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**ORGINAL ARTICLE**

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# On the role of the Chezy frictional term near the shoreline

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**Abstract** The behavior of the Chezy frictional term near the shoreline has been studied in detail. An asymptotic analysis valid for water depths going to zero clearly shows that the use of such a term implies a non-receding motion of the shoreline. This phenomenon is induced by a thin layer of water which, because of frictional forces, remains on the beach and keeps it wet seaward of the largest run-up. However, the influence of such a frictional layer of water on the global wave motion is very weak and practically negligible for most of the swash zone flow dynamics. The existence of a non-receding shoreline has led to some clarifications on the role of some ad-hoc tools used in numerical models for the prediction of the wet/dry interface.

**Keywords** Coastal engineering · Chezy frictional term · Gravity waves

## 1 Introduction

Wave propagation in the nearshore has always fascinated coastal scientists, the main reason being the incredible variety of physical mechanisms characterizing such a phenomenon. Those mechanisms associated to human activities and interests are also of great practical importance; therefore, scientists focus their attention particularly on problems like sediment transport, shore protection, and water/structure interactions. For all the above-cited cases, the swash zone plays an important role.

It is well-established that a large part of the nearshore sediment transport takes place in the swash zone, especially because of longshore drift mechanisms (e.g., Brocchini [1]), and that such a transport is fundamentally driven by low frequency motion, much of which generated within the swash zone (e.g., Elfrink and Baldock [2], Pritchard and Hogg [3,4], Brocchini and Baldock [5]).

Moreover, a sound description of the swash zone features is of fundamental importance for the prediction of beach inundation, maximum run-up, dynamical forces at the shoreline (e.g., Luccio et al. [6], Pritchard and Dickinson [7]; Antuono and Brocchini [8]) and, consequently, for wave/structure interactions. In this context, the modeling of frictional forces is one of the main goals since they strongly influence the swash zone dynamics (Grimshaw and Kamchatnov [9]).

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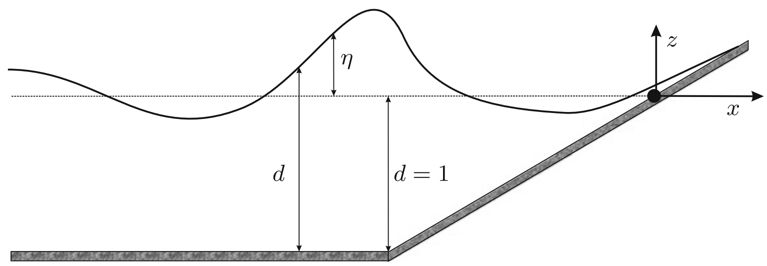


**Fig. 1** Sketch of geometry and flow dimensionless variables for the beach problem

According to the different kinds of equations used to describe the wave motion in the nearshore region, different schemes for frictional forces are used. The specific analysis that follows applies to the most widely used depth-averaged models, that is, the non-linear shallow water equations (NSWEs hereinafter) and the various Boussinesq-type equations available in the literature. However, since (see in the following) the ratio of frictional to gravitational forces increases as the water depth decreases, we focus on the model equations which better describe the wave dynamics for very small depths. We, thus, focus on the NSWEs even though the analysis applies to the Boussinesq-type equations as well.

The conservative and dimensionless NSWEs on a sloping frictional beach read:

$$\begin{cases} d_t + Q_x = 0, \\ Q_t + \left(\frac{d^2}{2} + \frac{Q^2}{d}\right)_x = h_x d - T_{\text{fric}} \end{cases} \tag{1}$$

where $Q = u\,d$ is the water volume flux, $d$ is the total water depth, $u$ is the onshore velocity, $h(x)$ gives the vertical position of the beach bottom, and $T_{\text{fric}}$ is the water–seabed frictional term. The axes origin is posed at the undisturbed shoreline; the $x-$coordinate gives the onshore direction and points in the landward direction, $(x, z)$ forming a right-handed Cartesian reference frame (see Fig. 1). The scale factors for vertical lengths, horizontal lengths, and times are, respectively, the still water depth at the seaward boundary $h_0$, $h_0/\alpha$, and $t_0 = \alpha^{-1}\sqrt{h_0/g}$, $\alpha$ being the beach slope. More details can be found in Brocchini and Peregrine [11]. For the analysis which follows, it is also useful to rewrite the system (1) in its non-conservative form:

$$\begin{cases} d_t + \left(ud\right)_x = 0, \\ u_t + uu_x + d_x = h_x - \frac{T_{\text{fric}}}{d} \end{cases} \tag{2}$$

where the usual dependence of the frictional term on $d^{-1}$ is shown.

The water–seabed frictional stresses $T_{\text{fric}}$ require some closure, i.e., they must be written as a function of the model variables which, for the mentioned equations, are the total water depth $d$ and a reference velocity $u$ (most often being the average over the water column of the flow). The analysis and the results provided in the present work hold true for a frictional law which has the following general structure:

$$T_{\text{fric}} = F(u, d) = \text{sign}(u)|F(u, d)|, \tag{3}$$

with:

$$\lim_{d \to 0} |F(u, d)| \not\equiv 0. \tag{4}$$

However, $T_{\text{fric}}$ is most often modeled through a special subclass of (3) so that it is taken as proportional to the square of the flow velocity through a Chezy-type friction factor. In other words a Chezy-type term (Bühler and Jacobson [10]) is used so that the system (1) becomes:

$$\begin{cases} d_t + Q_x = 0, \\ Q_t + \left(\frac{d^2}{2} + \frac{Q^2}{d}\right)_x = h_x d - C_f \frac{Q|Q|}{d^2}, \end{cases} \tag{5}$$

while the non-conservative system (2) reads:

$$\begin{cases} d_t + \left(ud\right)_x = 0, \\ u_t + uu_x + d_x = h_x - C_f \frac{u|u|}{d}. \end{cases} \tag{6}$$

With respect to this, we recall that the definition of the Chezy term is based on a rough-wall friction formulation for the case of uniform, steady motion [12]. This means that the flow is assumed to evolve through a sequence of steady and uniform states. Obviously, such hypotheses are rather restrictive. However, a large number of experimental and numerical studies testifies that the Chezy term gives both a simple and fairly good representation of the frictional forces not only in finite depth regions but also inside the swash zone (see, for example, Puleo and Holland [13]). These are the reasons that make the Chezy term to be widely used in depth-averaged nearshore hydrodynamics models leading to a very good agreement between experimental data and numerical results. Theoretical models of the swash zone dynamics also parameterized the seabed friction on the basis of a Chezy-type formulation (e.g., Archetti and Brocchini [14]). Finally, note that even if challenges are moved to the validity of a Chezy coefficient constant for both run-up and run-down phases, no major challenge is moved to the more general use of a velocity quadratic frictional term.

However, there is something which is, generally, hidden in each experimental and numerical study: nobody can actually verify the behavior of the Chezy term *up to the shoreline*, i.e., where the water depth becomes zero. For example, in the experiments of Puleo and Holland [13], a threshold value for the water depth has been used. A similar approach was also used by Packwood and Peregrine [15], their explanation being that the threshold depth gives account of various small-scale phenomena due to surface tension, bed friction, etc. This is particularly important in the case of breaking waves whose water surface is tangential to the beach (e.g., Peregrine and Williams [16]). It is also worthy to recall that in all the numerical solvers for the NSWEs, some ad-hoc expedients are, usually, implemented (e.g., a thin layer of water, a threshold water depth, a slotted beach, etc.).

The difficulty in handling the Chezy frictional term at the shoreline is a rather intuitive matter because of its dependence on $d^{-1}$. The first attempt at a correct handling of frictional terms at the shoreline dates back to Whitham [17]. A further contribution to this topic has been given by Hogg and Pritchard [18] which studied the dam break over a horizontal frictional plane. They proved that the $d^{-1}-$singularity might be balanced by a singularity of the same strength generated by the spatial derivatives of the flux term inside the momentum equation. This allows for a regular non-trivial solution up to the front curve, that is, the curve separating the wet part of the domain from the dry one. Because of the geometry considered by Hogg and Pritchard [18], the front curve was always advancing and never reversed.

However, if the geometry of the problem is given by a sloping beach, one would expect the front curve (i.e., the shoreline) to move back and forth. On the other hand, we show that the use of the Chezy frictional term implies a solution in which the shoreline is always non-receding. This due to the fact that the balance between the $d^{-1}-$singularity and the spatial derivatives of the flux term inside the momentum equation is no longer possible. As a consequence, this leads to the generation of a layer of water on the beach bottom which keeps the beach wet seaward of the largest run-up location during the whole fluid evolution.

In any case, the non-receding shoreline predicted by the analytical solution seems to be somehow “non-physical” since the layer of water has only a weak influence on the global wave motion and, therefore, the non-receding shoreline is not representative of the actual fluid evolution.

The behavior described above leads to some considerations on the numerical schemes used for the prediction of the wave motion near the shoreline. In fact, the numerical solvers that use quadratic-type frictional laws, like the Chezy law, should provide numerical solutions similar to the analytical solution proposed in the present paper and, consequently, should predict a non-receding shoreline. However, this does not occur since the numerical solvers, because of their ad-hoc corrections, predict a “physical-like” evolution of the shoreline, that is, a shoreline that moves back and forth over the beach face. This is a rather delicate matter. A numerical solver should provide a solution very close to any analytical one of the same *model equations* (in the present case the NSWEs), no matter the analytical solution is physically sound or not. The request for a physical solution must be addressed by the *model equations* at hand, while numerical schemes just have to approximate it.

Then, the ad-hoc corrections used by the numerical solvers and the water depth threshold value used for the comparison with the experimental measurements somehow seem to represent better the physical evolution of the shoreline than what would occur by using a numerically exact modeling of the Chezy frictional term.

It is evident that two different issues arise together: the behavior of the quadratic (Chezy-type) frictional terms near the shoreline and the influence of the numerical correction at the shoreline on the actual solution of the model equations. In the following, we deal with the first problem by providing a theoretically sound and detailed discussion of the matter and we propose a brief insight on the second issue.

## 2 Asymptotic analysis

We want to study the behavior of the NSWEs near the shoreline, i.e., the line separating the wet region of the beach from the dry one. Then, denoting the shoreline position through the symbol $x_s(t)$, the kinematic condition at the shoreline implies:

$$d\big(x_s(t), t\big) \equiv 0, \quad \dot{x}_s = u\big(x_s(t), t\big). \tag{7}$$

Since $d > 0$ inside the whole domain and it is equal to zero only at $x_s(t)$, the behavior of the NSWEs near the shoreline can be investigated through an asymptotic analysis for $d \to 0$. However, since $d$ is a dependent variable of (5), such an approach can be very difficult. To overcome this problem, we introduce the variable $\xi = -x + x_s(t)$ and rewrite (5) in the following way:

$$\begin{cases} d_t + \dot{x}_s d_\xi - Q_\xi = 0, \\ Q_t + \dot{x}_s Q_\xi - \left(\frac{d^2}{2} + \frac{Q^2}{d}\right)_\xi = -h_\xi d - C_f \frac{Q|Q|}{d^2}. \end{cases} \tag{8}$$

To be consistent with (7), the following limits must hold:

$$\lim_{d\to 0} d(x,t) = \lim_{x\to x_s(t)} d(x,t) = \lim_{\xi\to 0} d(\xi,t) = 0, \quad \lim_{d\to 0} u(x,t) = \lim_{\xi\to 0} u(\xi,t) = \dot{x}_s, \tag{9}$$

and, consequently, we can study the behavior of the NSWEs for $\xi \to 0$ instead of $d \to 0$. Then, starting from (9), we assume the following asymptotic expansions to hold near the shoreline (see Hinch [19] for details on asymptotic expansions):

$$d(\xi,t) = A(t) f(\xi) + o(f), \quad u(\xi,t) = \dot{x}_s(t) + B(t) g(\xi) + o(g), \tag{10}$$

where $f$ and $g$ go to zero as $\xi \to 0$ and:

$$\lim_{\xi\to 0} \frac{o(f)}{f} = 0, \quad \lim_{\xi\to 0} \frac{o(g)}{g} = 0. \tag{11}$$

Since $d > 0$ inside the domain, it is $A(t) > 0$ and $f(\xi) \geq 0$. For what concerns the analysis which follows, it is simple to show that only the first order of the expansion in (10) needs to be considered.

### 2.1 The continuity equation

Substituting (10) inside the continuity equation and rearranging, we obtain the following leading-order equation:

$$\frac{\dot{A}}{AB} = f^{-1} \frac{d(fg)}{d\xi}. \tag{12}$$

Since the right-hand side only depends on $\xi$ and the left-hand side only depends on $t$, it is possible to split the equation using a separation constant $\lambda$:

$$\dot{A} = \lambda AB, \quad \frac{d(fg)}{d\xi} = \lambda f. \tag{13}$$

We focus on the second equation and integrate once to obtain:

$$g(\xi) = \frac{\lambda}{f(\xi)} \int_0^{\xi} f(s) ds. \tag{14}$$

Under general assumptions of regularity for $f$, the previous solution implies that:

$$g(\xi) = c_0 \xi + o(\xi), \quad \text{for } \xi \to 0, \tag{15}$$

in which $c_0$ is a constant. This result confirms that $g$ goes to zero for $\xi \to 0$.

### 2.2 The momentum equation

After the general relationship between $f$ and $g$ has been obtained, we analyze the momentum equation. We prove that the use of the Chezy frictional term implies that the only bounded solution at the shoreline is that of no motion (i.e., $\dot{x}_s(t) \equiv 0$). However, before proceeding to the analysis, it is useful to rewrite the momentum equation in the following non-conservative form:

$$Q_t + \dot{x}_s Q_\xi - d d_\xi - 2u Q_\xi + u^2 d_\xi = -h_\xi d - C_f u|u|. \tag{16}$$

Using (10), Eq. (16) becomes:

$$Q_t + \dot{x}_s Q_\xi - d d_\xi - 2\big(\dot{x}_s + gB + o(g)\big) Q_\xi + \big(\dot{x}_s + gB + o(g)\big)^2 d_\xi = -h_\xi d - C_f \dot{x}_s |\dot{x}_s|.$$

Dropping the higher-order terms (it is simple to show that they do not influence the final result) and simplifying, we get:

$$Q_t - (\dot{x}_s + 2gB) Q_\xi - d d_\xi + \big(\dot{x}_s + 2gB\big)\dot{x}_s d_\xi = -h_\xi d - C_f \dot{x}_s |\dot{x}_s|.$$

Finally, using the continuity equation in (8), the previous expression becomes:

$$Q_t - (\dot{x}_s + 2gB) d_t - d d_\xi = -h_\xi d - C_f \dot{x}_s |\dot{x}_s|. \tag{17}$$

Now, substituting (10) into (17), we get:

$$- f g \dot{A} B + \ddot{x}_s f A + f g A \dot{B} - f \dot{f} A^2 = - h_\xi f A - C_f \dot{x}_s |\dot{x}_s|. \tag{18}$$

Since $f$ and $g$ go to zero as $\xi \to 0$, while $C_f \dot{x}_s |\dot{x}_s|$ is, generally, different from zero, the only way to balance Eq. (18) is to assume:

$$\lim_{\xi \to 0} f \dot{f} = c_1, \tag{19}$$

where $c_1$ is a positive constant (easily shown, for example, by direct integration of (19) and knowledge of $\xi$ being positive). This is equivalent to assume that the first derivative of $f$ is singular as $\xi \to 0$. In the specific case, (19) implies:

$$f(\xi) = \sqrt{2c_1 \xi} + o(\sqrt{\xi}). \tag{20}$$

Substituting (20) in (14), we immediately get:

$$g(\xi) = \frac{2}{3} \lambda \xi + o(\xi), \tag{21}$$

confirming the validity of (15). Now, we consider the limit for $\xi \to 0$ of Eq. (18) and obtain:

$$A^2(t) = \frac{C_f}{c_1} \dot{x}_s(t) |\dot{x}_s(t)|. \tag{22}$$

Since the left-hand side is always positive, the only acceptable solution is that the shoreline is always non-receding (that is, $\dot{x}_s \geq 0$). Such a condition is, generally, satisfied by a flow evolving on a horizontal plane. In this case, the shoreline is always advancing (that is, $\dot{x}_s > 0$) and it is possible to find some explicit asymptotic solutions (see, for example, Hogg and Pritchard [18]). However, the behavior is rather different when describing a flow over a sloping beach. In fact, the shoreline cannot be always advancing (that is, it is not possible that $\dot{x}_s > 0$ for all $t \in \mathbb{R}$) and, simultaneously, cannot recede (that is, $\dot{x}_s \geq 0$ for all $t \in \mathbb{R}$). This implies that after a finite time and unless large enough waves force the flow move upper over the beach, the shoreline must be still, that is, there exists a time $t_0$ such that $\dot{x}_s = 0$ for $t \geq t_0$. As shown in the following section, this is related to the generation of a thin layer of water which, however, seems to have only a minor influence on the global motion evolution. In any case, before proceeding to the description of the thin layer, we complete the asymptotic expansion during the run-up stage.

### 2.3 The run-up

Let us focus on a single swash event and denote by $t_0$ the instant when the maximum run-up occurs. As a consequence, $\dot{x}_s > 0$ for $t < t_0$ and $\dot{x}_s = 0$ at $t = t_0$. Then, from (22), we get:

$$A(t) = \sqrt{\frac{C_f}{c_1}}\, \dot{x}_s(t), \tag{23}$$

and, using the expressions in (10) and (13), we, finally, obtain:

$$d(\xi, t) = \sqrt{2C_f \xi}\, \dot{x}_s(t) + o(\sqrt{\xi}) \qquad u(\xi, t) = \dot{x}_s(t) + \frac{2}{3}\frac{\ddot{x}_s(t)}{\dot{x}_s(t)}\xi + o(\xi). \tag{24}$$

The latter expression has a singularity when $\dot{x}_s = 0$, that is, when $t = t_0$. Anyway, this singularity is out of the domain of validity of the asymptotic expansion itself. In fact, the condition $\dot{x}_s > 0$ for $t < t_0$ implies $u > 0$ in a neighborhood of the shoreline. Using the first-order solution for $u$ in (24), this corresponds to:

$$u(\xi, t) = \dot{x}_s(t) + \frac{2}{3}\frac{\ddot{x}_s(t)}{\dot{x}_s(t)}\xi > 0 \quad \Rightarrow \quad \xi < -\frac{3}{2}\frac{\dot{x}_s^2(t)}{\ddot{x}_s(t)} \tag{25}$$

Note that the bound above is well-posed since $\ddot{x}_s(t)$ is negative during the run-up. Further, it implies that the region of validity of the asymptotic expansion in (24) reduces to the shoreline when $\dot{x}_s$ goes to zero (that is, when $t \to t_0$). After this instant, a different asymptotic behavior is expected.

### 2.4 The thin layer

Following the reasoning suggested in Sect. 2.2, let us assume $\dot{x}_s \equiv 0$ for $t \geq t_0$ (i.e., $x_s$ = constant). Then, $u = gB + o(g)$ and the momentum Eq. (16) becomes:

$$gf\dot{A}B + gfA\dot{B} - f\dot{f}A^2 - g^2\dot{f}AB^2 - 2g\dot{g}fAB^2 = -h_\xi fA - C_f gB|gB|. \tag{26}$$

Assuming $h_\xi > 0$ (this is not a restrictive hypothesis on a sloping beach), two possible expansions are admissible for $f$. In fact, fractional powers are not allowed for a balanced equation; similarly, no powers higher than 2 are admissible because the frictional term is of order $O(\xi^2)$. The possible expansions are:

$$f(\xi) = c_1\xi + o(\xi) \quad \text{or} \quad f(\xi) = c_1\xi^2 + o(\xi^2). \tag{27}$$

The two expansions can be seen to characterize two different known solutions of the NSWE: a linear dependence of $f$ with $\xi$ characterizes the Carrier and Greenspan [20] solution (with a finite water surface at the shoreline), while a quadratic dependence characterizes the Shen and Meyer [21] solution (with a vanishing water slope at the shoreline).

Moreover, the analysis shows that the former (linear at leading order) expansion leads to a balance between the hydrostatic pressure gradient term ($dd_\xi$) and the "beach source term" ($dh_\xi$). Further, this is the only admissible expansion if no frictional terms appear in the momentum equation. In more detail, using the former expansion and taking the limit of (26) for $\xi \to 0$ ($t \geq t_0$) leads to:

$$A = \frac{h_\xi(x_s)}{c_1} = \text{constant}, \tag{28}$$

which, anyway, is not admissible since the first-order terms inside the asymptotic expansions for $d$ and $u$ do not depend on $t$.

Conversely, for $t > t_0$ the second expansion in (27), i.e., that quadratic at leading order, leads to the balance between "beach source term" ($dh_\xi$) of the momentum equation and the frictional term ($C_f u|u|$), that is:

$$0 = -h_\xi(x_s)c_1 A(t) - C_f c_0 B(t)|c_0 B(t)|. \tag{29}$$

The only way to get an admissible solution is to assume $c_0 B(t) \leq 0$ for $t > t_0$, that is, a negative velocity in the neighborhood of the shoreline. This immediately implies:

$$A(t) = \frac{C_f c_0^2 B(t)^2}{c_1 h_\xi(x_s)}, \tag{30}$$

which, substituted into the first expression of (13), gives:

$$B(t) = \frac{2B(t_0)}{2 - 3c_0 B(t_0)(t - t_0)} \tag{31}$$

which shows that $c_0 B(t) < 0,\ \forall t > t_0$ provided that $c_0 B(t_0) < 0$. The solutions in (30) and (31) predict that both the water depth and the velocity go to zero as $t \to \infty$. This feature along with the constancy of the shoreline position proves the generation over the beach bottom of a layer of water which, because of the water draining by gravity down the beach face, becomes thinner and thinner, unless a new swash event occurs.

Due to the balance between the source term of the momentum equation and the frictional term, the asymptotic solutions for $d$ and $u$ are linked by the following relation:

$$d(\xi, t) = \frac{C_f}{h_\xi(x_s)} u(\xi, t)^2 \quad \Rightarrow \quad F_s = -\sqrt{\frac{h_\xi(x_s)}{C_f}}, \tag{32}$$

where $F_s$ is the Froude number at the shoreline, and the negative sign has been chosen in accordance with the hypotheses made above. This relation is somehow similar to the solution obtained by using the NSWEs in open channels under the hypothesis of uniform and steady flow.

## 3 A hidden problem

As stated before, we here mainly aim at a theoretical analysis on the role of the Chezy frictional term near the shoreline. However, a brief and general discussion on the numerical aspects of modeling the Chezy term is useful and needed. Indeed, one could ask why no evidence of the results shown in Sect. 2.2 has been found in numerical simulations. The answer is hidden inside the specific numerical scheme at hand. However, some general points can be fixed.

First, a large number of numerical solvers are based on splitting-type techniques which allow to separately treat the homogeneous part of the equations and the source term contributions (Bermudez and Vazquez-Cendon [22], Toro [23]). This approach is based on the assumption that the influence of the source terms propagates slower than the homogeneous ones. However, as a consequence of the results shown in Sect. 2.2, this might be not true near the shoreline where the frictional term is not negligible in comparison with the homogeneous part of the momentum equation and strongly influences the structure of the solution.

A further problem is due to the deficiency of numerical solvers to model exactly the shoreline motion. To this purpose, three different approaches are, generally, used: in a first instance, a thin film of water is assumed to cover the whole beach while, in the second case, a minimum depth $d_m$ is introduced such that for $d < d_m$ a dry-bed condition is imposed (see for example, the review work of Balzano [24] and Brocchini et al. [25]), and finally, a slotted beach can be used so that water is present also inside what should be a rigid, impermeable beach (e.g., Yuan et al. [26]).

In all cases, the numerical schemes mentioned above do not solve the NSWEs at the shoreline (that is, where the water depth is exactly zero) but stop the solution procedure at a finite water depth. As we show in the following section, this is the reason why they all predict a shoreline motion which, at a first glance, seems very realistic and "physically based". However, since such a motion is not in agreement with the exact analytical solution shown in Sect. 2.2, it means that the mentioned numerical approaches significantly alter the structure of the NSWEs solution at the shoreline.

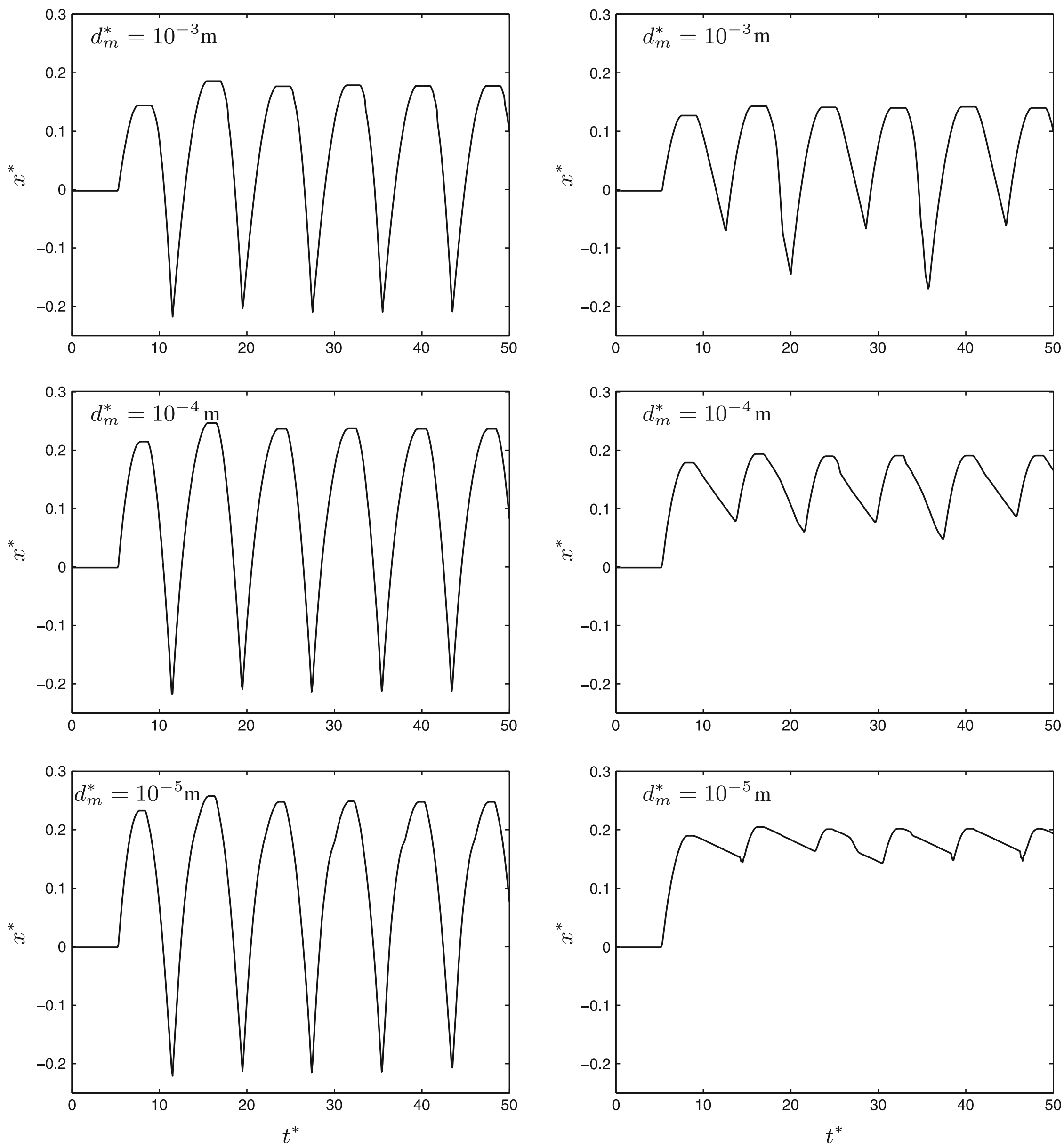


**Fig. 2** The shoreline evolution for periodic waves. Still water depth at the seaward boundary, $h_0^* = 1$ m, wave height at the seaward boundary, $H^* = 0.1$ m; wave period $T^* = 8$ s, beach slope, $\alpha = 0.1$. *Left column* $C_f = 0$. *Right column* $C_f = 0.01$. From *top to bottom*, $d_m^* = (10^{-3}, 10^{-4}, 10^{-5})m$

For the sake of brevity, we focus on one numerical solver available to use, and representative of the most widely used method for the shoreline evolution, which makes use of the threshold depth ($d_m$) approach to carry out the wetting and drying procedure. Figure 2 shows the shoreline evolution obtained through the NSWEs of Brocchini et al. [25]. This is a "shock-capturing" finite volume solver which employs a second-order "Weighted Averaged Flux" method (e.g., Toro [27]).

Periodic waves (wave height at the seaward boundary, $H^* = 0.1$ m; wave period $T^* = 2$ s, dimensional variables are starred in the following) have been propagated on a sloping beach with slope $\alpha = 0.1$ and offshore water depth $h_0^* = 1$ m. The grid was made of regular meshes of dimensions of 0.01 m.

The left panels of Fig. 2 show the shoreline evolution on a frictionless beach ($C_f = 0$) for $d_m^* = (10^{-5}, 10^{-4}, 10^{-3})$ m (from the bottom to the top). It is evident that only small changes occur in the shoreline shape. In the specific, as $d_m^*$ is decreased, the maximum run-ups slightly increase while the run-up peaks become more rounded. In all the cases, the run-up and run-down evolution is almost symmetrical.

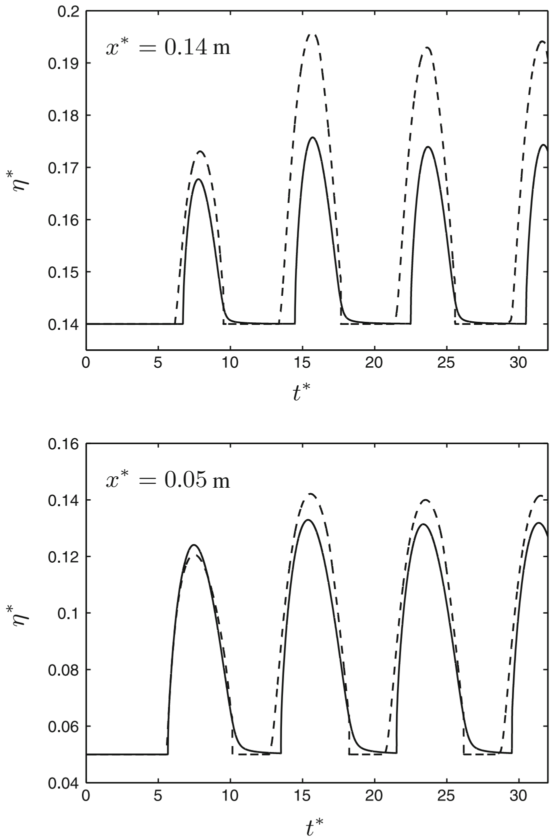


**Fig. 3** The free-surface evolution at fixed positions for $C_f = 0$ (*dashed lines*) and $C_f = 0.01$ (*solid lines*). The dimensional parameters are the same as in figure 2 and $d_m^* = 10^{-5}$ m

On the contrary, the shoreline motion radically changes when we introduce some seabed friction (right panels of Fig. 2). In this case, even if we have chosen a relatively small frictional coefficient (i.e., $C_f = 0.01$), the pattern of the run-downs is immediately altered. Further, a reduction of the threshold depth from $10^{-3}$ m to $10^{-5}$ m rapidly leads to a saturation of the run-downs while, on the contrary, the maximum run-ups are almost unaltered.

This is perfectly in agreement with the asymptotic analytical solution we have found in Sect. 2. Indeed, a shoreline motion can only occur when the shoreline is advancing (that is, when $\dot{x}_s > 0$). This is confirmed by the fact that all the initial run-ups of the simulations shown in the right panels of Fig. 2 are almost identical, notwithstanding the change in the critical depth. On the contrary, the run-down motion appears to be somehow impeded as $d_m^*$ is decreased. This is due to the fact that, for $d_m^*$ going to zero, no receding shoreline motion is allowed for and, therefore, the shoreline tends to become "still".

In Fig. 3, we show the comparison between the free-surface evolution at fixed cross-shore locations for $C_f = 0$ (dashed lines), $C_f = 0.01$ (solid lines) and $d_m^* = 10^{-5}$ m. As expected, in the frictional case, the free surface is damped with respect to the frictionless beach and the damping process increases as the depth decreases. However, the global behavior is somehow similar to that shown in the frictionless case since the swash events are always well defined in space and time. This means that use of the Chezy frictional term does not alter the water dynamics but simply introduces a damping effect. In any case, the largest difference shown by Fig. 3 is the generation of a thin layer of fluid during the run-down phase as predicted in the previous section on the basis of theoretical considerations.

In the next section, we give a further insight on the thin layer and on the influence of the threshold value on the prediction of the shoreline motion.

## 4 Further analysis on the thin layer

Let us focus on the flow behavior in the neighborhood of the shoreline after the run-up peak is reached (that is, for $t > t_0$ and $x_s$ constant). On the basis of the results shown in the previous sections, we assume that the fluid velocity is non-positive (that is, $u \leq 0$) and, for the sake of simplicity, that the beach is planar, that is, $h_x = -1$. Under these hypotheses, the momentum equations can be rewritten as follows:

$$u_t - u u_\xi - d_\xi = -\left[1 - \left(\frac{F}{F_s}\right)^2\right] \tag{33}$$

where $F = u/\sqrt{d}$ is the local Froude number. This form of the momentum equation enables an analysis of the magnitude of the frictional term with respect to the beach forcing term and, therefore, helps understand the behavior of the thin layer. It suggests that the frictional forces are negligible when the local Froude number is smaller than the Froude number at the shoreline, that is, when $(F/F_s)^2 \ll 1$, while a thin layer of water develops when $(F/F_s)^2 = \mathcal{O}(1)$.

To understand how the Froude number varies in the neighborhood of the shoreline, we rewrite the NSWEs using the change of variables $d = c^2$, $u = cF$ and $\xi = x_s - x$. We get:

$$\begin{cases} 2c_t - 3Fcc_\xi - c^2 F_\xi = 0 \\ cF_t - \left(2 - \frac{F^2}{2}\right) cc_\xi - \frac{c^2}{2} F F_\xi = -\left[1 - \left(\frac{F}{F_s}\right)^2\right]. \end{cases} \tag{34}$$

Now, on the basis of the results of the previous sections, we analyze the following asymptotic expansion near the shoreline:

$$c(\xi, t) = a_1(t)\xi + \mathcal{O}(\xi^2) \qquad F(\xi, t) = F_s + b_1(t)\xi + \mathcal{O}(\xi^2). \tag{35}$$

For $F_s \neq -2$ (i.e., $C_f \neq 1/4$), we find:

$$a_1(t) = \frac{2a_1(t_0)}{2 - 3a_1(t_0)F_s\,(t - t_0)} \qquad b_1(t) = \frac{F_s}{2}\left(2 - \frac{F_s^2}{2}\right) a_1^2(t), \tag{36}$$

while for $F_s = -2$, we get the following exact solution for the NSWEs:

$$c(\xi, t) = \frac{a_1(t_0)\xi}{1 + 3a_1(t_0)(t - t_0)} \quad F \equiv -2 \quad \Rightarrow \quad u(\xi, t) = -\frac{2a_1(t_0)\xi}{1 + 3a_1(t_0)(t - t_0)}. \tag{37}$$

Obviously, such solution cannot hold true for all times and, therefore, a transition to a different regime is expected. This can likely occur through a discontinuity in the derivatives (weak discontinuity) or a jump discontinuity (shock wave, bore). In any case, the special value $F_s = -2$ does not modify the analysis which follows.

The expansions above help understand the action of the threshold value $d_m$ on the prediction of the run-down motion. Indeed, the choice of a threshold value for the water depth corresponds to a cut of the thin layer which leads to the generation of a fictitious shoreline $x_f(t)$, i.e., a numerical shoreline that differs from the theoretical shoreline $x_s$. A first-order solution for $x_f(t)$ is easily obtained by using the first-order solution for $c$ in (35) and evaluating it at $x = x_f(t)$ (that is, at $\xi(t) = x_s - x_f(t)$). In this way, we find:

$$\sqrt{d_m} = \frac{2a_1(t_0)(x_s - x_f(t))}{2 - 3\,a_1(t_0)F_s(t - t_0)} \quad \Rightarrow \quad x_f(t) = x_s - \frac{\sqrt{d_m}}{a_1(t_0)} + \frac{3}{2}\sqrt{d_m}F_s(t - t_0). \tag{38}$$

This relation predicts a fictitious shoreline which at the leading order is linear with the time. This is confirmed by the numerical simulations (see the right panels of Fig. 2) where the run-down evolution on frictional beaches shows a linear profile. In this case, the term $\sqrt{d_m}/a_1(t_0)$ represents the initial gap $\Delta x$ between the actual shoreline $x_s$ and the fictitious shoreline $x_f(t)$.

The results shown above prove that the thin layer of water cannot be completely eliminated by the cut generated through the threshold depth. In fact, if we evaluate the first-order solution for $F$ at $x = x_f(t)$ (see Eq. (35)) and use the first-order solution in (38), we find:

$$F(\xi, t) = F_s + \frac{F_s}{2}\left(2 - \frac{F_s^2}{2}\right) a_1(t)\sqrt{d_m}. \tag{39}$$

This first-order solution converges to $F_s$ as $t$ goes to infinity and, therefore, leads to $(F/F_s)^2 = \mathcal{O}(1)$. This implies that the frictional forces along $x_f(t)$ become larger and larger as the time increases and that the generation of the thin layer still occurs in the neighborhood of $x_f(t)$. Note that a fictitious shoreline would be also achieved by frictionless computations because of the unavoidable presence of the threshold depth $d_m$. However, for $C_f = 0$, Eq. (38) would be replaced by something like $x_s(t) - x_f(t) = \mathcal{O}(d_m^p)$ with $p = 1/2, 1$ depending on the local flow conditions. This means that in the frictionless case, the difference between the theoretical and the fictitious shoreline always keeps infinitesimal while, and on the contrary, $x_s - x_f(t)$ grows linearly with time for $C_f \neq 0$.

Finally, the second-order solutions can be easily obtained by using the following expansions:

$$c(\xi, t) = a_1(t)\xi + a_2(t)\xi^2 + \mathcal{O}(\xi^3) \qquad F(\xi, t) = F_s + b_1(t)\,\xi + b_2(t)\xi^2 + \mathcal{O}(\xi^3). \tag{40}$$

These second-order solutions confirm the behavior studied above and, therefore, are not shown.

## 5 Conclusions

The results highlighted in Sects. 2.2 and 3 show that the usual Chezy formulation of the frictional forces within the depth-averaged models (that is, NSWEs and Boussinesq-type models) implies that the only acceptable solution is that represented by a constant shoreline regardless of the wave dynamics in the nearshore area.

This behavior is due to the generation of a thin layer of water after the swash event has reached the maximum run-up position; such a layer of water becomes thinner and thinner during the subsequent fluid evolution unless a new, larger swash event occurs leading to a novel run-up stage. In other words, the thin layer of water due to frictional effects extends seaward of the largest wave run-up over the beach face. Notwithstanding that, the frictional thin layer seems to have a weak influence on the water dynamics near the swash zone. As expected, the main influence of the Chezy frictional term is a global damping of the flow (water depth and velocity).

Finally, the influence on the shoreline evolution caused by the wet/dry threshold depth used in many NSWE solvers has been inspected. In this specific case, it has been found that the value of the threshold depth governs the generation and properties of a fictitious shoreline, i.e., a numerical shoreline that differs from the theoretical shoreline. The first-order solution predicts that such a shoreline depends linearly with time during the run-down stage.

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