Evaluation of Skewness and Kurtosis of Wind Waves Parameterized by JONSWAP Spectra

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ABSTRACT

The authors consider the deviations of wave height statistics from Gaussianity, manifested in higher statistical moments of random wind-wave fields, namely, in the nonzero values of the skewness and the kurtosis. These deviations are examined theoretically under the standard set of assumptions used in the established statistical theory of water waves, in particular in the derivation of the Hasselmann kinetic equation. P. Janssen proposed integral representations of the skewness and the kurtosis in terms of multidimensional integrals of wave spectra. However, the use of these representations for broadband wind-wave fields proved to be challenging; it requires substantial computational resources, which is unsuitable for applications. Using specially designed parallel algorithms to evaluate the integrals, the authors provide a comprehensive picture of the behavior of the kurtosis and the skewness of wind waves in the multidimensional parameter space of the most commonly used Joint North Sea Wave Project (JONSWAP) parameterizations of wind-wave spectra. Except for very narrow angular distributions where the overall picture is qualitatively different, the behavior of the higher moments proved to be not sensitive to the particular form of the directional spectrum. On this basis for the broad angular spectra typical of the ocean, the study puts forward simple parameterizations of the skewness and the kurtosis in terms of the JONSWAP peakedness parameter γ and in terms of the inverse wave age. These parameterizations can be used in operational wave forecasting and other applications.

1. Introduction

For practical applications, it is important to know the probability of wave height in seas and oceans at a given place and time. It is essential to predict the probability density function (PDF) of surface elevations, along with the meteorological forecasting (e.g., Goda 2000). If a wave field is linear, it obeys the Gaussian statistics, and the wave heights follow the Rayleigh distribution, under the additional assumption of the narrowbandedness of the energy spectrum (Rice 1954; Longuet-Higgins 1957; Goda 2000). The Rayleigh distribution captures qualitative features of the observed wave heights distributions. However, the real oceanic waves are neither linear nor narrowbanded, and the discrepancies between the Rayleigh distribution are important

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and, at present, poorly understood. The key uncertainty is in the behavior of the tails of the distribution. The knowledge of this behavior is crucial for the prediction of rare events, such as extreme waves, which remain a serious danger for ships and offshore structures. The observed statistics of such waves deviate significantly from the predictions based on the Rayleigh distribution (Stansell 2004; Mori and Janssen 2006). Even a slight difference in the distribution shape could result in huge disparities in the probability of freak waves and, correspondingly, significant scatter of "the highest wave"– type estimates required by the industry.

Over the last few decades, improvements of the simplest Gaussian model were carried out in several directions. For linear waves, the Rayleigh distribution was extended to narrow but finite width spectra (Longuet-Higgins 1980). Further efforts were concentrated on taking into account the effects of quadratic nonlinearity through bound harmonics, while retaining the assumption of narrowbandedness (Tayfun 1980; Forristall 2000; Fedele and Tayfun 2009). A number of semiempirical parameterizations of wave height distributions has been

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developed; in various ways they incorporate a finite spectral width and bound harmonic nonlinearity of the wave field [see references in Fedele and Tayfun (2009)]. A totally different line of research is based on the nonlinear Schrödinger equation and its generalizations (e.g., Onorato et al. 2001; Slunyaev 2006; Dysthe et al. 2008; Kharif et al. 2009; Mori et al. 2011) for narrowbanded spectra, but the domain of applicability of this approach to real oceanic broadband spectra is not clear.

Meanwhile, studies of oceanic broadband wave fields were mostly concentrated on the evolution of the energy or wave-action spectrum. Within the framework of the wave turbulence paradigm, where the wave field is considered a continuum of resonantly interacting random weakly nonlinear waves, the evolution of wave spectra is described by the kinetic (Hasselmann) equation derived from the first principles in 1960s (Hasselmann 1962). The approach applies to broad spectra only and was validated to be good in modeling the observed evolution of wave energy spectra (Komen et al. 1994; Janssen 2004, 2008). This theory underpins the present wave forecasting, and its predictions are being continuously tested around the globe (e.g., Komen et al. 1994; Janssen 2008). However, the spectrum is just the second statistical moment of a wave field. If the field is Gaussian, moments of all orders can be expressed in terms of the second moment. Nonlinearity causes spectral evolution and, at the same time, leads to a departure of wave field statistics from Gaussianity. This departure is manifested in higher moments of the wave field-skewness and kurtosis. Knowledge of these moments allows one to build the PDF of surface elevations, but the links between the modeled wave spectra evolution and wave height distributions are still poorly understood. It is also worth noting that the field measurements of the spectra are incomparably more numerous and reliable than those of higher moments, and the spectral evolution is also modeled reasonably well. The few available high-quality observations of wave statistics are mostly confined to wind tank measurements (Caulliez and Guérin 2012; Zavadsky et al. 2013), and although the spectra measured in the tanks look very similar to those observed in the ocean, it is not a priori clear to what extent the similarity holds for the higher moments.

For weakly nonlinear gravity wind waves, there are two primary causes of departure from Gaussianity manifested in nonzero skewness and kurtosis. The first contribution is caused by resonant nonlinear interactions. Because resonant three-wave interactions in a gravity wave field are absent, the quadratic nonlinearity of a wave field can be excluded by a suitable change of variables (canonical transformation) that eliminates bound harmonics. Here, the term "bound harmonics" refers to all harmonics that are generated by nonresonant nonlinear interactions and do not satisfy the linear dispersion relation, including the higher harmonics of Stokes waves, sum and difference harmonics of the free waves, and combinations of these harmonics with the free waves. Then the wave field in the transformed space is Gaussian to this order, and the lowest-order non-Gaussian effect is due to exactly and approximately resonant four-wave interactions. This non-Gaussianity is manifested in the nonzero value of kurtosis of the transformed wave field, but does not contribute to skewness. Following Annenkov and Shrira (2009b, 2013), we will refer to this component of kurtosis as "dynamic" and denote it as $C_4^{(d)}$. Strictly speaking, the value of $C_4^{(d)}$ cannot be determined from instantaneous wave spectra, but requires either the information about the phases of interacting waves or the knowledge of the history of spectral evolution (Annenkov and Shrira 2013). However, under the additional assumption that the evolution of the spectrum occurs on the time scale $O(\varepsilon^{-4})$, where ε is a small nondimensional measure of nonlinearity, $C_4^{(d)}$ can be estimated from the spectrum approximately. In this case, it can be expressed in terms of the instantaneous wave action spectrum $n(\mathbf{k})$ as an integral over the six-dimensional wavenumber domain with an oscillating kernel (Janssen 2003). Numerical evaluation of this integral is quite challenging; the integral has been first evaluated by Annenkov and Shrira (2013) for a number of typical spectra. These computations, complemented by direct numerical simulations (DNS) of the dynamic kurtosis by Annenkov and Shrira (2009a, 2013) have shown that in generic situations the dynamic kurtosis remains small both in the absolute value [being $O(10^{-2})$] and compared to the total value of kurtosis. The scale of required computations rules out this route as an option for operational wave forecasting and many other applications.

The second contribution to non-Gaussianity is due to bound harmonics. It manifests in both kurtosis and skewness and is present in any finite-amplitude wave field, even in the absence of resonant nonlinear interactions. The smallness of the dynamic kurtosis allows one to consider this contribution separately. Using the concept of wave height envelope rather than the usual trough-to-crest wave height, which is not well defined for broadband wave fields, Janssen (2009) has derived expressions for the second, third, and fourth moments of the envelope elevation in terms of wave spectra. Using these expressions, the formulation of the dynamic kurtosis in terms of spectra by Janssen (2003), and the DNS, Annenkov and Shrira (2013) have calculated both components of non-Gaussianity for various wave fields generated by stationary or fluctuating wind. It was found that while the dynamic kurtosis remains small during the wave field evolution, the bound harmonic kurtosis and skewness slowly decrease with time. The rate of this decrease was derived analytically for the known selfsimilar regimes of wave field evolution. These results provided a way to calculate higher moments of wave heights for the given spectra as well as a valuable insight into their evolution, but fell short from delivering a comprehensive picture of the higher moments' evolution. These calculations of skewness and kurtosis are very expensive computationally and therefore even in the long term cannot be made suitable for use in the operational forecasting and other applications.

The main purpose of this study is to translate the established understanding of the wind-wave spectral evolution into a quantitative description of higher moments of surface elevation. The specific aims are as follows:

- (i) employing the established theory of wave interactions to get a comprehensive picture of the behavior of kurtosis and the skewness of sea wind waves in the multidimensional parameter space for the very broad range of parameters, which includes all conceivable situations in the sea but extends further in order to capture the tendencies indiscernible otherwise;
- (ii) to examine the sensitivity of kurtosis and skewness to the shape of wind-wave spectra; and
- (iii) to find simple parameterizations of these moments, making use of the most common Joint North Sea Wave Project (JONSWAP) parameterization of the wave spectra and spectral evolution.

In this paper, we use the JONSWAP spectrum widely accepted as the design spectrum in the engineering community. Two different parameterizations of the angular spreading are used: the $\cos^N \theta$ model, where θ is the angle and N is the parameter of the spreading, and the sech² $\beta\theta$ model of Donelan et al. (1985), which does not contain an additional spreading parameter, but takes into account the dependence of β on ω/ω_p , the frequency normalized by the frequency of the spectral peak. A comprehensive picture of the dependence of skewness and kurtosis on the JONSWAP parameters is presented. It is found that for the broad angular spectra, the dependence is robust, which enables us to parameterize it. We also use the modification of the JONSWAP spectrum suggested by Donelan et al. (1985), which is based on the ω^{-4} shape of the spectral tail, supported theoretically and by observations. This version of the JONSWAP spectrum relates all spectral parameters to the wind forcing parameter U_{10}/c_p , where U_{10} is the wind speed at 10-m height in the direction of the wave propagation, and c_p is the phase speed at the spectral peak. We propose simple parameterizations of skewness and kurtosis in terms of U_{10}/c_p , which can be easily used in operational wave forecasting.

The paper is organized as follows. A brief theoretical background outlining the basic assumptions and key steps in the derivation of the integral expressions we employ for evaluation of skewness and kurtosis is given in section 2. Section 3 briefly reviews the JONSWAP parameterizations of wind-wave spectra and its modifications that we use throughout the paper. The numerical procedure employed to evaluate the integrals is briefly described in section 4. The results of simulations of skewness and kurtosis providing a detailed picture of their dependence on parameters are given in section 5. The conclusions are formulated and discussed in section 6.

2. Theoretical background

a. Basic equations

In most studies of random wind-wave fields, it is usually assumed that the evolution of statistical characteristics of a wave field is governed by the kinetic (Hasselmann) equation, which can be written in the form (e.g., Komen et al. 1994)

$$\frac{\partial n_0}{\partial t} = 4\pi \int T_{0123}^2 f_{0123} \delta_{0+1-2-3} \delta(\Delta \omega) \, d\mathbf{k}_{123} + S_f, \qquad (1)$$

where $n_0 \equiv n(\mathbf{k}_0, t)$ is the spectral density at wavevector \mathbf{k}_0 , $f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3)$, $\Delta \omega = \omega_0 + \omega_1 - \omega_2 - \omega_3$, T_{0123} is the interaction coefficient, and S_f is the forcing/dissipation term. Here and below, we use the compact notation that designates the arguments by indices, for example, $b_0 = b(\mathbf{k}_0, t)$, $T_{0123} = T(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$, $\delta_{0+1-2-3} = \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, and $d\mathbf{k}_{123} = d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$.

The spectral density $n(\mathbf{k})$ is the second-order correlator of the (complex) dynamical amplitude $b(\mathbf{k}, t)$, $\langle b_0^* b_1 \rangle = n_0 \delta_{0-1}$. Here and below angle brackets denote ensemble averaging. It is important to note that $b(\mathbf{k}, t)$ [and, hence, $n(\mathbf{k})$] is not a directly measurable quantity, but the result of a transformation of the physical amplitude $a(\mathbf{k}, t)$ in the form

$$a_{0} = b_{0} + \int A_{012}^{(1)} b_{1} b_{2} \delta_{0-1-2} d\mathbf{k}_{12} + \int A_{012}^{(2)} b_{1}^{*} b_{2} \delta_{0+1-2} d\mathbf{k}_{12} + \int A_{012}^{(3)} b_{1}^{*} b_{2}^{*} \delta_{0+1+2} d\mathbf{k}_{12} + \dots$$
(2)

Expressions for all kernels in (2), as well as the coefficient T_{0123} , can be found in Krasitskii (1994). The physical amplitude $a(\mathbf{k}, t)$ is related to Fourier-transformed physical variables $\zeta(x, y, t)$ (the elevation of the surface) and $\psi(x, y, t)$ (the velocity potential at the surface) as

$$\hat{\zeta}(\mathbf{k}) = \left[\frac{q(\mathbf{k})}{2\omega(\mathbf{k})}\right]^{1/2} [a(\mathbf{k}) + a^*(-\mathbf{k})] \text{ and}$$
$$\hat{\psi}(\mathbf{k}) = \left[\frac{\omega(\mathbf{k})}{2q(\mathbf{k})}\right]^{1/2} [a(\mathbf{k}) - a^*(-\mathbf{k})],$$

where the hat denotes Fourier transform, the asterisk means complex conjugation, $\omega(\mathbf{k}) = [gq(\mathbf{k})]^{1/2}$ is the linear dispersion relation, $q(\mathbf{k}) = |\mathbf{k}| = k$ for infinite depth, and g is the acceleration due to gravity. The canonical transformation (2) makes use of the fact that in the absence of capillarity, resonant three-wave interactions for gravity waves are not allowed by the dispersion relation, eliminating the nonresonant terms. Essentially, the dependent variables $b(\mathbf{k}, t)$ and $n(\mathbf{k})$ describe the free-wave part of the wave field, while the bound harmonics contribution up to the third order in nonlinearity is accounted for by the canonical transformation (2).

Resonant and nearly resonant four-wave interactions, present in the wave field, lead to the spectral evolution, which is described by (1), and to a departure of the field from Gaussianity, which, however, cannot be described by means of (1).

b. Dynamic non-Gaussianity

Consider first the non-Gaussianity of the transformed wave field $b(\mathbf{k}, t)$, denoting its statistical moments as m_j , where j = 2, 3, and 4. The second moment has the form

$$m_2 = \int \omega_0 b_0 b_0^* \, d\mathbf{k}_0,$$

and the third moment m_3 is identically zero, due to the absence of resonant three-wave interactions. The fourth moment m_4 in terms of the known wave field $b(\mathbf{k}, t)$ reads as (Janssen 2003)

$$m_4 = \frac{3}{4} \int (\omega_0 \omega_1 \omega_2 \omega_3)^{1/2} \langle b_0^* b_1^* b_2 b_3 \rangle \, d\mathbf{k}_{0123} + \text{c.c.}, \qquad (3)$$

where c. c. stands for complex conjugate. The kurtosis $C_4^{(d)}$ is defined in terms of the moments as

$$C_4^{(d)} = m_4/m_2^2 - 3.$$
 (4)

The superscript $^{(d)}$, which stands for dynamic, is used to emphasize that the kurtosis of a transformed wave field $b(\mathbf{k}, t)$ can be nonzero only in the presence of nonlinear resonant interactions.

The most straightforward way of calculating $C_4^{(d)}$ is by DNS, that is, by integrating the dynamical equations for the wave field numerically and averaging (3) over the realizations; this approach is prohibitively expensive and allows us to examine only a fraction of the parameter space. Another possibility is to use the statistical approach exploiting the proximity of wave statistics to Gaussian put forward by Janssen (2003). Under the same set of assumptions as employed in the standard derivation of the kinetic equation, this approach leads to the expression in the form of the following six-dimensional singular integral

$$C_4^{(d)} \approx -\frac{3}{2m_2^2} \int T_{0123}(\omega_0 \omega_1 \omega_2 \omega_3)^{1/2} \frac{f_{0123}}{\Delta \omega} \delta_{0+1-2-3} \, d\mathbf{k}_{0123},$$
(5)

where the Cauchy principal value of the integral is taken. The notable counterintuitive feature of the expression is that in contrast to the integrand of the kinetic equation, there is no delta function in frequency, that is, all the quartets, not only the resonant ones, make a contribution into dynamic kurtosis. A way to calculate the integral was first developed in Annenkov and Shrira (2013); it will be described below in section 4a.

c. Bound harmonic non-Gaussianity

Even if the wave field in the transformed variable $b(\mathbf{k}, t)$ is Gaussian, the finite-amplitude wave field in the physical space $a(\mathbf{k}, t)$ is not Gaussian. This additional source of non-Gaussianity is due to the presence of bound harmonics and is eliminated by the canonical transformation (2). For brevity, we will refer to it as bound harmonic non-Gaussianity and indicate the corresponding quantities by the superscript ^(b). It is described in the physical space in terms of the surface elevation and can be calculated from (2), provided that the dynamic component is small; otherwise, the clear cut separation of the two components is not possible.

Assuming that the dynamic non-Gaussianity is small, consider the statistical moments of the surface elevation:

$$\mu_i = \langle \zeta^j \rangle.$$

Janssen (2009) derived expressions for μ_j , where j = 2, 3, and 4, in terms of energy density defined as $E(\mathbf{k}) = \omega n(\mathbf{k})/g$. Note that

$$m_2 = \int \omega_0 b_0 b_0^* d\mathbf{k}_0 = g \int E_0 d\mathbf{k}_0$$

For the second statistical moment in physical space, Janssen (2009) obtained

$$\mu_2 = \langle \zeta^2 \rangle$$

= $\int E_1 dk_1 + \int (\mathcal{A}_{1,2}^2 + \mathcal{B}_{1,2}^2 + 2\mathcal{C}_{1,1,2,2}) E_1 E_2 d\mathbf{k}_{12}$

where expressions for coefficients $A_{1,2}$, $B_{1,2}$, and $C_{1,1,2,2}$ are given in Janssen (2009) and Annenkov and Shrira (2013), where it is also proved that the second integral is equal to zero due to the symmetry. Therefore,

$$\mu_2 = \langle \zeta^2 \rangle = \int E_1 \, d\mathbf{k}_1 = \frac{m_2}{g}.\tag{6}$$

The third and fourth moments have the form (Janssen 2009)

$$\mu_3 = \langle \zeta^3 \rangle = 3 \int (\mathcal{A}_{1,2} + \mathcal{B}_{1,2}) E_1 E_2 \, d\mathbf{k}_{12} \quad \text{and}$$
 (7)

$$\mu_{4} = \langle \zeta^{4} \rangle = 3 \int E_{1} E_{2} \, d\mathbf{k}_{12} + 12 \int \mathcal{J}_{1,2,3} E_{1} E_{2} E_{3} \, d\mathbf{k}_{123}.$$
(8)

The coefficients \mathcal{A} , \mathcal{B} , and \mathcal{J} can be found in Janssen (2009) and Annenkov and Shrira (2013). Then, we can write out the expressions for the bound harmonic components of skewness and kurtosis as

$$C_3^{(b)} = \frac{\mu_3}{\mu_2^{3/2}}$$
 and $C_4^{(b)} = \frac{\mu_4}{\mu_2^2} - 3.$ (9)

3. JONSWAP spectra

It is widely accepted that under nearly constant unidirectional wind and in the absence of swell the energy spectrum of wind waves has a universal shape. Its most common parameterization is the JONSWAP spectrum in the wavenumber and frequency domains (e.g., Young 1999),

$$E(\mathbf{k}) = 4\pi^2 \frac{\alpha}{2k^3} \exp\left[-\frac{5}{4}(k/k_p)^{-2}\right]$$
$$\times \gamma^{\exp\left[-\left(\sqrt{k/k_p}-1\right)^2/(2\sigma^2)\right]} D_k(\theta), \quad (10)$$

or in terms of frequency

$$E(\omega) = 4\pi^2 \frac{\alpha g^2}{\omega^5} \exp\left[-\frac{5}{4}(\omega/\omega_p)^{-4}\right]$$
$$\times \gamma^{\exp[-(\omega/\omega_p - 1)^2/(2\sigma^2)]} D_{\omega}(\theta), \qquad (11)$$

where parameter σ has the following two fixed values dependent on k: 0.07 for $k \leq k_p$ and 0.09 otherwise.

Thus, the one-dimensional spectrum is completely specified by setting just two parameters: α and γ , where α is a magnitude parameter proportional to the square of wave steepness ε , while γ is a shape factor characterizing the peakedness of the spectrum and position of the spectral peak. Functions $D_k(\theta)$ or $D_{\omega}(\theta)$ describe directional distribution of the two-dimensional spectrum; in one horizontal dimension $D_{\omega}(\theta) = D_k(\theta) = 1$. In this study, we mostly employ the commonly used parameterization for $D_{\omega}(\theta)$ proposed by Donelan et al. (1985):

$$D_{\omega}(\theta) = \frac{1}{2}\beta \operatorname{sech}^{2}(\beta\theta) \text{ and } D_{k}(\theta) = D_{\omega}(\theta)/k, \quad (12)$$

where the mean wave direction corresponds to $\theta = 0$, and

$$\beta = \begin{cases} 2.61(\omega/\omega_p)^{1.3} & \text{for } 0.56 < \omega/\omega_p < 0.95, \\ 2.28(\omega/\omega_p)^{-1.3} & \text{for } 0.95 < \omega/\omega_p < 1.6, \text{ and} \\ 1.24 & \text{otherwise.} \end{cases}$$

To have more flexibility for examining dependence on the angular spreading in parameter space, we also use the popular $\cos^{N}(\theta)$ model, which includes an additional parameter N,

$$D_{\omega}(\theta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(N/2+1)}{\Gamma(N/2+1/2)} \cos^{N}\theta, \qquad (13)$$

for $|\theta| \le \pi/2$, and 0 otherwise. Here, Γ is the gamma function.

Donelan et al. (1985) also proposed a modification of the one-dimensional spectrum (11), adopting the ω^{-4} shape of the spectral tail, which has a better theoretical and observational support. This parameterization has the form

$$E_{d}(\omega) = 4\pi^{2} \frac{\alpha_{d} g^{2}}{\omega^{5}} (\omega/\omega_{p}) \exp[-(\omega/\omega_{p})^{-4}]$$
$$\times \gamma_{d}^{\exp[-(\omega/\omega_{p}-1)^{2}/(2\sigma_{d}^{2})]} D_{\omega}(\theta), \qquad (14)$$

where the spectral parameters α_d , γ_d , and σ_d are rigidly linked to the inverse wave age parameter U_{10}/c_p , with U_{10} being the wind speed in the mean direction of wave propagation at the 10-m height, and c_p is the corresponding phase speed. The parameters have the following values:

$$\alpha_d = 0.006 (U_{10}/c_p)^{0.55}$$
 for $0.83 < U_{10}/c_p < 5$,

$$\begin{split} \sigma_d &= 0.08 [1 + 4/(U_{10}/c_p)^c] \quad \text{for} \quad 0.83 < U_{10}/c_p < 5, \\ \gamma_d &= \begin{cases} 1.7 & \text{for} \quad 0.83 < U_{10}/c_p < 1, \\ 1.7 + 6.0 \lg(U_{10}/c_p) & \text{for} \quad 1 \leq U_{10}/c_p < 5. \end{cases} \end{split}$$

We will use the modified JONSWAP spectrum (14) with the directional distribution (12); in this way, the twodimensional spectrum is completely specified by setting the single parameter U_{10}/c_p . Further on we will drop the subscript in U_{10} for brevity.

4. Numerical method

a. Dynamic kurtosis

The dynamic kurtosis $C_4^{(d)}$ is calculated using (5). The numerical grid used has 160 logarithmically spaced values of frequency ω in the range $0.5\omega_p \le \omega \le 3.0\omega_p$, where ω_p is the frequency of the spectral peak, and 75 uniformly spaced values of angle θ in the range $-\pi/2 \le$ $\theta \le \pi/2$. This (ω, θ) resolution was tested to be sufficient; there were no considerable changes to $C_4^{(d)}$ from further refining of the grid.

The main computational difficulty was because all resonant and nonresonant interactions needed to be taken into account. The total number of four-wave interactions on a 160 × 75 grid exceeds 5.5×10^{11} , so a special parallelized algorithm had to be designed. The computation for one spectrum required about 8 h on 64 Xeon X5650 cores, but actually groups of spectra were processed simultaneously to save the CPU the time required to compute the lengthy coefficients. The dynamic kurtosis was calculated in the large time limit, using the principal value integral (5). To avoid the loss of accuracy due to small denominators at the exact resonance, interactions with $\Delta\omega/\omega_{min} < 10^{-4}$ were excluded. The results were verified to be not sensitive to the chosen value of the cutoff in $\Delta\omega/\omega_{min}$.

b. Bound harmonics skewness and kurtosis

The bound harmonic skewness and kurtosis are calculated using (7), (8), and (9). Because of the robustness of the bound harmonics higher statistical moments, the computational grid was reduced to 81 values of ω and 41 values of θ . The results were verified not to depend on a further refinement of the computational grid.

5. Results

a. JONSWAP spectra with directional spreading (12)

First, we consider as a typical generic wind-wave spectrum using the classical JONSWAP parameterization (11) with the directional spreading function proposed by



FIG. 1. Dynamic kurtosis $C_4^{(d)}$ for JONSWAP spectra (a) vs the JONSWAP parameters (α , γ) and (b) vs wave steepness ε and the JONSWAP parameter γ . Directional spreading is according to (12).

Donelan et al. (1985) in the sech-squared form (12). Because gravity waves on deep water do not have a characteristic length scale, the wavenumber of the spectral peak is normalized to 1, without the loss of generality. Thus, the spectrum (11) with angular spreading (12) is characterized by two parameters α and γ .

The dynamic kurtosis $C_4^{(d)}$ as function of (α, γ) is shown in Fig. 1a. The value of $C_4^{(d)}$ can be positive or negative, but remains $O(10^{-2})$, except for large values of α and γ , which correspond to unphysical values of wave steepness, defined as

$$\varepsilon = \frac{1}{2} H_{\text{rms}} k_p$$
 and $H_{\text{rms}} = \frac{1}{\pi} \left[2 \int E(\mathbf{k}) \, d\mathbf{k} \right]^{1/2}$.

Because the steepness depends on both α and γ , and the range of realistic wave steepness is relatively narrow, it is instructive to plot $C_4^{(d)}$ as function of ε and γ , as shown in



FIG. 2. Bound harmonics kurtosis $C_4^{(b)}$ for JONSWAP spectra (a) vs the JONSWAP parameters (α , γ) and (b) vs wave steepness ε and the JONSWAP parameter γ . Directional spreading is according to (12).

Fig. 1b. For a certain value of γ , close to 3.4, the dynamic kurtosis is equal to zero for all values of wave steepness.

Bound harmonic kurtosis, shown in Figs. 2a and 2b in terms of (α, γ) and (ε, γ) , respectively, is positive and typically an order of magnitude larger than the dynamic kurtosis, except for large values of γ where the two components become comparable. The sum of the two components of the kurtosis is plotted in Figs. 3a and 3b. Parameters Π_1 , Π_2 , and the dashed curves are discussed in section 6 below.

Figure 4 shows the ratio of the dynamic and bound harmonics kurtosis as function of γ ; its absolute value remains $O(10^{-1})$, increasing up to ¹/₄ for large values of γ .

Because both components of the kurtosis are proportional to α (or ε^2) for a fixed value of γ , the total kurtosis C_4 is shown in Fig. 5 normalized by ε^2 , as function of γ . The kurtosis is approximated by the power fit



FIG. 3. Total kurtosis $C_4 = C_4^{(d)} + C_4^{(b)}$ for JONSWAP spectra (a) vs the JONSWAP parameters (α, γ) and (b) vs wave steepness ε and the JONSWAP parameter γ . Directional spreading is according to (12). Parameters Π_1 , Π_2 , and the dashed curves are discussed in section 6.

$$C_{A} = 12.6\gamma^{-0.328}\varepsilon^{2}$$

For comparison, the bound harmonic component of the kurtosis is shown by red dots. In Fig. 6, skewness C_3 is shown for the same range of γ , normalized by ε , and approximated by the power fit

$$C_3 = (0.0897 + 0.02\gamma^{-0.5})\varepsilon$$

b. JONSWAP spectra with $\cos^{N}(\theta)$ directional spreading

The dependence of kurtosis on the angular distribution of the JONSWAP spectrum is considered by assuming the $\cos^{N}(\theta)$ directional spreading (13) for values of N in the range $4 \le N \le 100$. Although wind-wave spectra with large values of N are unrealistic, it is useful to study the sensitivity of kurtosis to the angular spreading of the spectrum. The dynamic kurtosis is shown in Figs. 7a and 7b, where the dependence on γ considered in the



FIG. 4. Ratio of dynamic and bound harmonic kurtosis $C_4^{(d)}/C_4^{(b)}$ for JONSWAP spectra vs the JONSWAP parameter γ . Directional spreading is according to (12).

previous subsection is plotted for comparison. If the angular spectrum is not particularly narrow ($N \le 20$), the dynamic kurtosis remains close to the values obtained with the sech-squared-type directional spreading (12). For larger values of N, $C_4^{(d)}$ is negative for all values of γ , and a fixed value of ε weakly depends on γ . For very narrow spectra with N > 100, the angular resolution of 75 values of θ in the range $-\pi/2 < \theta < \pi/2$ becomes insufficient. Such spectra are studied separately in section 5d.

The bound harmonics kurtosis is shown in Figs. 8a and 8b. Its dependence on N is weaker for small N and virtually absent for N > 50.

c. Modified JONSWAP spectrum (14)

For the modified JONSWAP spectrum parameterization tightly linked to the wave age (14) with sechsquared directional spreading (12), the kurtosis and the skewness are functions of the single parameter U/c_p (inverse wave age). They are shown in Figs. 9 and 10, respectively, together with the power fits

$$C_4 = 0.04 + 0.022 (U/c_p)^{0.8/}$$
 and
 $C_3 = 0.153 (U/c_p)^{0.3}$. (15)

d. JONSWAP spectra with narrow directional spreading and the unidirectional limit

Although this study is primarily focused on realistic wind-wave spectra with wide directional spreading, it would be incomplete without an analysis of the limit of narrow angular distributions and of the transition to



FIG. 5. Bound harmonic kurtosis $C_4^{(b)}$ (red dots) and total kurtosis $C_4 = C_4^{(d)} + C_4^{(b)}$ (black dots) vs JONSWAP parameter γ , normalized by ε^2 , for directional spreading (12). Dashed curve shows the power fit $C_4 = 12.6\gamma^{-0.328}\varepsilon^2$.

quasi-one-dimensional spectra, often encountered in laboratory experiments. For a purely one-dimensional narrowband spectrum, the dynamic kurtosis is known to be positive (Janssen 2004). Moreover, in the experimental study of Shemer et al. (2010), the dynamic kurtosis of such spectrum was found to be dominant, while the contribution due to bound harmonics was insignificant. The results of sections 5a and 5b, however, show that the dynamic kurtosis is much smaller than the



FIG. 6. Skewness C_3 (black dots) vs JONSWAP parameter γ , normalized by ε , for directional spreading (12), and the power fit $C_3 = (0.0897 + 0.02\gamma^{-0.5})\varepsilon$, shown by the dashed curve.



FIG. 7. Dynamic kurtosis $C_4^{(d)}$ vs JONSWAP parameter γ for (a) $\alpha = 0.1$ and (b) $\varepsilon = 0.1$. Various models of directional spreading are used: the $\cos^{N}(\theta)$ family (11) with various values of N and sechsquared directional spreading (12), denoted by D and shown by the dashed curve.

bound harmonics component for spectra with a broad directional distribution, and when a spectrum is narrower in angle [say, for N = O(100) in the cos^N directional model], it is negative for all values of the JONSWAP parameter γ . To perform a more detailed study of the case of narrow angular spreading, in Fig. 11a the dynamic kurtosis is plotted versus γ for large values of N up to 1000. For the computation of kurtosis of such narrow spectra, a computational grid with a refined angular resolution is used, with 35 uniformly spaced values of angle θ in the range $-\pi/8 < \theta < \pi/8$ for N = 100 and N = 200 and in the range $-\pi/16 < \theta < \pi/16$ for $N \ge 300$; the frequency resolution is unchanged. For comparison, the dynamic kurtosis in the purely one-dimensional model with the same frequency resolution is shown in Fig. 11b. Contrary to expectations that the two-dimensional model with a narrow angular spreading should tend to the onedimensional case, it follows from Fig. 11a that when the spectrum is very narrow in angle, the dynamic kurtosis remains negative and increases in absolute value, with weak dependence on γ . Even for N = O(1000), no



FIG. 8. Bound harmonics kurtosis $C_4^{(b)}$ vs JONSWAP parameter γ for (a) $\alpha = 0.1$ and (b) $\varepsilon = 0.1$. Various models of directional spreading are used: the family $\cos^N(\theta)$ (11) with various values of N and sech-squared directional spreading (12), denoted by D and shown by the dashed curve.

tendency toward the one-dimensional results can be observed.

To resolve this apparent contradiction, in Fig. 12 we show the results of the study of JONSWAP spectra with very narrow directional spreading, using the model (12) with large values of β in the range $10 \le \beta \le 1000$. In this case, computations were performed with the same frequency resolution as before and 35 uniformly spaced values of angle θ in the range $-\Theta < \theta < \Theta$, with Θ chosen for each value of β to satisfy the condition that 99.9% of the spectral energy lies within the angle $\pm \Theta$. For narrow spectra with $10 \le \beta \le 70$, the dynamic kurtosis remains negative and continues to increase in absolute value with increasing β for all values of γ , reaching the values below -1 for β close to 70. For larger values of β , the kurtosis undergoes a fast change to positive values, which depend on the value of γ and are close to the corresponding values for the purely onedimensional spectrum with the same γ , shown as black dots in Fig. 12. It appears that the values of the dynamic





FIG. 9. Total kurtosis $C_4 = C_4^{(d)} + C_4^{(b)}$ (black dots) and bound harmonics kurtosis $C_4^{(b)}$ (red dots) for the modified JONSWAP spectrum (14) with the directional spreading (12) vs the inverse wave age U/c_p , and the power fit $C_4 = 0.04 + 0.022(U/c_p)^{0.87}$, shown by the dashed curve.

kurtosis obtained in the one-dimensional case remain valid for a two-dimensional spectrum only if the spectrum is very narrow in angle, with $\beta > 200$; this value of β corresponds to nearly all (99.9%) spectral energy contained within a 2° angle.

6. Discussion

In this work, we have evaluated the kurtosis and skewness for the most widely used family of parameterizations of the observed wind-wave spectra. The earlier observation (Annenkov and Shrira 2009a, 2013) that except for the qualitatively different situations with very narrow angular distributions the dynamic kurtosis is almost always much smaller than the bound harmonic contribution has been confirmed for a much wider range of parameters. An interesting feature of the dynamic kurtosis for broad angular distributions typical of the ocean is that it changes sign at $\gamma \approx 3.4$, that is, close to the most typical value $\gamma \approx 3.3$ found in the JONSWAP experiment (Young 1999). The main conclusion of this work is that the behavior of both the kurtosis and the skewness is robust in the wide range of parameters. This allows one to circumvent the need to perform very expensive simulations for each observed or simulated wave spectrum, because the already calculated dependencies can be parameterized.

We have found the behavior of both components of kurtosis in the larger than real three-dimensional parameter space (α , γ , N) and their sensitivity to approximations of the spectral shape. This provides a good idea of the possible degree of departure of wave statistics

FIG. 10. Skewness C_3 (black dots) for the modified JONSWAP spectrum (14) with the directional spreading (12) vs inverse wave age U/c_p , and the power fit $C_3 = 0.153(U/c_p)^{0.3}$, shown by the dashed curve.

from Gaussian for all kinds of wind-wave fields. Thus, the simple parameterizations found in this work can be used in a large variety of situations, including the operational wave forecasting. To the extent we can trust the Donelan et al. (1985) parameterization of the spectra (14) entirely controlled by the wave age U/C_p , we have expressed the higher moments in terms of U/C_p in the formulae of ultimate simplicity (15). By modeling evolution of U and c_p by means of the common operational models, it is straightforward to predict the evolution of the kurtosis and the skewness. These predictions might differ from those given in Annenkov and Shrira (2013), because the latter presumes complete self-similarity of the evolving spectra, which is justified only for mature wave fields with γ close to unity. The parameterizations obtained in this work are free from this restriction and hence can be used for predictions of probability distributions for a much wider range of situations. The found dominance of the bound harmonics component of the kurtosis over the dynamic one in a generic situation allows one to regard these parameterizations with a certain level of trust, because the bound harmonics kurtosis is very robust and relatively weakly depends on the details of the spectrum, in particular on the specific form of the JONSWAP spectral tail or of its angular spreading.

The question of how wide is the range of situations where these simple parameterizations are applicable needs a discussion. Recall that the approach is based on the same set of assumptions as the kinetic equation underpinning all wave models; among the assumptions there is a requirement that the spectra should be broad



FIG. 11. Dynamic kurtosis $C_4^{(d)}$ vs JONSWAP parameter γ for a JONSWAP spectrum with steepness $\varepsilon = 0.1$ (a) for twodimensional spectrum with $\cos^N(\theta)$ directional model (11) and for several values of N in the range $100 \le N \le 1000$ (b) for the onedimensional spectrum.

enough. Wave fields with too narrow spectra behave qualitatively differently; in contrast to a continuum of weakly interacting waves typical of the broadband spectra, there are interacting coherent patterns that cannot be captured by the classical kinetic description. The evolution of such wave fields is described by the nonlinear Schrödinger equation and its generalizations. Such regimes were studied in Mori et al. (2011) where the main conclusion is that the kurtosis is substantial and positive for strictly unidirectional waves and monotonically decays with the angular spreading. Our computations of wave spectra with narrow angular distributions revealed a different scenario. Although for strictly unidirectional and nearly unidirectional spectra with a pronounced peak (i.e., large γ), the dynamic kurtosis is positive and large, in accordance with earlier studies, but for a slightly wider angular spreading it changes sign and becomes large and negative (with virtually no dependence on γ). A sharp



FIG. 12. Dynamic kurtosis for a JONSWAP spectrum with $\varepsilon = 0.1$ for several values of the parameter γ and a very narrow angular spreading. The kurtosis is shown vs the angular spreading parameter β for the sech-squared directional model (12) and $10 \le \beta \le 1000$. Thick dots on the right correspond to the values of the kurtosis in the one-dimensional model.

change of the dynamic kurtosis occurs at a certain critical angular width, approximately corresponding to nearly all (99.9%) spectral energy contained within 2 angular degrees. This nonmonotonic behavior differs qualitatively from the results obtained by Mori et al. (2011). However, there is no contradiction here because there is no reason for the results for the two qualitatively different regimes based upon contrasting assumptions to coincide. The boundary between these two regimes is not sharply defined. For the lack of alternatives, to outline the boundary between the regimes we take as a proxy the boundary of the transition of random spatially homogeneous wave field with a given spectrum from stability to instability with respect to small spatially inhomogeneous perturbations. This instability is an analog of the Benjamin-Feir instability for a deterministic Stokes wave (Alber 1978). It should be stressed that the boundary is not sharp: it depends on the amplitude of the perturbations determined by the noise level (Shemer 2010). The dominance of nonlinear effects over dispersion results in instability and the subsequent emergence of coherent patterns. For the JONSWAP spectra with the $\cos^{N}(\theta)$ directional spreading (13), it was found that the stability domain, and hence the applicability of the adopted approach, is specified by the condition $\Pi_2 > 1.1$ (see Ribal et al. 2013), where

$$\begin{split} \Pi_2 &= \frac{\varepsilon}{\alpha \gamma} + \frac{\beta}{\varepsilon A_d}, \\ \beta &= 0.0256, \text{ and } A_d = \frac{\Gamma(N/2+1)}{\sqrt{\pi}\Gamma(N/2+1/2)}. \end{split}$$

For the idealized one-dimensional spectra the condition is less restrictive (Ribal et al. 2013):

$$\Pi_1 > 1$$
 and $\Pi_1 = \frac{\varepsilon}{\alpha \gamma}$.

These boundaries of validity are shown in Fig. 3 by dashed curves. Most of the parameter domain for realistic wave spectra is within the domain of validity. The domain of invalidity is confined to the top-right corner corresponding to the very young and very steep waves.

There is another unaccounted for factor with a potential to narrow the validity of our results, which has to be discussed. Our approach is based on the standard set of assumptions of weakly nonlinear theory. Crucially, in this context wave breaking is absent, or to be precise, its effect is parameterized in the kinetic equation (1) by the linear term S_{f} . This is probably justified for the evolution of wave spectra, but might lead to palpable distortions of the statistics, especially of the tails of the probability distribution function. We cannot quantify this effect at present, but the qualitative picture is clear: breaking flattens the distribution of wave heights and hence decreases the value of kurtosis; under sufficiently intense breaking kurtosis might even turn from positive to negative (Caulliez and Guérin 2012; Zavadsky et al. 2013). Probably this is one of the reasons why the wind tank observations yield negative kurtosis. The second plausible reason is the very narrow angular distributions. The only work so far, where the data analysis of wind tank observations enabled the authors to estimate both the full and dynamic kurtosis (Shemer et al. 2010), finds that C_4 is only slightly different from $C_4^{(d)}$. These results, obtained for nearly unidirectional waves, are in qualitative agreement with the present simulations for very narrow angular distributions. However, to make a quantitative comparison with tank measurements, the role of the tank walls has to be properly examined, which goes beyond the scope of the present paper aimed primarily at broadband wave fields with moderate and broad angular spreading. The impact of wave breaking on wave statistics as well as a more general question on the role of the higher-order nonlinearity also require a dedicated study. The present work is confined to modeling higher moments of sea states without swell; however, the approach can be extended to examine and parameterize the more complex sea states with coexisting wind and swell waves.

In this work, we do not attempt to derive the PDF of surface elevations on the basis of the obtained skewness and kurtosis. The notion of wave height is not uniquely defined for the broadband nonlinear wave fields; the issue is not trivial and needs to be studied in its own right. Acknowledgments. The work was supported by U.K. NERC Grant NE/I01229X/1. Computations were performed on the computer cluster at Keele University and on the ECMWF supercomputing facility, the access to which is gratefully acknowledged. We are grateful to Luigi Cavaleri and Lev Shemer for useful suggestions on the first draft of the manuscript.

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