On the Longuet-Higgins–Hasselmann theory for double-frequency microseisms,
and practical estimation of seismic noise from ocean wave model results

version β, not yet approved.

Fabrice Ardhuin, Ifremer, Brest, France
and
Eléonore Stutzmann, IPG, Paris, France

November 17, 2012

1 Preamble

All theories on the generation of double-frequency ocean noise and microseisms agree in their broad terms, namely they are all based on the nonlinear wave interaction first discovered by (Miche, 1944). Yet, the quantitative details of the vast litterature on the subject do not agree in general, with numerical factors like 2 or π arising here and there. Here we focus on the microseismic response, and not the the oceanic pressure field itself. For that other related problem, see Farrell and Munk (2008).

Also, there does not appear to have been a published formulation both correct and non-ambiguous that would give the microseismic vertical ground displacement power spectrum $F_{\delta}(f_s)$, where f_s is the seismic frequency, as a function of the directional-frequency spectrum $E(f, \theta)$, as computed by a spectral wave model. There is also no practical explanation of where the proper wave information can be found and how to use it. Those interested only in the practical calculation of the seismic response and use of the wave data can go directly to section 5, where they will find practical information and example applications.

Finally, the description of microseismic noise generation by random waves has been treated differently by Longuet-Higgins (1950) (hereinafter LH50) and Hasselmann (1963) (hereinafter H63). The former used a point force equivalence for a broad spectrum and is more complete on the details of the motion under waves, but it fails to give a general expression for random waves. The latter gives all the necessary results from the wave spectrum - without assumption on the wave spectral shape - but the interpretation is more difficult, and we clarify it here, with further details appearing in Ardhuin and Herbers (2012). Both approaches are found to be consistent, once numerical errors or ambiguities have been corrected. We also note that the proper correction for finite depth effects and the theory for seismic body waves has been derived by Ardhuin and Herbers (2012). From that last work, it appears that there are errors in both Tanimoto (2007), and Webb (2007).

2 Theory

We first make a remark on the notations. Because LH50 and H63 used different notations we have tried to be consistent with both, which is not always possible. Table 1 summarizes the notations when they differ.

The general method for obtaining the hydrodynamic and seismic response to the wave motion is to start from the irrotational equations of motion in the water, and solving it by successive approximations. At the lowest order of approximation, the linearized equations reduce to a harmonic oscillator equation, with solutions that are the "free waves": gravity or compression waves. "Free" means that these waves can persist in the absence of forcing. These waves obey the respective dispersion relations of such waves. Namely, for gravity waves the radian frequency σ is related to the wavenumber k via the dispersion relation (de Laplace, 1776),

$$\sigma^2 = gk \tanh(kD) \tag{1}$$

Table 1: Notations used in LH50, H63 and here when they differ

variable	LH50	H63	present work	reason for change
water depth	h	H	D	H used for waves height
water wave rad. freq.			σ	
seismic wave rad. freq.			ω	
interacting wavenumbers	k_1,k_2	k',k''	k_1,k_2	easier to use
seismic wavenumber			$\mathbf{K}$	avoid confusion with water waves
water density	ρ_s	ρ_1	ρ_w	

2.1 Equivalent pressure under monochromatic standing waves

Following LH50 we consider partially standing waves, with a surface elevation

$$\zeta = a_1 \cos(kx - \sigma t) + a_2 \cos(kx + \sigma t). \tag{2}$$

This wave pattern is the surperposition of two travelling waves of amplitudes a_1 and a_2, and it is a purely standing wave when the two amplitudes.

LH50 expressed the second-order velocity potential and pressure field from the compressible but irrotational equations of motion. The second order solution is a *forced* mixed gravity-compression wave (his pages 25 and 26). Close to the surface (at a distance much less than the acoustic wavelength) this forced motion contains pressure fluctuations that are uniform over the vertical. At greater depth, the acoustic standing wave pattern in prominent. In the general case, the u_1^2 and $\partial\phi_2/\partial t$ terms in equation LH50-(113) give a $\sinh(kD)$ terms, i.e.

$$p_2 = -\rho_w a_1 a_2 \sigma^2 \cos(2\sigma t) \left[2 + \frac{3}{4\sinh(kD)}\right], \tag{3}$$

where ρ_w is the density of sea water, (ρ_s in LH50 and ρ_1 in H63).

In the case of deep water waves, it does give LH50-(175),

$$p_2 = -2\rho_w a_1 a_2 \sigma^2 \cos(2\sigma t). \tag{4}$$

Unless explicitly mentionned otherwise, we will now consider deep water waves, and the result should be corrected by a factor $\{\tanh^2(kd)/[1 + 2kD/sinh(2kD)]\}$ for any water depth (Ardhuin and Herbers, 2012).

We can compute the variance of this equivalent pressure, and check it against the theory by Miche (1944), which was verified in the laboratory by Cooper and Longuet-Higgins (1951). In deep water it is,

$$\overline{p_2{}^2} = \rho_w^2 4 a_1^2 a_2^2 \frac{\sigma^4}{2} = 8\rho_w^2 \sigma^4 E_1 E_2 \tag{5}$$

where $E_1 = a_1^2/2$ and $E_2 = a_2^2/2$ are the surface elevation variances of the two interacting wave trains (expressed in m^2 when using S.I. units). For the present version of this report where we consider seismic waves of periods less than 10 s, the expressions for deep water waves will suffice because we shall see that the seismic response is a function of the water depth and vanishes for shallow water. We note, however, that this will not be the case for longer seismic waves which can be generated by waves for which 1000 to 5000 m is still shallow or intermediate depth. These long waves can be freely propagating infragravity waves radiated from the ocean's shorelines and which generate seismic noise via the exact same mechanism Herbers et al. (1995); Webb (2007).

2.2 Pressure spectrum under random waves

Here we start from the H63 paper. That work suffers from a number of misprints which are corrected below. H63, instead of solving for the full motion, makes the interesting remark that the 2nd order equations of motion H63-(2.10) are the same as the equations for a motion forced by a *surface* pressure[1] equal to

$$p_{2,\mathrm{eq}} = -\rho_w \frac{\partial}{\partial t} \left(\nabla \phi\right)^2 = -\rho_w \left(u^2 + v^2 + w^2\right) \quad \text{at} \quad z = 0. \tag{6}$$

One has to be very careful that this is not the real pressure (for that see Farrell and Munk, 2008), but an *equivalent* pressure that gives the same microseismic response.

In the last line of H63-(2.10) there is no factor 1/2 anymore because there are two non-linear terms that, in deep water, are both equal to $0.5 \left(\nabla \phi\right)^2$. Which means that the above expression is only valid for deep water waves, i.e. when $kD \gg 1$. The general form is given by Ardhuin and Herbers (2012).

Other than that, the expression for the second order equivalent pressure (6) is very general. If one looks at the pressure in the presence of two wave trains of wave numbers $\mathbf{k}_1$ and $\mathbf{k}_2$, and radian frequencies σ_1 and σ_2, the phase of the forcing for the second order motion is $[(\mathbf{k}_1 \pm \mathbf{k}_2)\mathbf{x} - (\sigma_1 \pm \sigma 2)t]$. This forcing (6) thus produces oscillations with $\mathbf{K} = \mathbf{k}_1 \pm \mathbf{k}_2$ and $\omega = \sigma_1 \pm \sigma_2$. These waves will be able to propagate as seismic (free waves) waves only if they obey a seismic dispersion relation with $\omega / |\mathbf{K}| = C_s$ where C_s is the seismic phase velocity. All other oscillations will nevertheless exist but will not be "free" to propagate on their own, and will be "forced" to follow the wave forcing. These forced motions only existing locally in the presence of this forcing..

One has to be very careful that spectra in H63 are single-sided (see eq. H63-(2.7)). This means that the variance of the pressure is recovered by summing only over positive frequency. By definition $F_p(\mathbf{K}, \omega) = 2 \left|dP_+\right|^2 / (d\mathbf{K} d\omega)$, where dP_+ is the 3D Fourier-Stieltjes transform of the second order pressure (3D means that the transform is over horizontal space and time). This is typically approximated by taking 3 consecutive FFTs in, x, y and t, producing a Fourier transform in the spectral space of coordinates k_x, k_y and f. Because this 3D Fourier transform is centrally symmetric around the origin, we simply forget the half-space $f < 0$ and double the variance for $f > 0$, hence the factor 2.

If one corrects for the missing $d\mathbf{K} d\omega$ in the LHS[2] of H63-(2.13), the expression given by H63 for monochromatic waves is identical to that of LH50. Namely $\overline{p_2{}^2} = F_p(\mathbf{k}, \omega) d\mathbf{k} d\omega$, and $F_\zeta(\mathbf{k}_1) d\mathbf{k}_1 = E_1$. We repeat here eq. H63-(2.13) neglecting the middle line (difference interactions) and keeping only the

[1] Some authors seem to have misunderstood that it is the bottom pressure, hence the emphasis on *surface*.

[2] Other misprints include the ρ_1 replaced by a ζ_1 and a $\mathbf{k}$ in the last line replaced by a k.

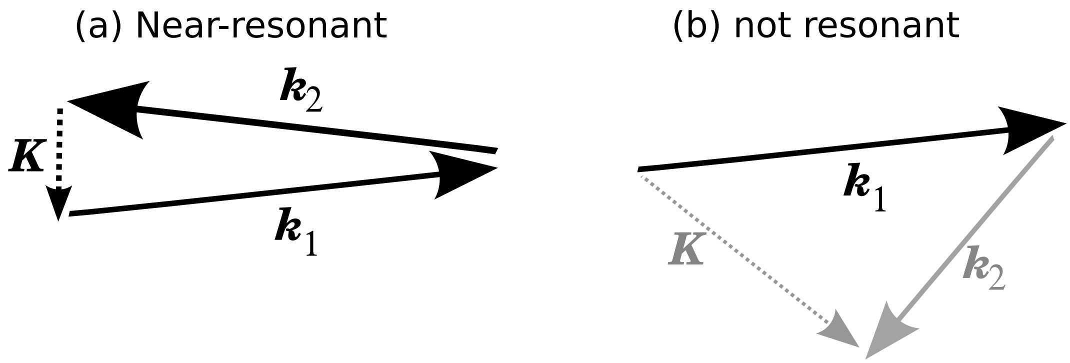

Figure 1: Geometry of the lowest order interactions between ocean surface gravity wave and seismic waves. Wave-wave interactions occur for any pair of wavenumber vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. (a) near resonant case in which K approaches a seismic wavenumber $K_s = 2\pi C_s(f_1 + f_2)$ with C_s a seismic phase speed. (b) However, in the case of the grey configuration, $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$ is very large and very far from the wavenumber $\mathbf{K}_s$ of any of the seismic modes with frequency $f_s = 2\pi(f_1 + f_2)$. Because the resonant $\mathbf{K}_s$ are 100 to 1000 smaller than $\mathbf{k}_1$, in practice there is only a significant seismic response for $\mathbf{k}_1 + \mathbf{k}_2 \simeq 0$ and thus $\sigma_1 = \sigma_2$ and $\omega = 2\sigma_1$.

last line (sum interactions)[3] for a few typographic errors we get,

$$F_p(\mathbf{K}, \omega) = \rho_w^2 g^4 \int\int\int\int F_\zeta(\mathbf{k}_1) F_\zeta(\mathbf{k}_2) \frac{4k_1^2 k_2^2}{\sigma_1^2 \sigma_2^2} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{K}) \delta(\sigma_1 + \sigma_2 - \omega) \mathrm{d}k_{1x} \mathrm{d}k_{1y} \mathrm{d}k_{2x} \mathrm{d}k_{2y}, \tag{7}$$

where δ is Dirac's function, σ_1 and σ_2 are the radian frequencies of the two interacting wave trains, and the four integrals are all from $-\infty$ to ∞.

For the general case of random waves, we obtain H63-(2.14) from the corrected H63-(2.13), by considering the Jacobian of the spectral coordinate transform from $\mathbf{k}_1$ to (ω, θ_1). Namely, in deep water, the wave group velocity is $C_{g1} = g/(2\sigma_1)$ and

$$d\mathbf{k}_1 = k_1 dk_1 d\theta_1 = k_1 d\sigma_1 d\theta_1 / C_{g1} = k_1 d\omega d\theta_1 / (g/\sigma_1) = \sigma_1^3 d\omega d\theta_1 / g^2. \tag{8}$$

We then get

$$F_p(\mathbf{K}, \omega) = \frac{4\rho_w^2 \sigma^7}{g^2} \int_0^{2\pi} F_\zeta(\mathbf{k}') F_\zeta(\mathbf{k}'') \mathrm{d}\theta' \tag{9}$$

which appears to be a factor of 2 larger than H63-(2.14), i.e. there is a factor 2 error in H63. This can be translated to

$$F_p(\mathbf{K} \simeq 0, \omega) = \rho_w^2 g^2 \sigma \int_0^{2\pi} E(\sigma, \theta) E(\sigma, \theta + \pi) \mathrm{d}\theta \tag{10}$$

which is equal to H63-(2.15) if we interpret $f_\zeta(\omega/2, \theta)$ as the spectral density in (σ, θ)-space. This is also exactly eq. (1) in Webb (2007) in the limit of deep water waves, when his coefficient G goes to 1.

Finally, we can also write this in term of the usual spectral density $E(f, \theta)$ as computed by numerical wave models, and the integral need only to be evaluated between 0 and π, giving,

$$F_p(\mathbf{K} \simeq 0, \omega) = \frac{\rho_w^2 g^2 \sigma}{2\pi^2} \int_0^{\pi} E(f, \theta) E(f, \theta + \pi) \mathrm{d}\theta. \tag{11}$$

[3]The " ! " in H63-(2.13) is clearly a typo for " ′ ".

Because it is more common to work with spectral densities in frequency, we can define $f_2 = \omega/(2\pi)$ and obtain eq. (1) in Ardhuin et al. (2011),

$$F_p(\mathbf{K} \simeq 0, f_2) = 2\pi F_p(\mathbf{K} \simeq 0, \omega) = \frac{\rho_w^2 g^2 \sigma}{\pi} \int_0^{\pi} E(f,\theta) E(f,\theta+\pi) \mathrm{d}\theta. \quad (12)$$

$$= \rho_w^2 g^2 f_2 \int_0^{\pi} E(f,\theta) E(f,\theta+\pi) \mathrm{d}\theta \quad (13)$$

2.2.1 Consistency of LH50 and H63

Now taking a monochromatic wave field with variance at two opposite wave numbers $\mathbf{k}_0$ and $-\mathbf{k}_0$,

$$F_\zeta(\mathbf{k}_1) = E_1 \delta(\mathbf{k}_1 - \mathbf{k}_0) + E_2 \delta(\mathbf{k}_1 + \mathbf{k_0}), \quad (14)$$

we can remove the integral over k_{1x} and k_{1y} in (7), because the integrand is now only non-zero for $\mathbf{k}_1 = \mathbf{k}_0$ or $\mathbf{k}_1 = -\mathbf{k}_0$, where $F_\zeta(\mathbf{k}_1)$ gives the surface elevation variance E_1 and E_2 respectively. Since we are only treating deep-water waves, σ^2 becomes $\sigma_0^2 = g|\mathbf{k}_0| = gk_0$, and we get

$$F_p(\mathbf{K},\omega) = 4\rho_w^2 g^4 \int\int \left[E_1 \delta(\mathbf{k}_0 + \mathbf{k}_2 - \mathbf{K}) + E_2 \delta(-\mathbf{k}_0 + \mathbf{k}_2 - \mathbf{K})\right] \frac{k_0^2 k_2^2}{\sigma_0^2 \sigma_2^2} F_\zeta(\mathbf{k}_2) \delta(\sigma_0 + \sigma_2 - \omega) \mathrm{d}k_{2x} \mathrm{d}k_{2y}. \quad (15)$$

We now also remove the integral over k_{2x} and k_{2y},

$$\begin{aligned} F_p(\mathbf{K},\omega) &= 4\rho_w^2 g^4 \frac{k_0^4}{\sigma_0^4} \{[E_1 E_2 \delta(\mathbf{k}_0 - \mathbf{k}_0 - \mathbf{K}) + E_2 E_1 \delta(-\mathbf{k}_0 + \mathbf{k}_0 - \mathbf{K})] \\ &\quad + \left[E_1^2 \delta(2\mathbf{k}_0 - \mathbf{K}) + E_2^2 \delta(-2\mathbf{k}_0 - \mathbf{K})\right]\} \delta(2\sigma_0 - \omega). \end{aligned} \quad (16)$$

Because there are no seismic waves with both $\mathbf{K} = \pm 2\mathbf{k}_0$ and $\omega = 2\sigma_0$, the second line is irrelevant and we get

$$F_p(\mathbf{K},\omega) = 8\rho_w^2 g^4 E_1 E_2 \delta(\mathbf{K}) \delta(2\sigma_0 - \omega) k_0^4/\sigma_0^4. \quad (17)$$

The pressure variance is now given by integrating over the spectral components of the pressure, which naturally removes the Dirac functions, and gk_0 can be replaced by σ_0^2 for deep water waves,

$$\begin{aligned} \overline{p_2{}^2(t)} &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty F_p(\mathbf{K},\omega) \mathrm{d}K_x \mathrm{d}K_y \mathrm{d}\omega \\ &= 8\rho_w^2 \sigma_0^4 E_1 E_2. \end{aligned} \quad (18)$$

We can now identify this expression with the variance from Longuet-Higgins's second order pressure (his equation 31),

$$\overline{p_2{}^2(t)} = \rho_w^2 4 a_1^2 a_2^2 \frac{\sigma^4}{2} = 8\rho_w^2 \sigma^4 E_1 E_2 \quad (19)$$

where $E_1 = a_1^2/2$ and $E_2 = a_2^2/2$ are the surface elevation variances of the two interacting wave trains (expressed in m^2 when using S.I. units). This variance of the second order pressure was verified in the laboratory by Cooper and Longuet-Higgins (1951).

3 Microseismic response to random waves

3.1 Microseismic response on a flat Earth, using the approach by H63

This part of the theory is where H63 and LH50 most differ in their approach. While H63 views the problem as one of mode-coupling similar to the wave-wave interaction (Hasselmann, 1962) or wave-bottom reflection (Ardhuin and Magne, 2007), LH50 expresses the solution of the stationary displacement field using Green's functions.

Because the H63 approach is one of radiative transfer (computing the change of energy of wave modes as they propagate through a medium), it includes, ab initio, the effect of refraction and shoaling (the change of energy density related to a change in group velocity: like waves entering shallow water). It also easily includes the effect of the Earth sphericity: on a curved surface the energy balance equation of the wave modes contains a pseudo-refraction term due to the turning of the geodesics relative to a fixed direction. Obviously the two approaches are equivalent in stationary conditions.

Let us consider the situation of a flat Earth with a stationary wave forcing and without damping. In this case, the seismic spectrum has accumulated energy as the seismic wave packets propagated through the region of forcing (H63, figure 1). The generic energy balance for mode n given by H63-(1.13) reduces to H63-(1.15). For the vertical ground displacement spectrum of mode n, this is,

$$F_\delta^{(n)}(\omega, \theta) = \int \widetilde{T}_\delta^{(n)} F_p(\mathbf{K}_n, \omega) ds, \tag{20}$$

where s is the coordinate along the seismic ray, and $\widetilde{T}_\delta^{(n)}$ is a transfer function, detailed below, that transforms the equivalent surface pressure into vertical ground displacement.

$$F_\delta^{(n)}(\omega) = \int \int \widetilde{T}_\delta^{(n)} F_p(\mathbf{K}_n, \omega) ds d\theta \tag{21}$$

For a wave forcing confined to a finite domain of area Λ and no wave forcing outside of it, we can re-arrange the integral as a sum over the area,

$$F_\delta^{(n)}(\omega) = \int_\Lambda \widetilde{T}_\delta^{(n)} F_p(\mathbf{K}_n, \omega)/R(x,y) dx dy, \tag{22}$$

where R is the distance between the observation point where $F_\delta^{(n)}(\omega)$ is measured, and the location of the source $F_p(\mathbf{K}_n, \omega)$.

If the forcing is homogeneous in space and confined to a small area, then $F_p(\mathbf{K}_n, \omega)/R(x,y)$ is constant and one gets eq. H63-(1.17).

Now, the displacement transfer function $\widetilde{T}_\delta^{(n)}$ is related to the amplitude $\left|G_\delta^{(n)}\right|$ of the Green function for mode (n), namely,

$$\widetilde{T}_\delta^{(n)} = \left[2\pi a_\delta^{(n)}\right]^2 = \left[2\pi a_\delta^{(n)}\right]^2 = \left[2\pi \sqrt(R) \left|G_\delta^{(n)}\right|\right]^2 = R \left[2\pi W^{(n)}\right]^2 \tag{23}$$

where $W^{(n)}$ is the W of LH50 for mode (n) only, given by eq. LH50-(183). For many modes the coefficient c is simply the root mean square of the various c_n. On a flat Earth we thus have

$$W^2 = \frac{\omega c^2}{2\pi \rho_s^2 \beta^5 R} \tag{24}$$

where R is the distance between the seismic source and the recording station, and c is a coefficient that combines all the modes.

As a result, eq. H63-(1.17) can be re-written as

$$F_\delta(\omega) = \Lambda W^2 \rho_w^2 g^2 \omega \int_0^\pi E(f,\theta) E(f,\theta+\pi) \mathrm{d}\theta, \tag{25}$$

If the wave forcing is not homogeneous over the area Λ, one can simply revert the the more general form (22) with the integral. This will be re-written below for the case of a spherical Earth.

3.2 Comparison with LH50

LH50 considered the case of two interacting wave groups with their energies E_1 and E_2 uniformly spread over the spectral domains Ω_1 and Ω_2. In this case the spectral densities are $E(f,\theta) = E_1/(\Delta_{f1}\Delta_{\theta 1})$ for the Ω_1 domain and $E(f,\theta) = E_2/(\Delta_{f2}\Delta_{\theta 2})$ for Ω_2. This is true if the spectrum is *one-sided*. This is the case in LH50 due to the normalization LH50-(188), which says that all the variance of the sea surface elevation is contained in the full wavenumber plane.

We can define the microseismic power spectral density by taking LH50-(198) and divide it by Δ_ω, giving,

$$F_\delta(\omega)_{LH50} = \frac{|\delta|^2}{2} \times \frac{1}{\Delta_\omega} = \tag{26}$$

$$= 32\Lambda \rho_w^2 W^2 \pi^2 E_1 E_2 \sigma^4 \frac{\Omega_{12}}{\Omega_1 \Omega_2 \Delta_\omega} \tag{27}$$

Taking $\Delta_f = \Delta_{f1} = \Delta_{f2} = \Delta_\omega/(4\pi)$, and $\Delta_\theta = \Delta_{\theta 2} = \Delta_{\theta 1}$, with the Ω_1 and Ω_2 domains perfectly symmetric, we have the areas of the spectral domains $\Omega_1 = \Omega_2 = \Omega_{1,2} = k\Delta_k\Delta_\theta$. In deep water the wave dispersion relation is $\sigma^2 = gk$ which, by differentiation gives $2\sigma\Delta_\sigma = g\Delta_k = 4\pi\sigma\Delta_f$. This gives,

$$F_\delta(\omega)_{LH50} = \frac{\delta^2}{\Delta\omega} = 32\Lambda \rho_w^2 W^2 \pi^2 E_1 E_2 \sigma^4 \frac{1}{k\Delta k \Delta_\theta 4\pi \Delta f} \tag{28}$$

$$= 32\Lambda \rho_w^2 W^2 \pi^2 E_1 E_2 \frac{g\sigma^4}{k 16\pi^2 \sigma \Delta_\theta \Delta_f^2} \tag{29}$$

$$= \Lambda W^2 \rho_w^2 g^2 2\sigma E_1 E_2/(\Delta_\theta \Delta_f^2) \tag{30}$$

We can see that this is the same as the solution according to H63, and re-written as (25). Indeed, in this case we have $E(f,\theta) = E_1/\Delta_\theta\Delta_f$ in the spectral region Ω_1 and $E(f,\theta+\pi) = E_2/\Delta_\theta\Delta_f$ in the spectral region Ω_2, hence

$$E_1 E_2/(\Delta_\theta \Delta_f^2) = E(f,\theta) E(f,\theta+\pi) \mathrm{d}\theta. \tag{31}$$

3.3 Solution on the spherical Earth

As stated by H63, the spectral densities of seismic waves in **k**-space are not changed by propagation, this is also a well known property for ocean waves (Longuet-Higgins, 1957). We can thus write the the directional integration of the local seismic spectrum at point O, represented by $F_e(k,\theta)$ as an integral over the source area. For each direction θ, the directional spectrum is the sum of the sources along the seismic ray that arrives at O from direction θ, and which has a direction θ' at the source points P. We assume a stationary seismic source in H63's eq. 1.14, and a constant spatial decay rate μ for the seismic energy[4]. Assuming that $\mu < 1/(\pi R_E)$, with R_E is the Earth radius, we can neglect the accumulation of the sources and sum over only one circumference,

$$F_e(k,\theta) = T_e(k) \int_0^\pi F_p(k,\omega,\Delta',\theta') \mathrm{e}^{-\mu R_E \Delta} \frac{R_E}{U} \mathrm{d}\Delta. \tag{32}$$

[4]Without decay, the seismic energy would not be stationary and increase with time as the waves go several times round the Earth.

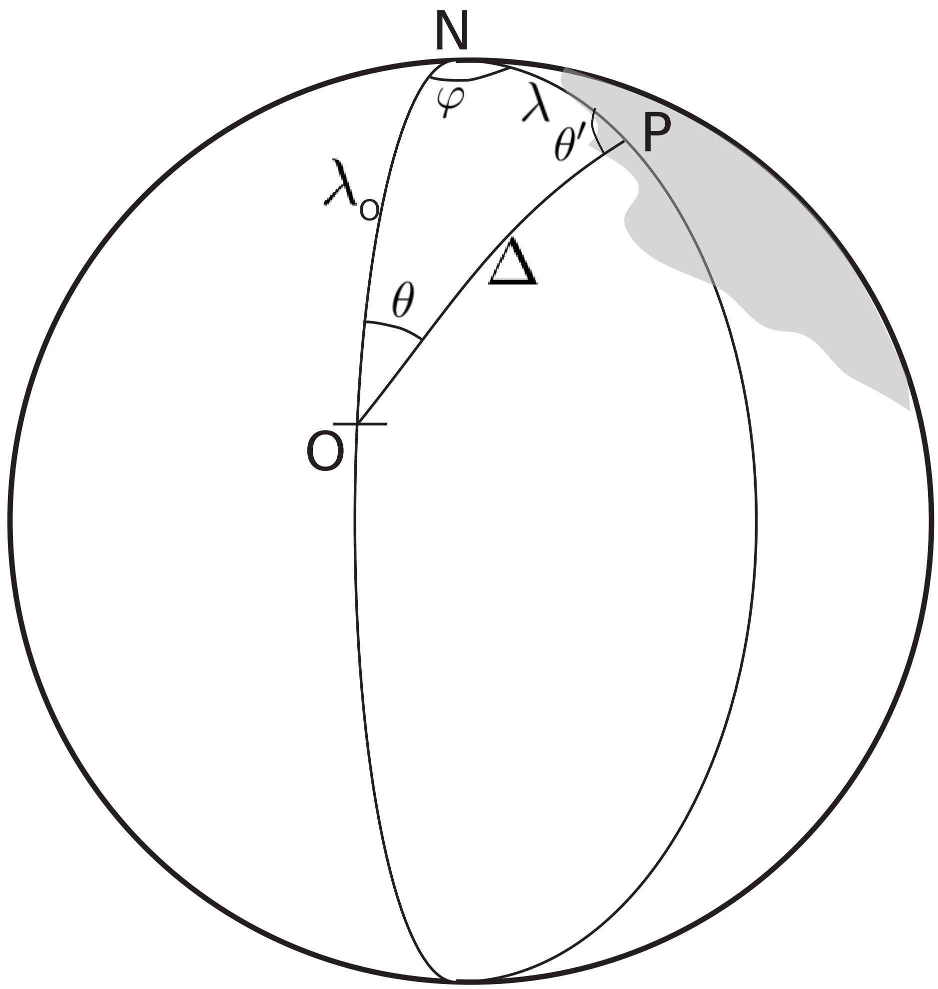

Figure 2: Geometry of the generation area (shaded) and observation conditions. Any point P of colatitude λ and longitude φ inside of the storm, generates seismic waves that are observed at point O of colatitude λ_O and longitude 0. The observed seismic waves that come from P have direction θ at P, relative to the North, which gives a direction $\pi - \theta'$ at O. In the triangle OPN the angles λ, θ' and φ are related to the distances Δ, λ_O and λ by the usual spherical trigonometry relationships.

Now the sum over all directions at O gives,

$$\int_0^{2\pi} F_e(k,\theta)\mathrm{d}\theta = T_e(k) \int_0^{2\pi} \int_0^{\pi} F_p(k,\omega,\Delta',\theta')\mathrm{e}^{-\mu R_E \Delta} \frac{R_E}{U} \mathrm{d}\Delta \mathrm{d}\theta \tag{33}$$

where Δ is the angular distance between the source point and the observation point, and U is the group speed of the seismic waves. Since the local source F_p is isotropic, it does not depend on the angle θ'.

So far the integral is similar on a flat or spherical Earth. On a spherical ocean surface the elementary area is $dA = R_E^2 |\sin\varphi| \mathrm{d}\varphi \mathrm{d}\lambda$, when using the usual (λ, φ) coordinates, but it is also equal to $dA = R_E^2 |\sin\Delta| \mathrm{d}\theta \mathrm{d}\Delta$, when using using coordinates defined with a pole on O instead of N. This equality thus gives,

$$\int_0^{2\pi} F_e(k,\theta)\mathrm{d}\theta \quad = \quad T_e(k) \int_{\text{Ocean}} F_p(k,\omega,\lambda,\varphi) \frac{\mathrm{e}^{-\mu R_E \Delta} R_E}{U \sin\Delta} dA \tag{34}$$

On a flat Earth we would have obtained (H63-eq. 1.17),

$$\int_0^{2\pi} F_e(k,\theta)\mathrm{d}\theta \quad = \quad T_e(k) \int F_p(k,\omega,\varphi,\theta) \frac{\mathrm{e}^{-\mu R}}{UR} dA \tag{35}$$

with R the distance between O and P. The difference between the two expressions is simply a factor $\Delta/\sin(\Delta)$, in the integrand, which goes to 1 in the limit of small spherical distances Δ' (the Earth is indeed locally flat). This correction factor gives a singularity at the exact antipode, but, because the storm is not a point source, the measure of the sources that contribute to this singularity is zero and the seismic response is always finite. However, when estimating this expression on a discretized Earth surface, one has to be careful to properly treat this singularity int the integrand.

In the absence of seismic wave attenuation and for a spherical Earth one has

$$F_\delta(\omega) = \rho_w^2 g^2 \omega \int \widetilde{W}^2 \int_0^{\pi} E(f,\theta)E(f,\theta+\pi)\mathrm{d}\theta\mathrm{d}A, \tag{36}$$

where $\widetilde{W}^2 = W^2 R/R_E \sin\Delta$, namely,

$$\widetilde{W}^2 = \frac{2\sigma c^2}{2\pi\rho_s^2\beta^5 R_E \sin\Delta}. \tag{37}$$

4 Practical use of the theory

The necessary ingredients fore estimating the seismic spectrum are clearly:

- the directional wave spectrum $E(f,\theta)$ at every point of the ocean. This can be replaced by a knowledge of $F_p(\mathbf{K} \simeq 0,\omega)$ only, which considerably reduces the volume of data to be handled. In practice we have chosen to compute $F_p(\mathbf{K} \simeq 0,\omega)$ inside of the wave model so that only this quantity is stored as an output of the wave model. To be more precise we store $F_p(\mathbf{K} \simeq 0,\omega)/(\rho_w^2 g^2)$ which has units of m^4, and we store it on the grid of the wave model frequencies. In order to save disk space, only the frequencies between 0.04 and 0.17 Hz have been stored, corresponding to seismic frequencies 0.08 to 0.34 Hz. We also recall that our wave model uses a geometrically increasing frequency grid, with $f_{i+1} = 1.1 f_i$.

- The spatial damping coefficient μ. Seismologists usually take $\mu = 2\pi f_s/(2UQ)$ with f_s the seismic wave frequency, U the group speed of the seismic waves, and Q a non-dimensional quality factor. For seismic waves in the frequency range 0.1–0.6 Hz, Q may vary between 40 and 1200, while U is of the order of 2 km s^{-1} depending on crust properties (Mooney et al., 1998).

Here we describe the most simple seismic noise model which assumes uniform damping rates given by $V_g Q$. A variable damping / scattering of seismic waves can also be used, but this requires to introduce a sink term in eq. (20).

The computation of

$$\widetilde{F_p}(f) = \left[F_p(\mathbf{K}=0,f_2)\right]/(\rho_w^2 g^2) \tag{38}$$

is performed by the routine `w3iogomd.ftn` and was introduced in version 3.14-Ifremer currently developed in collaboration between Ifremer and NOAA/NCEP. This has now been merged in the trunk of the development version at NCEP and should thus be part of the next release of the code; The model result is post-processed in NetCDF format using `ww3_ounf.ftn`, which generates files named `*_p2f.nc`. The `Fp3D` variable stored in these files corresponds to the integral,

$$Fp3D(\lambda,\phi,f,t) = f_2 \int_0^{\pi} E(f,\theta)E(f,\theta+\pi)\mathrm{d}\theta. \tag{39}$$

NetCDF is a common self-describing format for Earth System data and can be read with many different tools, including Matlab. Reading functions are provided at the following URL: `http://tinyurl.com/2wqwk89`. On regular latitude-longitude grids, the pressure spectrum takes the form of a three-dimensional variable[5] $Fp3D(i,j,l,m)$ where i is the longitude index, j, is the latitude index, l is the frequency index, and m is the time index. As normalized, the variance of the second order pressure forcing is simply given by the sum of $Fp3D(i,j,l,m)\Delta f(l)$, with $\Delta f(l) =$`freq(l)*0.5*(1.1-1/1.1)`.

Because these files contain one field for each frequency, they are large. Even stored as short integers with a scale factor, one month of a global grid at 0.5 degree resolution every three hours and with 15 frequencies takes 1.6 Gb. Fortunately, the sources are very sparse and these files compress very well (60 Mb). The latest version of the NetCDF format (version 4) supports compression and tests are underway to produce the noise source files in this format. Unfortunately, only the latest version of Matlab supports NetCDF4 (Matlab version 2010b) whereas NetCDF4 is supported by IDL versions 8.0 and following.

For practical application (see below) one may use the frequency-integrated result $\widetilde{F_p}(\mathbf{K}=0)$ given in the smaller *_p2s.nc file, together with T_{ps} the frequency of the local maximum of $\widetilde{F_p}(f_s)$. If one assumes a narrow spectrum for the pressure forcing term, then the integral over the frequencies can be restricted to $\widetilde{F_p}(f_s)$, giving,

$$
\begin{aligned}
< \delta^2 > &= \int\int_{f_{\mathrm{s,min}}}^{f_{\mathrm{s,max}}} \left(\frac{g\rho_w}{\rho_s}\right)^2 \frac{\sigma c^2}{\beta^5 R_E \sin\Delta} \widetilde{F_p}(\mathbf{K}\simeq 0, f_s)\mathrm{d}f_s \mathrm{e}^{-\mu\Delta R_E}\mathrm{d}f_s\mathrm{d}A, & (40)\\
&\simeq \int \left(\frac{g\rho_w}{\rho_s}\right)^2 \frac{\sigma_p c(f_p)^2}{\beta^5 R_E \sin\Delta} \widetilde{F_p}(\mathbf{K}\simeq 0)\mathrm{e}^{-\mu\Delta R_E}\mathrm{d}A. & (41)
\end{aligned}
$$

5 Example calculation and comparison with Kedar (2008)

Using the very good example detailed by Kedar et al. (2008), we will outline here the result from the LH50 and H63 approaches. Here the wave model result are taken as spectra from the location 53 N 42 W in the global 0.5 degree WAVEWATCH III model configuration that is described as TEST441b in Ardhuin et al. (2010). The spectrum file is at `ftp://ftp.ifremer.fr/ifremer/cersat/products/gridded/wavewatch3/pub/HINDCAST/GLOBAL05_NOC/2004/SPEC/mww3.W042N53.spec`. One example spectrum is shown in figure (3).

The spectral density of the equivalent surface pressure (ESP), $F_p(\mathbf{K}\simeq 0,\omega)$, is given by eq. (12). Following H63, this is expected to be white in wavenumer space and only the variation with ω matters, since the spectral density that drives the microseisms will naturally be selected at the $\mathbf{k}$ that do match an ω of an acoustic or Rayleigh wave mode. We can thus plot the ω dependence of this function, which is done in figure (4). Most of the ESP is contained in the domain of periods from 5 to 6 s.

Now, we can integrate across the seismic frequencies, to get the spectral density in wavenumber space only, i.e.,

$$F_p(\mathbf{k}) = \int F_p(\mathbf{k},\omega)\mathrm{d}\omega = \int F_p(\mathbf{k},\omega)2\pi\mathrm{d}f \qquad (42)$$

which here gives, $F_p(\mathbf{k})/\rho_w^2 = 194$ m^6 s^{-4} rad^{-2}.

If we had used the LH50 approach, the equivalent quantity is

$$\widetilde{F_p}(\mathbf{k}) = 16\pi^2 E_1 E_2 \sigma^4 \frac{\Omega_{12}}{\Omega_1\Omega_2} \qquad (43)$$

We can get E_1 and E_2 from a partitioning of the spectra. Here the wind sea E_1 from the south-east has a variance $E_1 = 1.53$ m^2 and the swell from the north-west has a variance $E_1 = 0.57$ m^2. If

[5]In later versions the factor $\rho_w^2 g^2$ will likely be included in the variable in order to provide quantities in Pa^2 m^2 s, more exotic but normal for a spectral density of pressure in three dimensions.

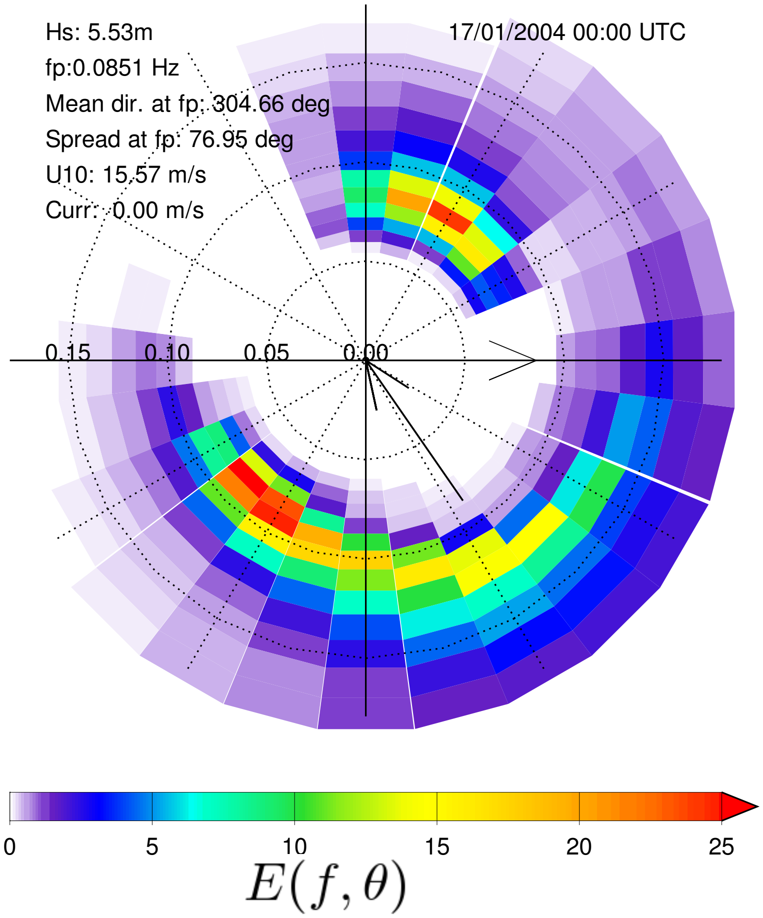

Figure 3: Wave spectrum used in the example, corresponding to the same time and location as the one showed by Kedar et al. (2008) in their figure 2.a.i. The units for the colorbar are m^2/Hz. The two arrows show the wind and current direction. Because no current is used in this wave model, the current arrow points to the East. The wind speed is given on the plot.

we had taken a uniform spectral density over the wave frequency range 0.08 to 0.13 Hz, this gives us wavenumbers $k_1 = (2\pi \times 0.08)^2/g = 0.0258$ rad/m, and $k_1 = (2\pi \times 0.13)^2/g = 0.0680$ rad/m. Assuming that the energy is evenly distributed over a 180 degree sector, the value of the area in (k_x, k_y) space is $\Omega_{12} = \Omega_1 = \Omega_2 = \pi \left(k_2^2 - k_1^2\right) = 0.0125$ rad^2/m^2, which eventually gives $F_p(\mathbf{k})/\rho_w^2 = 1600$ m^6 s^{-4} rad^{-2}. This is 8 times the value calculated here, which shows the importance of using a full directional spectrum.

What was actually done by Kedar et al. (2008) is to replace this simple interaction of the two wave trains by the actual integral over the entire spectrum, consistent with the H63 expression. Doing this, the areas in the spectral domain Ω_1, Ω_2 and Ω_{12} are all equal. As a result the factor $\Omega_{12}/\Omega_1\Omega_2$ can be rewritten as as $1/\Omega_{12}$. If we correct a few typos in K08[6], their equation (1.6) is,

$$\delta^2(\omega) = 64\pi^2\sigma^4 \sum_i \left(\Lambda \frac{E_1 E_2}{\Omega_{12}}\right) \rho^2 W^2. \tag{44}$$

[6]There appears to be a missing π and the Ω_{12} term should be in a denominator.

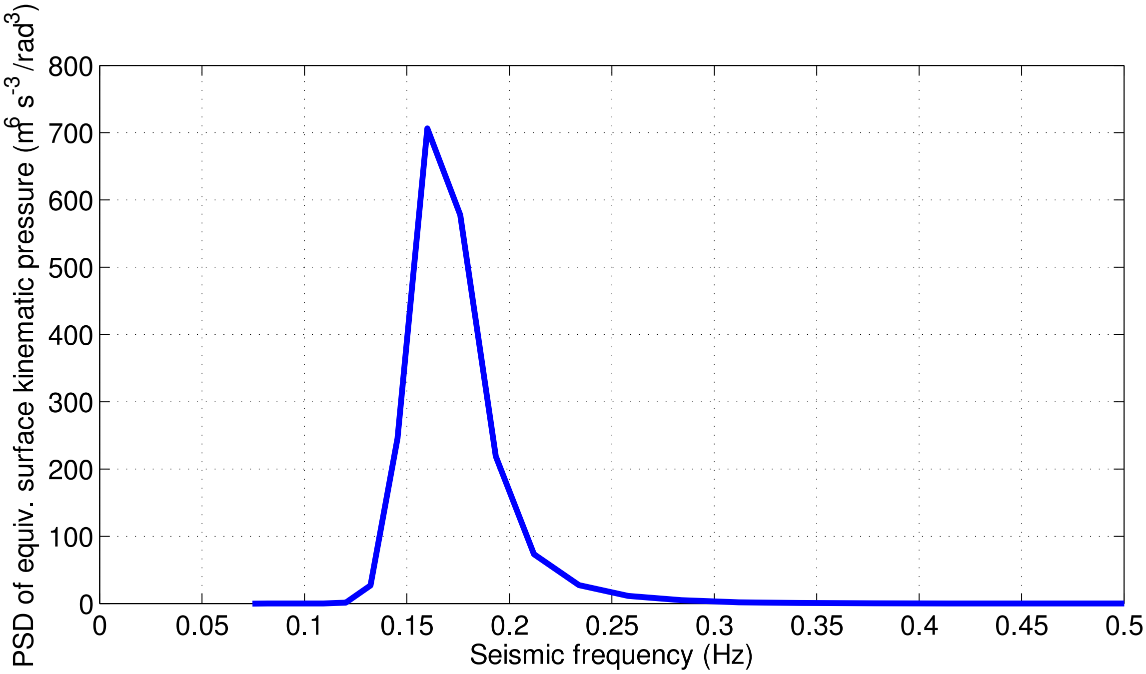

Figure 4: Power spectral density (single-sided) of the equivalent kinematic surface pressure, $F_p(\mathbf{K} \simeq 0, \omega)/\rho_w^2$.

Using our notations, this becomes a sum over the discretized wave spectrum.

$$\delta^2(\omega) = 64\pi^2\sigma^4 W^2 \rho_w^2 \Lambda \sum_i \frac{E(f,\theta_i)E(f,\theta_i+\pi)\Delta_f^2\Delta_\theta^2}{\Omega_{12}}. \tag{45}$$

We can replace Ω_{12} by $k\Delta_k\Delta_\theta$, which is, in deep water, $\Omega_{12} = k\Delta_\theta 2\pi\Delta_f/C_g = 4\pi\sigma^3\Delta_\theta\Delta_f/g^2$, giving,

$$\delta^2(\omega) = 8\pi\sigma g^2 W^2 \rho_w^2 \Lambda \sum_i E(f,\theta_i)E(f,\theta_i+\pi)\Delta_f\Delta_\theta. \tag{46}$$

and replacing the sum in the limit of a continuous spectrum this is,

$$\delta^2(\omega) = 8\pi\Delta_f\sigma g^2 W^2 \rho_w^2 \Lambda \int E(f,\theta_i)E(f,\theta_i+\pi)\mathrm{d}\theta. \tag{47}$$

which gives, according to K08, the spectral density of ground displacement, $F_\delta(\omega) = \delta^2(\omega)/\Delta_\omega$, where $4\pi\Delta_f = \Delta_\omega$,

$$F_\delta(\omega) = \omega g^2 W^2 \rho_w^2 \Lambda \int E(f,\theta_i)E(f,\theta_i+\pi)\mathrm{d}\theta. \tag{48}$$

which is exactly the expression in eq. (25) given by H63 for a steady wave field. Thus K08 actually used H63 theory without knowing that he does so. Obviously his use of a flat Earth solution for W is not quite appropriate. On the sphere, the distance D on the sphere should be replaced by $R_E \sin\Delta$ where R_E is the Earth radius and Δ is the spherical distance.

Anyway, one can start looking at time series as shown in figure 5. If the sources were uniform over the ocean we should get the second half on the time series in K08 (their figure 6). The result is somewhat sensitive to the integration range for the seismic frequencies. Taking the same range as K08 (3 to 7 s

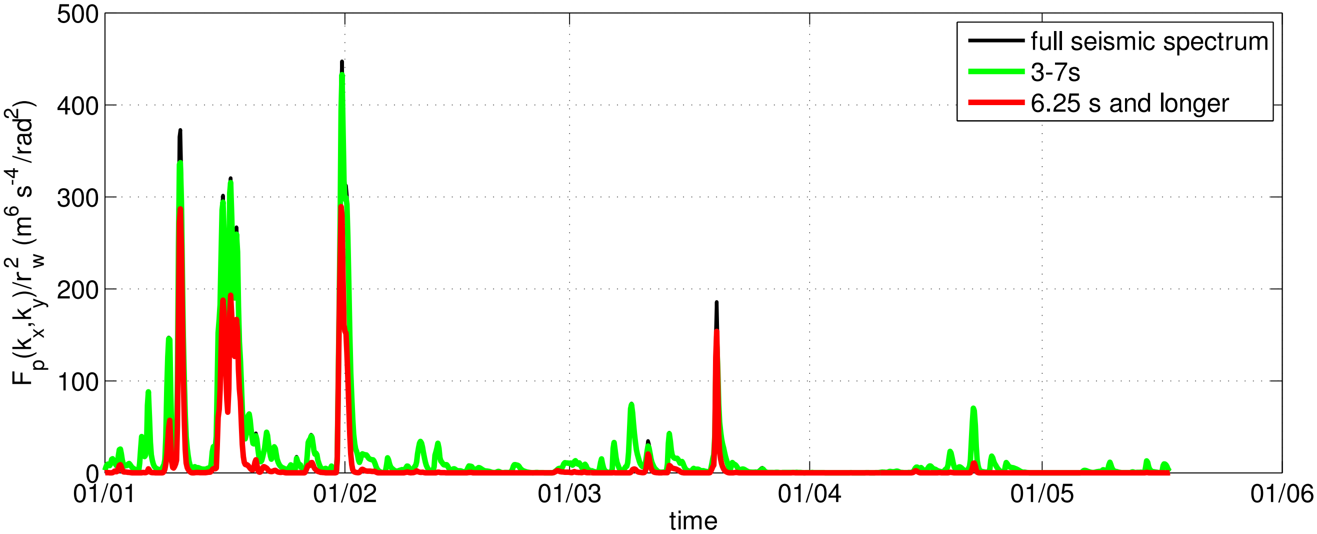

Figure 5: Time series of $F_p(\mathbf{k})/\rho_w^2$.

for the wave periods) gives similar peaks. However the magnitude seems to be off. Indeed, the variable[7] $\Psi = E_1 E_2/\Omega_{12}$, is only of the order of 8.5 m^6 for the wave spectrum showed in figure 3. The map of $F_p(\mathbf{k})/(\rho_w g)^2$ suggests that the value of $F_p(\mathbf{k})/(\rho_w g)^2$ is not larger than 5 m^4 at this same time. For an average value of $\sigma = 2\pi/10$ this gives $\Psi = F_p(\mathbf{k})/(16\sigma^4) = 192$ m^6. The values of Ψ up to 1000 m^2 in K08, seem rather high compared to the estimates given here.

A map of the wave peak period contribution to $F_p(\mathbf{k})$ is added in figure 7. This gives an indication of the range of important wave period for deep-water microseismic generation, and their spatial variability.

6 Practical use of p2s or p2l files for the computation of the seismic response

Here we present the practical calculation of the vertical ground displacement variance E_v at a location with latitude λ and longitude ϕ. This variance is given by the integration of the frequency spectrum, as estimated from a seismometer. Because the seismic response is computed from the wave spectra, we choose to use the wave frequency f as the spectral coordinate (and not the seismic frequency $2f$)

$$E_v = \int_{f_1}^{f_2} F_v(f) \mathrm{d}f \tag{49}$$

f_1 and f_2 are the integration bounds.

The seismic spectrum due to the wave double frequency mechanism is given in the form

$$F_\delta(f) = \int S(f,\lambda',\phi') A(f,\lambda',\phi') \sin(\lambda') \mathrm{d}\lambda' \mathrm{d}\phi' \tag{50}$$

where S is the local source, and A is an attenuation factor.

[7]Here again we have corrected for a likely typo in K08, namely there should be a division by Ω_{12} and not a multiplication in order to get the right units of m^6 for Ψ.

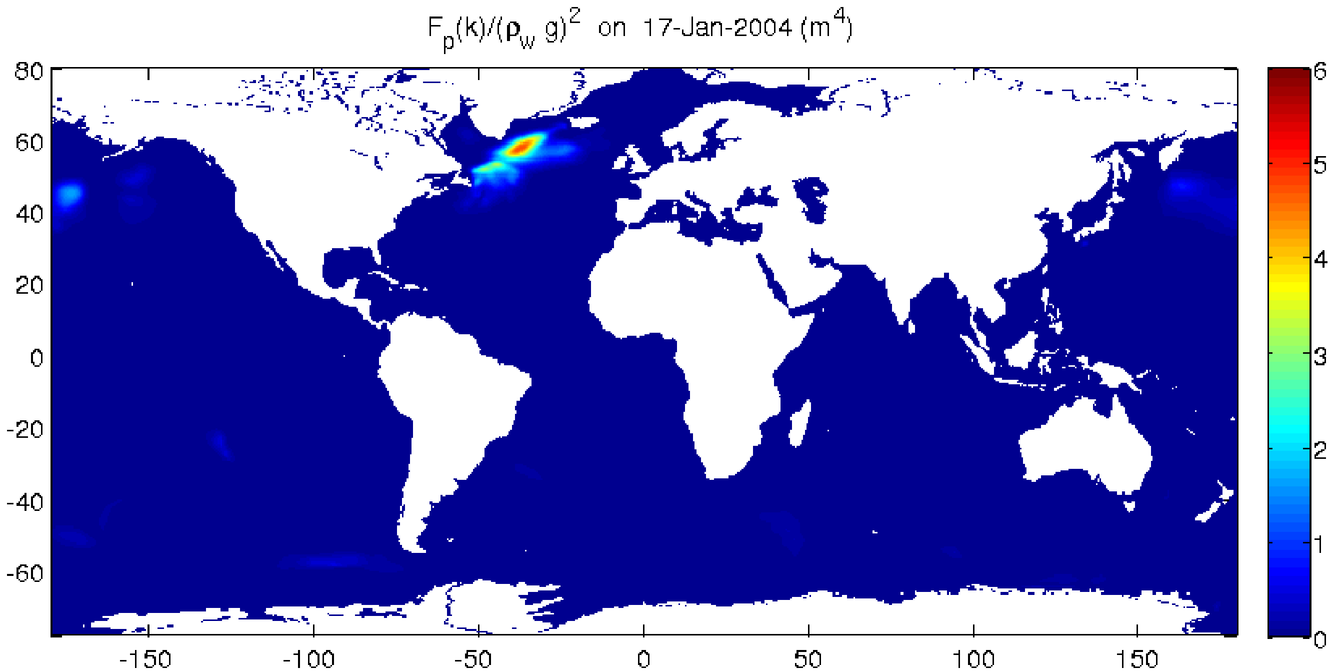

Figure 6: Map of $F_p(\mathbf{k})/(\rho_w g)^2$.

6.1 Calculation from p2l files

The integration of sources on the sphere, including attenuations can be performed with the matlab script `seismic_response_spectral_from_logmap_v4.m` which can also be found at this URL: `http://tinyurl.com/iowagaftp/pub/TOOLS/MATLAB`. This scripts will read the $\widetilde{F}_p(f)$ from the netCDF files and combine them to produce seismic spectra at any location.

References

Ardhuin, F. and T. H. C. Herbers, 2002: Bragg scattering of random surface gravity waves by irregular sea bed topography. *J. Fluid Mech.*, **451**, 1–33.

Ardhuin, F. and T. H. C. Herbers, 2012: Double-frequency noise generation by surface gravity waves in finite depth: gravity, acoustic and seismic modes. *J. Fluid Mech.*, URL `http://hal.archives-ouvertes.fr/hal-00711326/`, in press.

Ardhuin, F. and R. Magne, 2007: Current effects on scattering of surface gravity waves by bottom topography. *J. Fluid Mech.*, **576**, 235–264.

Ardhuin, F., E. Stutzmann, M. Schimmel, and A. Mangeney, 2011: Ocean wave sources of seismic noise. *J. Geophys. Res.*, **116**, C09 004, doi:10.1029/2011JC006952, URL `http://wwz.ifremer.fr/iowaga/content/download/48407/690392/file/Ardhuin_etal_JGR2011.pdf`.

Ardhuin, F., et al., 2010: Semi-empirical dissipation source functions for wind-wave models: part I, definition, calibration and validation. *J. Phys. Oceanogr.*, **40 (9)**, 1917–1941.

Cooper, R. I. B. and M. S. Longuet-Higgins, 1951: An experimental study of the pressure variations in standing water waves. *Proc. Roy. Soc. Lond. A*, **206**, 426–435, doi:10.1098/rspa.1951.0079.

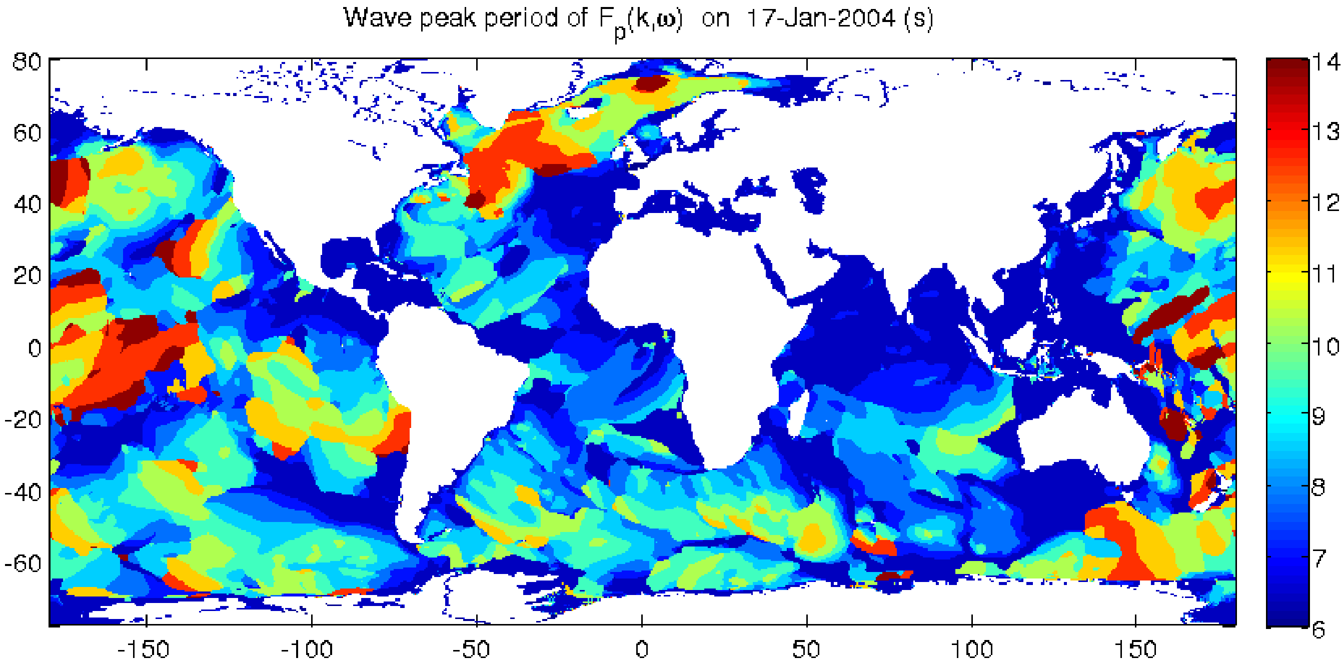

Figure 7: Map of the peak wave period contribution to $F_p(\mathbf{k})$.

de Laplace, P. S., 1776: Suite des recherches sur plusieurs points du système du monde (XXV–XXVII). *Mém. Présentés Acad. R. Sci. Inst. France*, 542–552.

Farrell, W. E. and W. Munk, 2008: What do deep sea pressure fluctuations tell about short surface waves? *Geophys. Res. Lett.*, **35 (7)**, L19 605, doi:10.1029/2008GL035008.

Hasselmann, K., 1962: On the non-linear energy transfer in a gravity wave spectrum, part 1: general theory. *J. Fluid Mech.*, **12**, 481–501.

Hasselmann, K., 1963: A statistical analysis of the generation of microseisms. *Rev. of Geophys.*, **1 (2)**, 177–210.

Herbers, T. H. C., S. Elgar, and R. T. Guza, 1995: Infragravity-frequency (0.005-0.05 Hz) motions on the shelf. part II: free waves. *J. Phys. Oceanogr.*, **25**, 1063–1079, URL `http://journals.ametsoc.org/doi/pdf/10.1175/1520-0485%281995%29025%3C1063%3AIFHMOT%3E2.0.CO%3B2`.

Kedar, S., M. Longuet-Higgins, F. W. N. Graham, R. Clayton, and C. Jones, 2008: The origin of deep ocean microseisms in the north Atlantic ocean. *Proc. Roy. Soc. Lond. A*, 1–35, doi:10.1098/rspa.2007.0277.

Longuet-Higgins, M. S., 1950: A theory of the origin of microseisms. *Phil. Trans. Roy. Soc. London A*, **243**, 1–35.

Longuet-Higgins, M. S., 1957: On the transformation of a continuous spectrum by refraction. *Proceedings of the Cambridge philosophical society*, **53 (1)**, 226–229.

Miche, A., 1944: Mouvements ondulatoires de la mer en profondeur croissante ou décroissante. Première partie. Mouvements ondulatoires périodiques et cylindriques en profondeur constante. *Annales des Ponts et Chaussées*, **Tome 114**, 42–78.

Mooney, W., G. Laske, and G. Masters, 1998: Crust 5.1: A global crustal model at 5 x 5 degrees. *J. Geophys. Res.*, **103**, 727–747.

Tanimoto, T., 2007: Excitation of normal modes by non-linear interaction of ocean waves. *Geophys. J. Int.*, **168**, 571–582, doi:10.1111/j.1365-246X.2006.03240.x.

Webb, S. C., 2007: The earth's 'hum' is driven by ocean waves over the continental shelves. *Nature*, **445**, 754–756, doi:10.1038/nature05536.