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Non-normal stability analysis of a shear current under surface gravity waves

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The stability of a horizontal shear current under surface gravity waves is investigated on the basis of the Rayleigh equation. As the differential operator is non-normal, a standard modal analysis is not effective in capturing the transient growth of a perturbation. The representation of the stream function by a suitable basis of bi-orthogonal eigenfunctions allows one to determine the maximum growth rate of a perturbation. It turns out that, in the considered range of parameters, such a growth rate can be two orders of magnitude larger than the maximum eigenvalue obtained by standard modal analysis.

1. Introduction

A long-standing problem in fluid mechanics is the stability of a shear flow bounded by long-crested gravity waves. After Burns (1953), this topic received wide attention both from a theoretical (see Yih 1972; Shrira 1993; Longuet-Higgins 1998; Miles 2001*a*, *b*) and a numerical point of view (Caponi *et al.* 1991; Morland, Saffman & Yuen 1991). The instability is explained in terms of the critical-layer theory developed by Miles (1957) to study the generation of waves by a shear wind.

From a physical point of view, as the wind starts blowing on a flat interface between air and water, shear stress transfers part of the momentum to a surface drift. Shemdin (1972) estimated that the surface drift is about 3 % of the asymptotic wind. Therefore, the direct formation of waves through the Miles mechanism is simultaneous to the formation of a concave shear current right under the interface which, if unstable, can contribute to the generation of waves.

For the coupled wind–current system (including surface tension), growth rates and phase speeds have been studied numerically by Valenzuela (1976). Kawai (1979) extended the work by combining numerical and experimental results, considering a reference wind speed ranging from 4 to 8 m s^{-1}. He measured growth rates, phase speeds and frequencies when wavelets start appearing on the surface. Numerical results were consistent with the measurements. A further contribution was given by van Gastel, Janssen & Komen (1985) who found that the growth rate is very sensitive to the shape of the wind profile, while the influence of changes in the current profile is much smaller.

In the present study, we restrict ourselves to the de-coupled problem: we neglect the generation of waves by wind and assume that a stationary concave shear current is established and maintained. On the basis of a large amount of recent literature, we suspect that the instability of such a base flow has been overlooked in the past and new results on the growth rates can be discussed. More specifically, we investigate the relevance of a non-normal mode analysis in the determination of the growth rate of perturbations of an unstable shear flow. In transient growth of shear flows, the perturbations that exhibit the largest growth rates may not correspond to the largest eigenvalues of the linearized operator.

The starting point of our analysis is the Rayleigh equation, obtained from the Euler equations, assuming small perturbations of a base shear flow. A classical linear stability analysis leads to the determination of the spectrum of the linearized operator (Morland *et al.* 1991). A positive imaginary part of the phase velocity for some wavenumber is the signature of instability (Drazin & Reid 2004). Such a standard linear analysis provides an insight into the stability properties of the system (with the exception of the neutral case) and allows the most rapidly growing component of the perturbation to be identified; in general, for sufficiently long time, the eigenfunction corresponding to the largest imaginary eigenvalue will dominate.

This approach, depending on the form of the evolution operator, may not furnish information on the most rapidly growing perturbation at initial time. In the specific case of interest, a shear flow is characterized by the non-orthogonality of the eigenfunctions that compose the perturbation (Schmid & Henningson 2001). The energy of the perturbation, while being a quadratic function of the velocity components, depends on the mutual product of the eigenfunctions corresponding to different wavenumbers. These wavenumbers may combine, giving rise to a transient behaviour different from the asymptotic one (Farrell & Moore 1992).

The literature concerning non-normal analysis of shear flow is quite extended in the case of flow between rigid walls, where the classical theorems by Rayleigh and Fjørtoft apply (see for instance Schmid & Henningson 2001). Much less is known when a free surface bounds a fluid subject to gravity. Olsson & Henningson (1994) considered a viscous free-surface fluid flowing down an inclined plate. The fluid layer is very shallow and the gravity force establishes a parabolic current. Using a velocity–vorticity formulation, they showed that, for moderate times, the transient growth dominates over the exponential growth and that its characteristics are similar to the transient growth found in other shear flows. Farrell & Ioannou (2008) addressed the problem of wind-wave generation by considering an atmospheric baseflow of logarithmic type without current. The Rayleigh equation describes the dynamics of the air-flow perturbation, thus suggesting non-orthogonality of eigenfunctions. A non-normal analysis reveals that the largest growth rates may be much greater than the maximum value predicted by modal analysis.

In this paper, we address the stability of a smooth concave monotonic free-surface shear flow by non-normal analysis of the Rayleigh equation. The aim is to compare the maximum eigenvalue provided by single-mode analysis with the maximum growth rate of a non-modal perturbation at time $t = 0$. We show that, exploring the space of parameters that characterize the base flow, the maximum growth rate of the perturbation at initial time can be much larger than the maximum eigenvalue.

The paper is organized as follows. In § 2, the mathematical problem is formulated introducing the Rayleigh equation; its eigenfunctions are introduced and it is shown that they are not mutually orthogonal. In § 3, a method for evaluating numerically the maximum growth rate of a perturbation at time $t = 0$ is described; the results are compared with the maximum eigenvalue for a given base velocity profile in § 4. Some basic notions on non-normal systems of eigenvectors are reported in the Appendix.

2. The Rayleigh equation

Our starting point is the Euler equations in two dimensions in the presence of a gravitational field:

$$u_t + uu_x + vu_y + \frac{p_x}{\rho} = 0, \tag{2.1}$$

$$v_t + uv_x + vv_y + \frac{p_y}{\rho} = -g, \tag{2.2}$$

where $u(x, y, t)$ and $v(x, y, t)$ are the horizontal and vertical components of the velocity field, respectively; ρ is the density of the fluid, $p(x, y, t)$ is the pressure and g is the acceleration due to gravity. Subscripts denote differentiation. We choose a coordinate system where the flat bottom is located at $y = -h$ and the undisturbed water elevation is $y = 0$. The following boundary conditions apply:

$$\zeta_t + u\zeta_x = v, \qquad y = \zeta, \tag{2.3}$$

$$p = 0, \qquad y = \zeta, \tag{2.4}$$

$$v = 0, \qquad y = -h, \tag{2.5}$$

where $y = \zeta(x, t)$ is the free-surface elevation.

It is well known that a shear velocity $U(y)$ with flat free surface is a solution of the equations above. We linearize around such an equilibrium state taking $u = U(y) + \hat{u}(x, y, t)$ and restricting ourselves to small perturbations of the hydrostatic pressure: $p = \rho g y + \hat{p}(x, y, t)$. After introducing the streamfunction, $u = \psi_y$, $v = -\psi_x$, we look for solutions of the form $\psi(x, y, t) = \hat{\psi}(y; k)\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}$, $\zeta(x, t) = \hat{\zeta}(k)\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}$; after some calculations, dropping the superscript, the Rayleigh equation is found:

$$\psi_{yy} - \left(k^2 + \frac{U''}{U - c}\right)\psi = 0, \tag{2.6}$$

where $c(k) = \omega(k)/k$ is the phase velocity. Boundary conditions at the free surface are obtained using the continuity of the pressure at the interface (equation (2.4)). From the linearized horizontal component of the momentum equation, (2.1), the kinematic boundary condition (2.3) can be rewritten as:

$$-\mathrm{i}\omega(k)\zeta(k) + \mathrm{i}kU\zeta(k) = -\mathrm{i}k\psi, \qquad y = 0, \tag{2.7}$$

and, disregarding non-hydrostatic contributions of the pressure, we obtain:

$$(U - c)^2\,\psi_y - (U'(U - c) + g)\psi = 0, \qquad y = 0, \tag{2.8}$$

$$\psi = 0, \qquad y = -h. \tag{2.9}$$

The Rayleigh equation (2.6), with boundary conditions (2.7)–(2.9), reads as an eigenvalue problem. The Rayleigh equation can have a singular point where the baseflow velocity is equal to the phase velocity ($\mathrm{Re}\, c(k)$). However, we are interested in instability and we look for wave speeds with non-zero imaginary part.

For a given wavenumber k in the x-direction and a given shear profile $U(y)$, the eigenvalue problem (2.6)–(2.9) can be solved numerically, after fixing the uniqueness of the eigenfunction solution by a normalization condition. We follow Morland *et al.* (1991) taking

$$\psi_y(0; k) = 1. \tag{2.10}$$

The phase velocity of the kth component is fixed by the boundary condition (2.8):

$$c(k) = U(0) - \frac{U'(0)\psi(0;k)}{2} \pm \tfrac{1}{2}\sqrt{(U'(0)\psi(0;k))^2 + 4g\psi(0;k)}. \tag{2.11}$$

Eigenfunctions of the Rayleigh equation corresponding to different wavenumbers are not mutually orthogonal. This can be shown by considering the following integration by parts:

$$\int_{-h}^{0} \psi_y(y;k_1)\psi_y^*(y;k_2)\,\mathrm{d}y = -\int_{-h}^{0} \psi(y;k_1)\psi_{yy}^*(y;k_2)\,\mathrm{d}y + [\psi(y;k_1)\psi_y^*(y;k_2)]_{-h}^{0}. \tag{2.12}$$

Similarly,

$$\int_{-h}^{0} \psi_y(y;k_1)\psi_y^*(y;k_2)\,\mathrm{d}y = -\int_{-h}^{0} \psi_{yy}(y;k_1)\psi^*(y;k_2)\,\mathrm{d}y + [\psi_y(y;k_1)\psi^*(y;k_2)]_{-h}^{0}. \tag{2.13}$$

Subtracting the latter expressions and using (2.6) with its boundary conditions, it follows that

$$\int_{-h}^{0} \left[\left(k_2^2 + \frac{U''}{U + c^*(k_2)}\right) - \left(k_1^2 + \frac{U''}{U + c(k_1)}\right)\right] \quad \psi(y;k_1)\psi^*(y;k_2)\,\mathrm{d}y = \psi(0;k_1) - \psi^*(0;k_2) \neq 0. \tag{2.14}$$

Therefore, the eigenfunctions of the Rayleigh equation are not mutually orthogonal. Note that this characterization is due to the boundary conditions at the free surface.

For a given shear profile $U(y)$, the phase velocity $c(k)$ defines the linear stability of the system: if $\mathrm{Im}(c(k)) > 0$ for some k, the system is unstable. Nevertheless, because the operator is non-normal, such a dispersion relation provides no information about the largest perturbation growth rate, which is due to be determined by a different kind of approach.

3. Stability and growth rate

Some remarks on the relation between stability and normality can be stated effectively in the framework of dynamical systems, i.e. in finite-dimensional spaces. The strategy we use for finding the maximum growth rate is introduced in such a simple framework first and then applied to the Rayleigh equation in the next section.

Given an ordinary differential equation, we consider its linearized form around the equilibrium state $w_0 \in \mathbb{R}^n$:

$$\dot{w}(t) = A(w(t) - w_0). \tag{3.1}$$

where A is the resulting linear operator. The equilibrium configuration is stable in the sense of Lyapunov if $\forall\epsilon > 0, \exists\delta$ such that (see, for instance, Drazin & Reid 2004)

$$|w(0) - w_0| < \delta \Rightarrow |w(t) - w_o| < \epsilon. \tag{3.2}$$

A closer look at the definition above reveals that it provides no information about the behaviour of $|w(t) - w_o|^2$ at short times: at $t = 0$, the solution can drift away from w_o even though the equilibrium is stable.

Linear stability analysis provides a sufficient condition for stability: if all the eigenvalues of A have negative real part, the equilibrium point w_o is stable. For a normal operator, this implies that the energy norm of the solution decreases monotonically in time, but this statement is not true for non-normal operators, because of the lack of orthogonality: even for a stable configuration, there may exist perturbations that trigger a transient growth of the energy of the solution. As the time of growth of such a perturbation can be large compared to the typical observation time of the physical system, it is not possible to extrapolate information on finite-time behaviour on the basis of standard modal analysis. The modal analysis is a tool to predict stability (or instability) according to the definition above, which is intrinsically of an asymptotic nature and no information is provided about the transient regime.

As the energy is the basic physical quantity of interest, we focus on the time evolution of a suitable norm of the unknown. Given the differential problem (3.1), the energy is defined as

$$E(t) = \frac{\langle w(t), w(t) \rangle}{2}, \tag{3.3}$$

where angle brackets indicate a suitable internal product.

We are here interested in maximizing the energy growth at $t = 0$. The time derivative of the energy is

$$\begin{aligned} 2\dot{E}(t) &= \langle \dot{w}(t), w(t) \rangle + \langle w(t), \dot{w}(t) \rangle \\ &= \langle Aw(t), w(t) \rangle + \langle w(t), Aw(t) \rangle \\ &= \langle (A + A^*) w(t), w(t) \rangle, \end{aligned} \tag{3.4}$$

where A^* denotes the adjoint operator of A. Determining the maximum energy growth at initial time corresponds to finding the maximum eigenvalue of $(A + A^*)/2$ (see Schmid & Henningson 2001).

Our strategy for investigating the spectrum of $A + A^*$ consists in using directly the spectrum of the original operator A, without the explicit introduction of the adjoint operator. The maximum eigenvalue of $A + A^*$ is related to the eigenvectors and eigenvalues of A in a form that enlightens the geometric nature of non-normality. In fact, let $w = w_i r_i$, where $\{w_i\}$ are the vector components in the basis of the eigenvectors $\{r_i\}$. The convention of sum over repeated indixes applies. Equation (3.4) can be rewritten as:

$$\begin{aligned} \langle (A + A^*) w, w \rangle &= \langle Aw, w \rangle + \langle A^* w, w \rangle \\ &= \langle A w_i r_i, w_j r_j \rangle + \langle A^* w_i r_i, w_j r_j \rangle. \end{aligned} \tag{3.5}$$

Using (A 9) in the Appendix, connecting the eigenvectors $\{r_j\}$ of A to the eigenvectors $\{\ell_j\}$ of the adjoint operator, the latter can be rewritten as:

$$\begin{aligned} \langle (A + A^*) w, w \rangle &= \langle w_i A r_i, w_j a_{jk} \ell_k \rangle + \langle A^* w_i a_{ik} \ell_k, w_j r_j \rangle \\ &= \langle w_i \lambda_i r_i, w_j a_{jk} \ell_k \rangle + \langle w_i a_{ik} \lambda_k^* \ell_k, w_j r_j \rangle \\ &= w_i \lambda_i w_j^* a_{jk}^* \langle r_i, \ell_k \rangle + w_i a_{ik} \lambda_k^* w_j^* \langle \ell_k, r_j \rangle. \end{aligned} \tag{3.6}$$

The bi-orthogonality condition (see (A 6) in the Appendix) allows us to simplify (3.6) as follows:

$$\begin{aligned} \langle (A + A^*) w, w \rangle &= w_i \lambda_i w_j^* a_{ji}^* + w_i a_{ij} \lambda_j^* w_j^* \\ &= w_j^* (\lambda_i a_{ji}^* + a_{ij} \lambda_j^*) w_i, \end{aligned} \tag{3.7}$$

where Re(.) indicates the real part of the argument. Therefore, the maximum eigenvalue of $A + A^*$ is the maximum eigenvalue of the Hermitian matrix

$$\langle r_i, r_j \rangle (\lambda_i + \lambda_j^*), \tag{3.8}$$

(no sum on i and j) corresponding to the largest eigenvalue of A for normal matrices (that is when $\langle r_i, r_j\rangle = 2\,\mathrm{Re}(\lambda_i)\delta_{ij}$).

Matrices of the type (3.8) are typically ill-conditioned: eigenvalues can differ for order of magnitude. However, our major interest is in determining the maximum eigenvalue, so the numerical difficulties that can arise in characterizing the whole spectrum can be disregarded in the present context.

The simplest way to obtain numerically the maximum eigenvalue of a matrix is by the power method (see, for instance, Quarteroni, Sacco & Saleri 2007). Defining

$$w_i^{n+1} = \langle r_i, r_j\rangle(\lambda_i + \lambda_j^*)w_j^n, \tag{3.9}$$

the maximum eigenvalue of the matrix (3.8) is given by

$$\nu_{max} = \lim_{n\to+\infty} \frac{w_i^n\left(w_i^{n+1}\right)^*}{w_j^n\left(w_j^n\right)^*}. \tag{3.10}$$

Summarizing, the illustrated methodology allows us to determine the maximum growth rate of a perturbation at the initial time as the maximum eigenvalue of the linear operator (3.8), calculated from the eigenfunctions and eigenvalues of the original operator. Note that a suitable internal product $\langle . \rangle$ must be specified, possibly on the basis of physical arguments. Moreover, this approach leads to the determination of the maximum eigenvalue of $A + A^*$, which is our goal, without giving information on the corresponding eigenvector; designing the optimal perturbation calls for a different approach not addressed herein (see Farrell & Ioannou 2008).

4. Maximum growth rate: results and discussion

The methodology outlined in the section above can be applied to determine the maximum growth rate of a perturbation of a shear current. For a given wavenumber k in the x-direction and a given velocity profile $U(y)$, the eigenvalue problem (2.6)–(2.9) is solved numerically, after fixing the uniqueness of the eigenfunction solution by the normalization condition (2.10). Adopting a second-order implicit finite-difference centred scheme to discretize (2.6) with boundary conditions (2.9) and (2.10), the resulting tridiagonal system provides a solution for a given value $c(k)$. The phase velocity is fixed by the boundary condition (2.11). Using the resulting $c(k)$ in the differential problem (2.6)–(2.9) iteratively yields the solution.

There is some degree of freedom in investigating the behaviour in time of the kinetic or the potential as well as the total energy of the perturbations; every choice corresponds to choosing a specific norm of the solution. Driven by physical motivations, here we concentrate on the maximum growth in time of the total energy of the perturbation, that is

$$2E = \rho \int_{-h}^{0} (|\psi_y|^2 + |\psi_x|^2)\,\mathrm{d}y + \rho g \zeta^2. \tag{4.1}$$

The time evolution of the kth component of the solution in $x = 0$ can be written as follows:

$$\psi(y, t; k) = \psi_k(y)\mathrm{e}^{-\mathrm{i}\omega(k)t}. \tag{4.2}$$

Therefore,

$$\frac{\partial}{\partial t}\psi(y, t; k)|_{t=0} = -\mathrm{i}\omega(k)\psi_k(y). \tag{4.3}$$

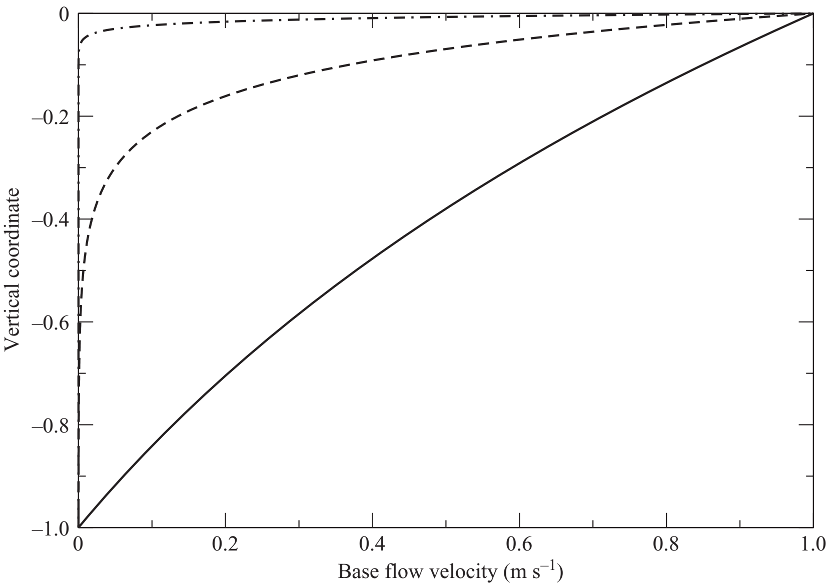

FIGURE 1. The unperturbed current profile with $U_o = 1\,\mathrm{m\,s^{-1}}$ for: —, $\alpha = 1$; – – –, 0.1; · - ·, 0.01.

The maximum growth rate of a perturbation at time $t = 0$ is determined by the largest eigenvalue of the form (3.7), after the identification of eigenvectors, eigenvalues and the scalar product of the problem at hand. Using the linearized kinematic condition (2.7), we must calculate the maximum eigenvalue of the Hermitian operator:

$$a(k_1, k_2) = (\omega^*(k_2) - \omega(k_1)) \int_{-h}^{0} (\psi_y(y, k_1)\psi_y^*(y, k_2) + k_1 k_2 \psi(y, k_1)\psi^*(y, k_2))\,\mathrm{d}y$$
$$+ (\omega^*(k_2) - \omega(k_1)) g \frac{\psi(0, k_1)}{c(k_1) - U(0)} \frac{\psi^*(0, k_2)}{c^*(k_2) - U(0)}. \quad (4.4)$$

As a specific example of shear flow, we consider the exponential profile:

$$U(y) = U_o \frac{\mathrm{e}^{(y+1)/\alpha} - 1}{\mathrm{e}^{1/\alpha} - 1}, \quad (4.5)$$

where U_o represents the surface velocity while α is a parameter controlling the shear-layer thickness. Figure 1 shows a plot of the base flow for $U_o = 1\,\mathrm{m\,s^{-1}}$ and different values of the parameter α. This profile resembles the one proposed by Morland *et al.* (1991) for $\alpha \ll 1$. It has no inflection point, as the Rayleigh theorem does not hold for free-surface flow. Modal growth rates clearly depend on the specific base flow; however, the examples reported by Morland *et al.* (1991) suggest that, for a given vorticity, the stability characterization weakly depends on the specific smooth shear function. Similar results have been obtained by van Gastel *et al.* (1985) who adopt a linear-logarithmic, exponential or error-function profile.

Eigenvalues and growth rates are computed by dividing the depth (1 m) into a finite number of intervals. The eigenvalue problem (2.6) with corresponding boundary conditions is then solved by an implicit finite-difference method with $k_{min} \leqslant k \leqslant k_{max}$. The nonlinearity due to the dependence of $c(k)$ on $\psi(0; k)$ (see the dispersion relation (2.11)) is handled iteratively by fixed-point iterations. Very slow convergence is achieved when the vertical coordinate is near to the depth y such that $U(y) = \mathrm{Re}(c)$, a

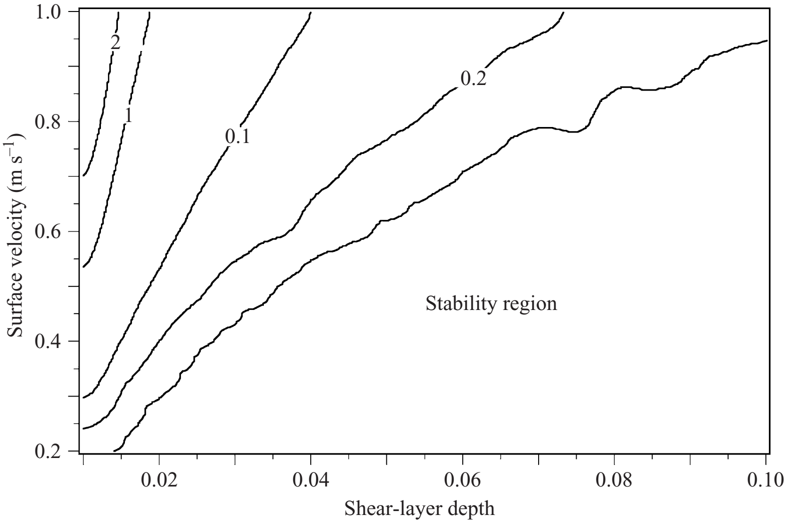

FIGURE 2. Maximum imaginary eigenvalue of the perturbation at time $t = 0$ for a given shear flow of exponential type. For the stable region, all eigenvalues have null positive imaginary part up to computing accuracy.

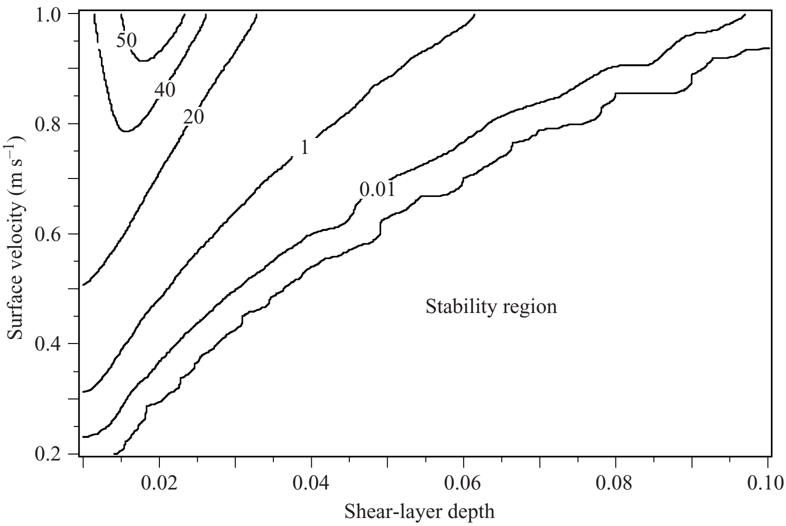

FIGURE 3. Maximum growth rate of an initial perturbation at time $t = 0$ for a given shear flow of exponential type.

difficulty reported also by Valenzuela (1976) and Morland *et al.* (1991). The maximum wavenumber k_{max} corresponds to the wavelenght such that capillarity, not resolved in the present model, starts to play a role (2 cm), while k_{min} is sufficiently small that all frequencies with non-null positive imaginary part are retained. The maximum eigenvalue of (4.4) is obtained by the power method outlined in § 3 where finite differences approximate derivatives.

Numerical results are shown in figures 2 and 3: the maximum value of $\mathrm{Im}(\omega(k))$ (figure 2) should be compared with the maximum perturbation growth rate (figure 3) as a function of the parameter α that controls the shear-layer thickness and surface velocity U_o.

The maximum surface velocity ranges between $0.2\,\mathrm{m\,s^{-1}}$, the minimum value exhibiting instability for some value of α, and $1\,\mathrm{m\,s^{-1}}$, a maximum reasonable physical value. The parameter α ranged in the computations between $\alpha = 0.01$ and $\alpha = 1$, corresponding approximately to a shear thickness between 0.05 and 1. For large α, the flow becomes stable, thus in figure 1, we restrict ourselves to the more significant range $0.01 \leqslant \alpha \leqslant 0.08$. The stability region corresponds to $\mathrm{Im}(\omega(k)) \leqslant 0$ (up to round-off error). For small surface velocity ($U_o \leqslant 0.2$) the flow is always stable. For large surface velocity, instability appears if the shear layer is thin enough, i.e. for large enough vorticity of the base flow.

In the instability region, maximum imaginary eigenvalues are located at the top left-hand corner (see figure 2), where the vorticity of the base flow is maximum. Conversely, the maximum transient growth rate is predicted around $\alpha = 0.024$ (figure 3), thus confirming that non-normality plays a role. More important, transient growth rate is larger than the maximum eigenvalue in any point of the considered portion of the (α, U_o)-plane. For values of α around 0.024 and high surface velocity, such a ratio raises up to 60.

From a physical point of view, a shear current is typically established by the wind action and the velocity at the surface is of the order 3 % of the asymptotic wind. Therefore, we expect that non-modal analysis is relevant for wind speed larger than $15\,\mathrm{m\,s^{-1}}$.

Our results confirm that the knowledge of the spectrum of the Rayleigh operator may yield poor information about maximum growth rate of the perturbations at short time and a non-normal analysis is required. This conclusion is along the lines of many other findings in shear flows (Schmid & Henningson 2001).

Appendix

In this section, we summarize some notions on a bi-orthogonal basis. The theory is restricted to finite-dimensional spaces, a simpler framework that nevertheless allows us to point out the crucial aspects of non-normality: no issues of convergence and completeness are addressed in the present work.

Given two normed spaces X, Y and an operator $A : X \rightarrow Y$, its adjoint $A^* : Y \rightarrow X$ is defined by

$$\langle w_2, Aw_1 \rangle_Y = \langle A^* w_2, w_1 \rangle_X, \qquad \forall w_1 \in X, w_2 \in Y. \tag{A 1}$$

where $\langle w_2, Aw_1 \rangle_Y$ and $\langle A^* w_2, w_1 \rangle_X$ denote the scalar product in Y and X, respectively.

An operator is normal if it commutes with its adjoint. For any A, the operator A^*A is self-adjoint and is therefore normal. The eigenvectors $\{e_i\}$ of a normal operator with distinct eigenvalues μ_i are mutually orthogonal. In fact

$$\begin{aligned} \langle A^*Ae_i, e_j \rangle &= \langle \mu_i e_i, e_j \rangle = \mu_i \langle e_i, e_j \rangle \\ &= \langle e_i, AA^* e_j \rangle = \langle e_i, \mu_j^* e_j \rangle = \mu_j \langle e_i, e_j \rangle. \end{aligned} \tag{A 2}$$

In general, the eigenvalues and eigenvectors of A^*A differ from those of A (and A^*). If λ_i is an eigenvalue of A, λ_i^* is an eigenvalue of A^*. In fact, let r_j, ℓ_i be eigenvectors of A and A^*, respectively, then

$$\langle \ell_i, Ar_j \rangle = \langle \lambda_i^* \ell_i, r_j \rangle = \langle A^* \ell_i, r_j \rangle, \qquad \forall r_j. \tag{A 3}$$

Therefore

$$(A^* - \lambda_i^*)\ell_i = 0, \tag{A 4}$$

and the eigenvectors of the adjoint operator form a biorthogonal basis:

$$\langle A\ell_i, r_j\rangle = \lambda_i\langle \ell_i, r_j\rangle = \langle \ell_i, A^* r_j\rangle = \lambda_j\langle \ell_i, r_j\rangle, \tag{A 5}$$

and it follows that

$$\langle \ell_i, r_j\rangle = c_i\delta_{ij}. \tag{A 6}$$

The adjoint basis $\{\ell_i\}$ can be suitably normalized so that $c_i = 1$.

Thanks to the biorthogonality condition, the relation between the two bases can be simply calculated. In fact, if

$$r_i = a_{ij}\ell_j, \tag{A 7}$$

multiplying both sides by r_k, it follows that

$$\langle r_i, r_k\rangle = \langle a_{ij}\ell_j, r_k\rangle = a_{ij}\langle \ell_j, r_k\rangle = a_{ik}\langle \ell_k, r_k\rangle, \tag{A 8}$$

and therefore,

$$a_{ik} = \langle r_i, r_k\rangle, \tag{A 9}$$

where the normalization condition has been used.

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