} = \frac{\partial S_{xz}}{\partial \varsigma}. \quad (10)$$

[1] Note that Mellor (2003) uses a slightly different notation, namely, $\tilde{p}$ in Mellor (2003) is equal to $p + gD\varsigma$ in Mellor (2015).

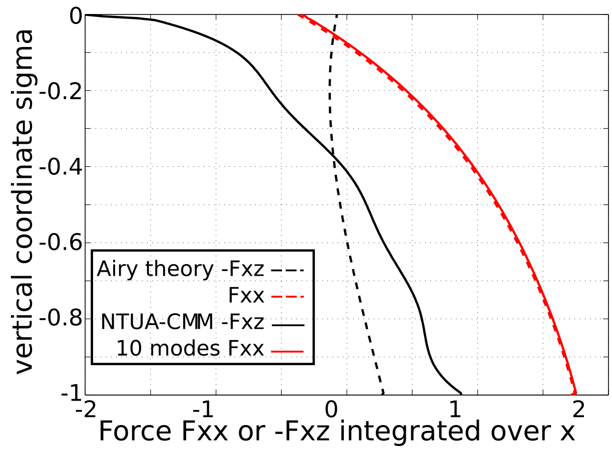

FIG. 1. Illustration of the profile of forces F_{xx} and $-F_{xz}$ for the case of waves shoaling over a slope proposed by Ardhuin et al. (2008a) using two different approximations for the wave motion, that is, Airy wave theory keeping the ε_2 terms from the horizontal gradients of a and the NTUA Coupled Mode Model with 10 modes (Athanassoulis and Belibassakis 1999).

The term S_{xx} is easily approximated to order $\varepsilon_2^0\varepsilon_1^2$ using Airy wave theory, and thus the force F_{xx} is of order $\varepsilon_2\varepsilon_1^2$ thanks to the horizontal gradient, which is thus the leading order of the momentum balance. For consistency, we need to have F_{xz} at the same order. Because the vertical derivative $\partial/\partial\varsigma$ does not change the order in ε_2, it means that $\overline{p\partial s/\partial x}$ must be obtained at order $\varepsilon_2\varepsilon_1^2$; hence, both $\tilde{p}$ and $\tilde{s}$ must be evaluated to first order in ε_2, meaning that the approximations such as (6) are insufficient for a consistent estimation of the 3DRS.

This inconsistency occurs at the leading order in all cases that have vertical fluxes of wave momentum. It was exposed by Ardhuin et al. (2008a) for the particular case of waves propagating over varying topography. In that case, it was shown that a numerical solution of the full Laplace equation valid for any bottom slope (this requires a specific non-Airy model; e.g., Chandrasekera and Cheung 1997; Athanassoulis and Belibassakis 1999) could provide consistent estimates of the 3DRS, as illustrated in Fig. 1 for the case of waves shoaling over a slope without any dissipation.

However, if the vertical flux is ignored, as is the case in Mellor (2015), the force F_{xx} cannot be balanced at all depths by the hydrostatic pressure gradient associated with the free-surface slope. Using Airy wave theory approximation for F_{xz} increases that imbalance. As an alternative, we used the National Technical University of Athens Coupled Mode Model (NTUA-CMM), implemented here with $n = 10$ modes.

This model expands the velocity potential ϕ on a basis of solutions with different vertical profiles. These include a flat-bottom mode $\phi_0 = \cosh(kz + kh)$, a sloping-bottom mode ϕ_{-1} (the only one with nonzero vertical velocity at the bottom), and n evanescent modes $\phi_n = \cos(k_n z + k_n h)$ that decay exponentially with horizontal distance, with k_n as the solutions to $(2\pi f_w)^2 = gk\tan(kD)$. All these modes are coupled through the surface and boundary conditions. In the limit $n \to \infty$, the coupled solution is a solution to Laplace and both bottom and surface boundary conditions. Such a model provides a better approximation of F_{xz}, but spurious oscillations remain in the vertical profile of F_{xz}, due to the finite number of modes (Fig. 1).

The alternative use of a momentum equation for the current only (e.g., McWilliams et al. 2004) removes this difficulty because the derivation includes effects of the currents on the waves en route to deriving the effects of the waves on the currents; the result is that the latter can be evaluated from the solutions of usual phase-averaged spectral models, which themselves do not include this more complete representation of the wave–current interaction.

This is because the problematic flux $S_{\alpha z}$ is a flux of wave momentum, which adjusts the vertical profile of wave properties to their waveguide as determined by the current and depth variations and has no dynamical effect on the mean flow. For example, as shown in Fig. 2, the momentum that is located at $x = -200$ m, $z = -5$ m, is progressively pushed up the water column as waves propagate over the slope, giving a different profile at $x = 200$ m. This change in profile is due to the combination of $S_{\alpha z}$ and the hydrostatic pressure gradient associated with the setdown; the transport is increased by 22%, but the surface Stokes drift increases by 69%.

4. Consistency of depth-integrated equations and current-only equations and related issues

On his last page, Mellor (2015, p. 1463) writes ''in L-HS and Phillips, $W_S = 0$, [the vertical component of Stokes drift] as discussed in section 10. Conversely, after vertical integration of the equation of McWilliams and Restrepo (1999), Ardhuin et al. (2008[b]), or Bennis et al. (2011) and use of (B3), there seems to be no way to bring them into agreement with those of Phillips [(1977)] or Smith (2006).'' We beg to disagree, and it is essentially a question of asymptotic assumptions. Indeed, the depth-integrated equations of Longuet-Higgins and Stewart (1962) and Smith (2006) neglect the effect of vertical current shear on the wave kinematics. This statement in Mellor (2015) is related to two other claims on the vertical Stokes drift and the existence of consistent 3D theories on which we do not agree with Mellor (2015).

a. Consistency of the total and current-only form of the momentum equations

Mellor (2016, p. 4475) wrote, ''If the 'radiation stress' and 'vortex force' theories are both correct, then one should be able to derive one from the other.'' We fully agree, and indeed, this was done by Andrews and McIntyre (1978a), with a very general definition of the Stokes drift as the wave pseudomomentum. As mentioned in the previous section, there is no known analytical form for the vertical fluxes of wave momentum, and hence we cannot express simply the 3DRS tensor. However, we can see that the same 3DRS tensor shows up in the total momentum equation [(8.7a) in Andrews and McIntyre 1978a] and the 3D wave momentum equation given in Andrews and McIntyre (1978b). Hence, subtracting the wave momentum equation from the total momentum equation yields the current-only momentum equation.

b. Consistency with depth-integrated equations

The relationship between the two families of equations, one for the current momentum and the other for the total momentum, is summarized in Table 1.

The agreement of (9.15) of McWilliams et al. (2004) or (55) of Ardhuin et al. (2008b) with (3.11) of Garrett (1976) or (2.29) of Smith (2006) was shown in (47) of Lane et al. (2007) and (84) of Ardhuin et al. (2008a), the only difference being the additional terms due to the vertical current shear because Garrett (1976) and Smith (2006) neglected the effect of vertical shear on $\tilde{u}$, $\tilde{w}$, and $\tilde{p}$. Recall that Smith (2006) is an extension to finite depth of Garrett (1976) and that the total momentum equation of Longuet-Higgins and Stewart (1962) is obtained by adding the current-only equation and the wave momentum equation (Smith 2006; Ardhuin 2006).

We also recall that Ardhuin et al. (2008a) is consistent with McWilliams et al. (2004) to first order in the vertical current shear, but they differ when the curvature of the current profile or finite current shears are considered.

Hence, once the effects of vertical shear are ignored, the 3D momentum equation can be simplified from (1) in Uchiyama et al. (2010) or (11) in Bennis et al. (2011). For the x component, of the quasi-Eulerian current $(\hat{u}, \hat{v}, \hat{w})$, it is

$$\frac{\partial \hat{u}}{\partial t} + \hat{u}\frac{\partial \hat{u}}{\partial x} + \hat{v}\frac{\partial \hat{u}}{\partial y} + (\hat{w} + W_s)\frac{\partial \hat{u}}{\partial z} - f\hat{v} + \frac{1}{\rho}\frac{\partial p^H}{\partial x} \simeq \left[f + \left(\frac{\partial \hat{v}}{\partial x} - \frac{\partial \hat{u}}{\partial y}\right)\right]V_s - \frac{\partial J}{\partial x} + \hat{F}_x + \hat{F}_x^w, \quad (11)$$

where f is the Coriolis parameter, p^H is a hydrostatic pressure, (U_s, V_s, W_s) are the 3D components of the

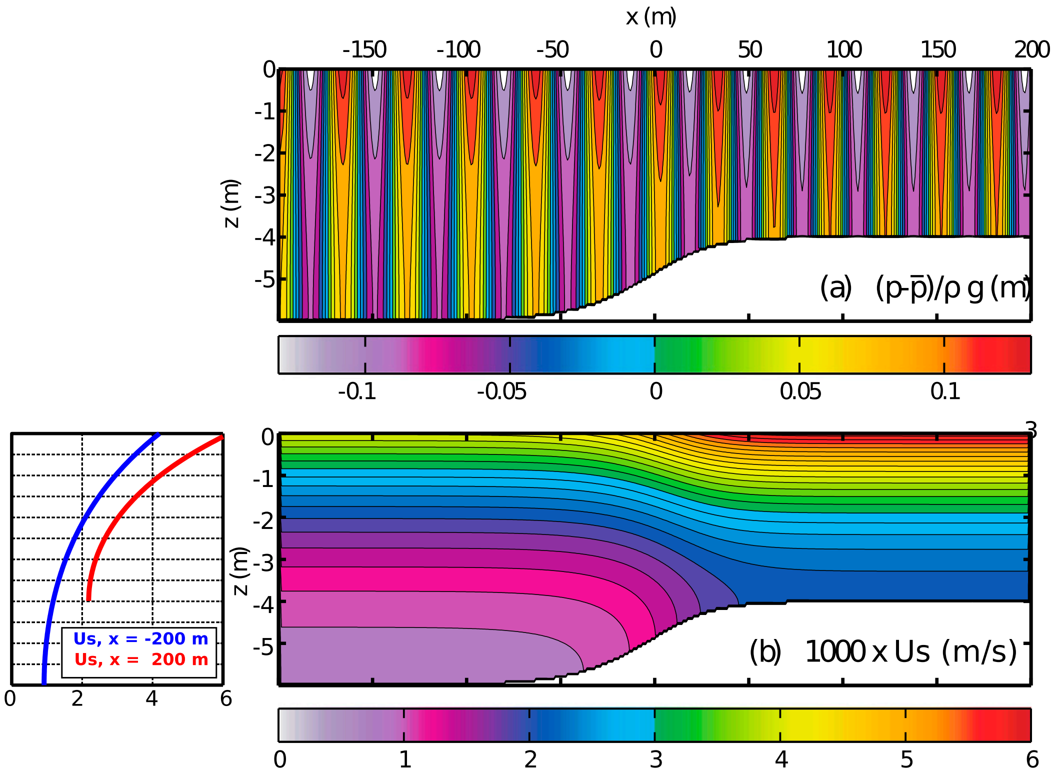

FIG. 2. Example of (a) snapshot of nonhydrostatic pressure field in waves shoaling over a slope and (b) associated Stokes drift. The Stokes drift profile adjusts from (left panel) the blue profile to the red profile, as the water depth changes from 6 to 4 m, as waves propagate from $x = -200$ to 200 m. This adjustment is made possible by both vertical and horizontal fluxes of wave momentum and by the setdown of the wave-averaged surface elevation.

Stokes drift velocity, $\hat{F}_x$ is the nonwave nonconservative force, $\hat{F}_x^w$ is the wave-induced nonconservative force (e.g., due to wave breaking), and J is the Jerry form of the Bernoulli head, as used with the same notation by Smith (2006):

$$J = g\int_0^{2\pi}\int_0^{\infty} \frac{kE(f_w, \theta)}{\sinh(2kD)}\, df_w\, d\theta. \qquad (12)$$

Here, $D = h + \overline{\zeta}$ is the mean water depth, k is the wavenumber related to the wave frequency $f_w = \sigma/2\pi$ by the surface gravity wave dispersion relation, and $E(f_w, \theta)$ is the spectrum of the surface elevation variance associated with waves, distributed across frequencies f_w and propagation direction θ. The vertical integration of (11) from $z = -h$ to $z = \overline{\zeta}$ gives, with a careful manipulation of the vertical Stokes drift component as in section 3 of Ardhuin et al. (2008a), (2.28) of Smith (2006), with the addition of the Coriolis force.

c. The vertical Stokes drift component

Starting from (78) in Ardhuin et al. (2008a), the first four terms of (11) recombine to give

$$\frac{\partial}{\partial t}(\rho\hat{u}) + \frac{\partial}{\partial x}(\rho\hat{u}^2) + \frac{\partial}{\partial y}(\rho\hat{u}\hat{v}) + \frac{\partial}{\partial z}[\rho\hat{u}(\hat{w} + W_s)] - \hat{u}\frac{\partial W_s}{\partial z}. \qquad (13)$$

In these, the last term can be rewritten as

$$-\hat{u}\frac{\partial W_s}{\partial z} = \hat{u}\left(\frac{\partial U_s}{\partial x} + \frac{\partial V_s}{\partial y}\right). \qquad (14)$$

Hence, the vertical Stokes drift component is a key term to recover depth-integrated equations from the 3D equations, with the vertical integral of (14) giving the $U(\nabla \cdot M^w)$ in (2.29b) of Smith (2006), in which U is the surface current and M^w is the depth-integrated Stokes drift.

So what is this vertical Stokes drift, and why is it so little discussed?

In McWilliams et al. (2004), W_s was defined to be compatible with 3D incompressibility for Stokes drift, as a complement to 3D incompressibility of $\hat{\boldsymbol{u}}$. The term W_s is small compared to the horizontal Stokes drift by a factor of ε_2; that is, it is associated with Stokes drift variations on a horizontal scale larger than the wave scale. A physical interpretation follows from Ardhuin et al. (2008b), who found that this W_s agreed, at the lowest order, with the vertical component of the wave pseudomomentum vector defined by (3.1) of Andrews and McIntyre (1978a). Neglecting the Coriolis effect on wave kinematics has each component $\mathbf{p}_i$ given by

$$\mathbf{p}_i = -\overline{\frac{\partial \xi_j}{\partial x_i} u_j^l}, \tag{15}$$

where ξ_j is the j component of the generalized Lagrangian mean (GLM) displacement vector, and u_j^l is the Lagrangian perturbation of the velocity component. This pseudomomentum is a priori different from the Stokes drift velocity component, given by (2.27) of Andrews and McIntyre (1978a):

$$U_{si} = \overline{\xi_j \frac{\partial u_i'}{\partial x_j}} + \frac{1}{2}\overline{\xi_j \xi_k}\frac{\partial^2 \overline{u}_i}{\partial x_j x_k} + O(a^3), \tag{16}$$

where u_i' is the Eulerian perturbation of the velocity component. Since (16) is based on a Taylor expansion, the partial differentiation should read $u_{i,j}' \equiv \partial u_i'/\partial \Xi_j$, where Ξ_j represents the Eulerian coordinates in Andrews and McIntyre (1978a). A related and detailed manipulation appears in appendix D of Aiki and Greatbatch (2013). In the GLM framework, the Eulerian perturbation is estimated from the Lagrangian perturbation using an approximation $u_i' = u_i^l - \xi_j \overline{u}_{i,j} + O(a^2)$.

In many simple cases the pseudomomentum vector and the Stokes drift velocity do coincide, as discussed in Phillips (2001) and Phillips et al. (2010). This coincidence holds to fourth order in the wave steepness for irrotational waves, but it is not true in general. When they coincide, the vertical component of $\mathbf{p}$ is equal to W_s and this wave-induced drift of water particles, hence a vertical Stokes drift component that has the same interpretation as the horizontal Stokes drift component. For example, Fig. 3 in Ardhuin et al. (2008a) shows that component for waves shoaling over a slope. In the absence of such a vertical drift, water particles would cross the bottom, a clearly unphysical situation.

It is interesting to note that, in general, the 3D nondivergent (U_s, V_s, W_s) defined by McWilliams et al. (2004) may not always correspond to the pseudomomentum $\mathbf{p}$, which may itself differ from a true drift. See also the discussion on the quasi-Stokes velocity in Aiki and Greatbatch (2012).

5. Conclusions and recommendations

The wave-averaged total momentum equation by Mellor (2015) formulated as a function of the wave-induced pressure $\tilde{p}$ and vertical displacement $\tilde{s}$ was shown to be inconsistent because of a missing term that is the divergence of vertical flux $\overline{(\tilde{p}-g\tilde{s})\tilde{s}_\alpha}$, which integrates to zero over the depth. That term was missed because of an incorrect inference of an equality of integrands [(30)] from an equality of integrals [(28)], and it is in general a leading-order term in the total momentum (Ardhuin et al. 2008b; Aiki and Greatbatch 2013). When this term is added, the wave-induced forcing is consistent with Mellor (2003). The radiation stresses in Mellor (2003) are correct if his wave variables represent all wave-induced fluctuations. However, both in Mellor (2003) and Mellor (2015), these fluctuations are approximated with wave-induced pressure proportional to cosh(kz + kh), as given by Airy theory. This neglects important nonhydrostatic pressure perturbations. As a result, the radiation stresses are incorrectly evaluated.

The fundamental problem with the three-dimensional radiation stresses (3DRS) is that a consistent estimation of the vertical momentum flux $\overline{(\tilde{p}-g\tilde{s})\tilde{s}_\alpha}$ requires a wave-induced forcing that must be accurate to first order in the normalized amplitude gradient $\varepsilon_2 = ka \times \partial a/\partial x$. Such a forcing cannot be determined from the wave spectrum alone and generally requires a solution of an elliptic wave equation (e.g., Athanassoulis and Belibassakis 1999; Chandrasekera and Cheung 1997). This makes the use of equations for the total momentum much less practical than the equations for the current momentum only (e.g., McWilliams et al. 2004; Ardhuin et al. 2008b; Aiki and Greatbatch 2013). A similar remark can be made about the need to sufficiently include the effects of the currents on the waves to fully represent the consequences in the 3DRS. We have read or reviewed papers by many different authors over the last 15 years that have attempted to derive analytical 3DRS expressions, and Mellor (2015) is the latest in the series. Until somebody finds an analytical solution to the wave motion to order ε_2, these attempts are bound to fail.

Whatever the wave–current coupling approach, for the full momentum or the current momentum, there is a clear need for a hierarchy of reference solutions. As insisted upon by Mellor (2015), the depth-integrated equations of Smith (2006), both for the total momentum or current-only momentum, are important guidelines, but they provide no constraint on the vertical profile of the wave-induced forcing nor do they account for effects of vertical current shear. The adiabatic shoaling case of Ardhuin et al. (2008a) is a first test of vertical profiles. For other adiabatic effects, there is a clear value in defining test cases for vertical current shears, modulation on the scale of wave groups (e.g., McWilliams et al. 2004), or subwavelength modulations introduced by partially standing waves (Ardhuin et al. 2008a, their section 4.1). A second class of cases should consider turbulent closures in the presence of waves (e.g., Olabarrieta et al. 2010; Sullivan and McWilliams 2010).

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