THE TWO-FREQUENCY MICROWAVE TECHNIQUE FOR MEASURING OCEAN-WAVE SPECTRA FROM AN AIRPLANE OR SATELLITE

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Abstract. The two-frequency microwave technique at slanting incidence for the measurement of ocean wave spectra, first proposed by Ruck *et al.* (1972), is investigated in more detail with respect to its applicability in aircraft and space vehicles. It is shown that by carrying out signal processing in the frequency domain the system-inherent signal-to-noise ratio can be increased considerably, making the operation of the system from air- and space-borne platforms feasible.

1. Introduction

The two-frequency microwave technique at slanting incidence angles first proposed by Ruck *et al.* (1972), (cf. also Ransone and Wright, 1972; Hasselmann, 1972) may prove to be a feasible, simple method for measuring two-dimensional ocean-wave spectra from aircraft and satellites. However, its applicability from air- or space-borne platforms has occasionally been questioned (e.g., Plant, 1977). In this paper, we investigate this problem in more detail.

For fixed difference frequency, incidence angle and azimuth the technique measures a single wave component of the long surface wave field determined by the Bragg resonance condition for the 'beat wave'. By varying the difference frequency of the two microwave signals, together with the azimuth of the antenna axis, the full two-dimensional gravity wave spectrum can be determined. The method makes use of the fact that the short surface waves (capillary and ultra-gravity waves) which scatter the microwaves via the Bragg mechanism are modulated by the longer waves on which they propagate.

In order to relate the backscattered microwave signals to the surface wave spectrum, the modulation transfer function must be known. This can be calculated using the two-scale, or composite-wave model. The modulation of the backscattered power arises from the change in local depression angle induced by the long waves, as calculated by Barrick (1972), and from the hydrodynamic interactions between the short Bragg waves and the longer surface waves. This latter effect has not always been fully considered by previous authors when discussing the two-frequency technique (Ruck *et al.*, 1972; Barrick, 1972; Jackson, 1974; Plant,

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1977). However, Keller and Wright (1975) have taken hydrodynamic interactions into account when calculating the modulation of the backscattered microwave power from short wind-generated waves in a wave tank in the presence of a longer plunger-generated wave. More recently, this interaction has also been studied by Plant *et al.* (1977) in the ocean for shoaling waves.

In this paper the basic equations for the two-frequency technique are rederived in the frequency domain using a rather different analysis from that of previous investigators. It is shown that by carrying out signal processing in the frequency domain the system-inherent signal-to-noise ratio can be increased considerably. Under typical conditions an effective signal-to-noise ratio of order 10 : 1 is estimated for both airplane and satellite applications, and it is concluded that in both cases the two-frequency technique does indeed represent a useful, simple method for measuring the two-dimensional surface-wave spectrum.

2. The Theory of the Two-Frequency Technique

We consider a radar system which emits two continuous waves (CW) of slightly differing frequencies f_1, f_2. In practice, the implementation of the two-frequency technique may be achieved by a scatterometer which emits successive pairs of pulses at the two frequencies. We assume that in this case the time separation between subsequent pulses of different frequency is so small that aliasing effects can be neglected and the analysis reduces to the CW case.

The antenna axis is assumed to be directed at the ocean surface at an oblique incidence angle such that the backscattering of the microwaves by the rough sea surface is caused by Bragg scattering. Typically, the incidence angles should lie between 30° and 70°, where the upper limit is determined by the shadowing effects of long waves near glancing incidence and the lower limit by the breakdown of the small-amplitude conditions for the long Bragg-scattering waves encountered near vertical incidence (zero incidence angle).

To relate the backscattered signal to the surface-wave field we apply the two-scale or composite-wave model, as introduced by Wright (1968), Barrick and Peak (1968) and Bass *et al.* (1968). According to this model, the surface-wave spectrum is divided into two regions of different wavelength scales. The short waves represent the Bragg-scattering waves. The long waves are represented locally by tangent surface elements, or 'facets', of dimension D small compared to the wavelength $\hat{\lambda}$ of the long waves, but large compared to the wavelength λ of the Bragg waves,

$$\lambda \ll D \ll \hat{\lambda} . \tag{1}$$

It is then assumed that Bragg theory can be applied in the local reference system of the moving, inclined facet.

The amplitudes $B^{(s)}$ $(s = 1, 2)$ of the two backscattered microwave signals can be represented as the superposition of the contributions from all facets j within the footprint of the antenna. For simplicity of notation, we assume that the emitted waves both have the same amplitude. In the reference system of the antenna, moving with the platform velocity $\mathbf{V}$, the (complex) backscattered amplitudes can then be expressed in the form

$$B^{(s)} = \sum_j A_j \, e^{i\varphi_j^{(s)}}. \tag{2}$$

Here A_j represents a (complex) scattering amplitude for the facet j, and the phase $\varphi_j^{(s)}$ is given by

$$\varphi_j^{(s)} = 2\mathbf{x}_j \mathbf{k}_j^{(s)},$$

where $\mathbf{x}_j$ is the average position of the facet j relative to the antenna, located at $\mathbf{x} = 0$, and $\mathbf{k}_j$ is the wave-number vector of the backscattered wave, pointing from the facet j to the antenna.

Although the complex amplitude A_j contains a phase factor by definition, we have retained in A_j only the phase contribution which is the same for both microwave frequencies. The frequency-dependent part, arising from the different phase shifts along the propagation path to and from the facet for each frequency, has been factored out in the term $e^{i\varphi_j^{(s)}}$. This separation is well-defined if the facet size is small compared with the horizontal wavelength $\lambda_h = 2\pi/|\Delta \underline{k}|$ of the beat wave, where $\Delta \mathbf{k} = \mathbf{k}^{(1)} - \mathbf{k}^{(2)}$ represents the beat wave number and $\Delta \underline{k}$ its horizontal projection. In this case the backscattered field from a given facet can be regarded as identical for each frequency, except for the relative phase shift between the two waves.

Because we have chosen the platform as reference system, the velocity of the facet j is given by the sum of the platform velocity $\mathbf{V}$ and the orbital velocity $\hat{\mathbf{U}}_j$ of the long waves at the local facet position. Thus for small times

$$\varphi_j^{(s)} = \varphi_j^0 + 2\mathbf{k}^{(s)} \cdot (\hat{\mathbf{U}}_j + \mathbf{V})t$$

or, alternatively,

$$\varphi_j^{(s)} = \varphi_j^0 + 2\underline{k}_j^{(s)} \underline{x}_j + 2\mathbf{k}^{(s)} \cdot (\hat{\boldsymbol{\xi}}_j + \mathbf{V}t), \tag{3}$$

where $\hat{\boldsymbol{\xi}}_j(t) \equiv (\hat{\underline{\xi}}_j, \hat{\zeta}_j)$ represents the horizontal and vertical displacements of the facet by the long-wave orbital motion relative to its mean position $\mathbf{x}_j$, corresponding to the surface at rest (the Stokes drift may be neglected or included in $\mathbf{V}$).

The Doppler shift induced by the phase velocity of the short Bragg-scattering waves relative to the facet velocity appears in this model in the phase of the scattering factor A_j, since within the approximation (1) it is the same for each frequency.

The mean square modulus of the scattering amplitude A_j is proportional to the local cross section σ_j per unit surface area, the proportionality factor $G(\mathbf{x}_j)$ depending on the distance of the platform from the facet, the antenna pattern and the areas of the horizontal projections of the facet surfaces. The facet sizes can be defined such that the latter are constant for all j and t, and can therefore be taken as Unity.

Thus,

$$\overline{|A_j|^2} = G(\mathbf{x}_j)\sigma_j\,. \tag{4}$$

The averaging bar denotes here the ensemble average over the short waves for a given long-wave-field realisation.

Because of the electromagnetic and hydrodynamic modulation, the cross section σ_j will be a function of the position of the facet with respect to the long-wave field. In fact, σ_j depends not only on the facet position, but also on the facet velocity and other properties of the long-wave field (cf. Appendices A and B). For linear (i.e., weak) modulation this dependence may be expressed by a linear transfer function $R_{\hat{k}}$, and Equation (4) may be written

$$\overline{|A_j|^2} = \sigma_0 G(\mathbf{x}_j)\left[1 + \int (R_{\hat{k}} z_{\hat{k}}\, \mathrm{e}^{i(\hat{k}\mathbf{x}_j - \hat{\omega} t)} + \text{c.c.})\, \mathrm{d}\hat{\underline{k}}\right], \tag{5}$$

where σ_0 is the average cross section over the footprint, $z_{\hat{k}}$ represents the Fourier transform of the surface elevation $\hat{\zeta}$ associated with the long carrier waves

$$\hat{\zeta} = \int [z_{\hat{k}}\, \mathrm{e}^{i(\hat{k}\cdot \mathbf{x} - \hat{\omega} t)} + \text{c.c.}]\, \mathrm{d}\hat{\underline{k}} \tag{6}$$

with $\hat{\omega} = (g|\hat{\underline{k}}|)^{1/2}$, $g = 9.81\ \mathrm{m}\cdot\mathrm{s}^{-2}$, and c.c. stands for complex conjugate.

We consider now the complex product

$$y(t) = B^{(1)}B^{(2)*} = \sum_{jj'} A_j A_{j'}^*\, \mathrm{e}^{i(\varphi_j^{(1)} - \varphi_{j'}^{(2)})} \tag{7}$$

and its frequency power spectrum

$$P_y(f) \equiv P_{B_1 B_2^*}(f) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-2\pi i f \tau} \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, y(t) y^*(t+\tau)\,. \tag{8}$$

It will be shown that $P_y(f)$ contains a peak which is proportional to the long-wave spectrum. This provides the basis for the dual frequency technique.

Since the surface wave field is both statistically stationary and homogeneous, the time average in Equation (8) can be replaced by a spatial average $\langle \cdot\cdot\cdot \rangle$ over a large number of different, statistically independent surface patches:

$$P_y(f) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-2\pi i f \tau} \langle y \cdot y^*(\tau)\rangle\,. \tag{9}$$

This is the ensemble average that a moving platform, such as an aircraft or spacecraft, in effect carries out when performing time averages of signals received at the platform. It may therefore be expected that the number of degrees of freedom involved in the estimate (9) is governed only by the distance the platform travels, which determines the number of statistically independent patches, and is not dependent on the platform speed. The sampling time needed to estimate a spectrum with given statistical stability is accordingly inversely proportional to the platform velocity.

Consistent with this result, we shall find that the characteristic frequencies of the spectrum $P_y(f)$ scale with the platform velocity. This result implies that the feasibility of the two-frequency method is not dependent on the speed of the instrument carrier and applies equally for air- and space-borne platforms.

Inserting Equation (7) in Equation (9) we obtain

$$\begin{aligned} P_y(f) &= \int_{-\infty}^{\infty} \mathrm{d}\tau \left\langle \sum_j \overline{|A_j|^2} \right\rangle^2 \mathrm{e}^{-2\pi i f\tau} + \\ &+ \int_{-\infty}^{\infty} \mathrm{d}\tau \left\langle \sum_{j,k} \overline{|A_j|^2}\,\overline{|A_k|^2}\, e^{i(\varphi_j^{(1)}-\varphi_j^{(2)}-\varphi_k^{(1)}+\varphi_k^{(2)})} \right\rangle e^{-2\pi i f\tau} \qquad (10) \\ &\equiv P_1(f)+P_2(f)\,. \end{aligned}$$

In deriving this equation we have assumed that the complex scattering amplitudes for different facets are statistically independent and have uniform phase distributions. Thus when calculating the ensemble average of quadruple products only index combinations containing the index pairs indicated in (10) survive.

The first term is the frequency spectrum $P_{|B_1|^2}$ of $|B^{(1)}|^2 \approx |B^{(2)}|^2$.

For complex Gaussian $B^{(s)}$ $(s = 1, 2)$ (which in the present case follows from the Central Limit Theorem) it can be expressed as the convolution of the Doppler spectrum of $B^{(s)}$ with its complex conjugate:

$$P_1 = \int_{-\infty}^{\infty} \mathrm{d}f' P_{B_s}(f') P_{B_s^*}(f-f')\,. \qquad (11)$$

According to the two-scale model (cf. Wright, 1968; Hasselmann and Schieler, 1971) $P_{B_s}(f)$ consists of the usual two Bragg lines, displaced by the Doppler shift f_D due to the platform velocity, and broadened by the Doppler shifts due to the random orbital wave motion and the variable Doppler shift of the platform motion within the footprint. For aircraft and satellite applications, the platform motions are dominant, and the two Bragg lines are merged into a single bell-shaped distribution, depending in detail on the antenna pattern, with a maximum at the mean platform Doppler shift f_D and a width of order $(1/\pi)|\mathbf{V}| \cdot |\mathbf{k}| \sin\theta\,\delta\theta$, where θ is the angle between the platform velocity $\mathbf{V}$ and the radar wave number $\mathbf{k}$, and $\delta\theta$ is the beam width.

The convolved spectrum $P_1(f)$ then has a peak at $2f_D$ and at zero frequency, each peak being of order $\sqrt{2}$ broader than the original peaks of $P_{B_s}(f)$ (cf. Figure 1).

The second term has a more interesting structure. Substituting Equation (3) and (5) into the second term of Equation (10), and replacing the sum over j and k by integrals, we obtain

$$P_2(f)=\sigma_0^2\int_{-\infty}^{\infty} d\tau\left\langle\left|\int d\underset{\sim}{x}G(\mathbf{x})\,e^{2i[\Delta \underset{\sim}{k}\underset{\sim}{x}+(\Delta \underset{\sim}{k}\cdot \underset{\sim}{V}-\pi f)\tau]}\times\right.\right.$$

$$\left.\left.\times\left\{1+\int d\hat{\underset{\sim}{k}}(S(\hat{\underset{\sim}{k}})z_{\hat{\underset{\sim}{k}}}\,e^{i(\hat{\underset{\sim}{k}}\underset{\sim}{x}-\hat{\omega}(\hat{\underset{\sim}{k}}))\tau}+\text{c.c.})\right\}\right|^2\right\rangle, \tag{11}$$

where $S(\hat{\underset{\sim}{k}})=R(\hat{\underset{\sim}{k}})+2i\,\Delta k_3-2\,\Delta\underset{\sim}{k}\cdot\hat{\underset{\sim}{k}}/|\hat{\underset{\sim}{k}}|$ and $\Delta\underset{\sim}{k}$ is the projection of $\Delta\mathbf{k}$ onto the horizontal plane.

The two additional terms in the transfer function $S(\hat{\underset{\sim}{k}})$ arise from the expansion of the phase term (cf. Equation (3)) in the form

$$e^{2i\Delta\mathbf{k}\hat{\boldsymbol{\xi}}}=1+2i(\Delta k_3-i\,\Delta\underset{\sim}{k}\,\hat{\underset{\sim}{k}}/|\hat{\underset{\sim}{k}}|)\hat{\zeta}, \tag{12}$$

which is valid for beat wavelengths large compared with the height of the long waves.

Performing first the $\underset{\sim}{x}$-integration and neglecting at this stage the variation of the Doppler shift $2\Delta\mathbf{k}\cdot\mathbf{V}$ within the footprint, we obtain

$$P_2(f)=4\pi^2\sigma_0^2\int d\tau\, e^{2i(\Delta\underset{\sim}{k}\cdot\underset{\sim}{V}-\pi f)\tau}\left\{\left|D(2\,\Delta k)\right|^2+\right.$$

$$+\tfrac{1}{2}\int d\underset{\sim}{k}|S|^2[|D(\hat{\underset{\sim}{k}}+2\,\Delta\underset{\sim}{k})|^2\,e^{-i\hat{\omega}\tau}+ \tag{13}$$

$$\left.+|D(-\hat{\underset{\sim}{k}}+2\,\Delta\underset{\sim}{k})|^2\,e^{+i\hat{\omega}\tau}]F(\hat{\underset{\sim}{k}})\right\},$$

where the filter function $D(\underset{\sim}{k})$ is the Fourier transform of the antenna footprint

$$D(\underset{\sim}{k})=\frac{1}{2\pi}\int d\underset{\sim}{x}G(\mathbf{x})\,e^{ikx} \tag{14}$$

and $F(\hat{\underset{\sim}{k}})$ is the spectrum of the long waves:

$$\langle z_{\hat{\underset{\sim}{k}}}z_{\hat{\underset{\sim}{k}}'}\rangle=\tfrac{1}{2}\delta(\hat{\underset{\sim}{k}}-\hat{\underset{\sim}{k}}')F(\hat{\underset{\sim}{k}}). \tag{15}$$

For a large footprint, $D(\underset{\sim}{k})$ is sharply peaked at $\underset{\sim}{k}=0$ and can be approximated by a δ-function,

$$|D(\underset{\sim}{k})|^2=\alpha^2A_f\,\delta(\underset{\sim}{k}), \tag{16}$$

where α is proportional to the incident radiation intensity at the center of the footprint and $A_f=\alpha^{-2}\int d\underset{\sim}{x}\,G^2(\mathbf{x})$ is a weighted footprint area. For example, for a Gaussian antenna pattern we have

$$G^2(\mathbf{x})=\alpha^2\exp[-\underset{\sim}{x}^2/\underset{\sim}{x}_0^2] \tag{17}$$

$$D^2(\underset{\sim}{k})=\frac{\alpha^2A_f}{2\pi\underset{\sim}{k}_0^2}\exp[-\underset{\sim}{k}^2/2\underset{\sim}{k}_0^2], \tag{18}$$

where

$$|\underline{k}_0| = \frac{1}{\sqrt{2}}\frac{1}{|x_0|} \tag{19}$$

and

$$A_f = \pi x_0^2 . \tag{20}$$

Taking the δ-function approximation ($A_f \to \infty$) for the second term of Equation (13), we obtain

$$\begin{aligned} P_2(f) = 4\pi^2\sigma_0^2\Big\{ & \delta\Big(-f + \frac{1}{\pi}\Delta\mathbf{k}\cdot\mathbf{V}\Big)|D(2\,\Delta\underline{k})|^2 + \\ & + \tfrac{1}{2}\alpha^2 A_f\Big[\delta\Big(-f + \frac{1}{\pi}\,\Delta\mathbf{k}\cdot\mathbf{V} - \hat{f}\Big)|S_-|^2 E(-2\,\Delta\underline{k}) + \\ & + \delta\Big(-f + \frac{1}{\pi}\,\Delta\mathbf{k}\cdot\mathbf{V} + \hat{f}\Big)|S_+|^2 E(-2\,\Delta\underline{k})\Big]\Big\} \end{aligned} \tag{21}$$

with

$$S_- = S(-2\,\Delta\underline{k}), \qquad S_+ = S(+2\,\Delta\underline{k}) \quad \text{and} \quad \hat{f} = \frac{1}{2\pi}\hat{\omega} .$$

The first term represents a d.c. contribution at essentially zero frequency and zero beat wave number, which is independent of the long-wave modulation and not of interest in the present context. The integral of this term over f for $\Delta\underline{k} = 0$ can be shown to be equal to

$$\begin{aligned} \int \mathrm{d}f\, P_1(f) &= \sigma_0^2\Big(\int \mathrm{d}x\, G(\mathbf{x})\Big)^2 \\ &= 4\sigma_0^2 A_f^2 \alpha^2 . \end{aligned} \tag{22}$$

For $\Delta\underline{k} \neq 0$, Equation (21) shows that $P_2(f)$ consists of two Bragg lines at the surface-wave frequencies $\pm\hat{f}$ corresponding to the Bragg components $\underline{k} = \pm 2\,\Delta\underline{k}$ of the beat wave number, with superimposed Doppler displacements due to the platform velocity:

$$f = \pm\hat{f} + \frac{1}{\pi}(\Delta\mathbf{k}\cdot\mathbf{V}) . \tag{23}$$

The peaks are directly proportional to the two-dimensional wave spectral density at the corresponding Bragg beat wave numbers, with a proportionality factor which is given by the modulation transfer function and the general radiation geometry.

If the variability of the platform-induced Doppler shift $2\,\Delta\mathbf{k}\cdot\mathbf{V}$ within the footprint is taken into account, a straightforward calculation shows that the Bragg lines are simply broadened by an amount of the order of

$$\delta f_{\text{res}} = \frac{1}{\pi} |\mathbf{V}| \cdot |\Delta \mathbf{k}| \, \delta\theta \,. \tag{24}$$

(The line broadening associated with the indeterminateness of the frequency as a result of the finite footprint size is small for the cases considered here. However, for small footprints and stationary platforms this effect can become dominant and was responsible for the finite width of the resonance peaks observed by Plant, 1977.)

3. Signal-to-Noise Ratio

The feasibility of extracting the surface wave spectrum from the Bragg peaks of $P_2(f)$ depends critically on the ratio of the energy in the peaks to the spectral density of the broad 'noise' background $P_1(f)$ in the vicinity of the peaks (Figure 1).

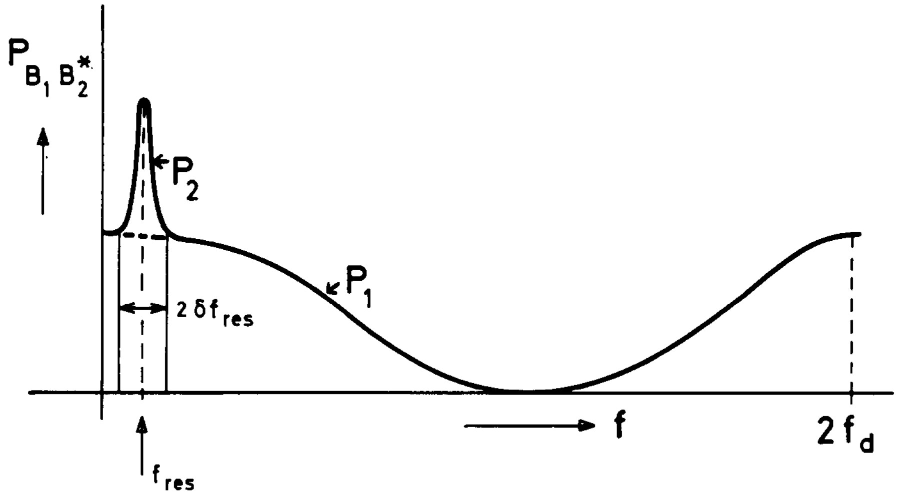

Fig. 1. Schemetic graph of the Doppler spectrum of the product signal $B^{(1)} \cdot B^{(2)*}$.

For the optimal case of a narrow band filter equal to twice the width δf_{res} of the peak, the signal-to-noise ratio is given by

$$\text{SNR} = \frac{\int_{f_{\text{res}}-\delta f_{\text{res}}}^{f_{\text{res}}+\delta f_{\text{res}}} \mathrm{d}f \, P_2(f)}{2\,\delta f_{\text{res}} P_1(f_{\text{res}})}$$

$$= \frac{2\pi^2 \sigma_0^2 \alpha^2 A_f \{|S_-|^2 F(-2\,\Delta \underset{\sim}{k}) + |S_+|^2 F(+2\,\Delta \underset{\sim}{k})\}}{2\,\delta f_{\text{res}} P_1(f_{\text{res}})} \,. \tag{25}$$

For satellite and most aircraft applications the Bragg frequency shifts $\pm\hat{f}$ are masked by the Doppler broadening δf_{res}, so that (25) must be expressed for the combined signal of both lines.

To obtain an estimate for SNR we insert in Equation (25) a typical value for the modulation transfer function as calculated from theory and measured during JONSWAP 1975 (to be published),

$$|R(\mp 2\,\Delta \underline{k})/2\,\Delta \underline{k}| \approx 13\,. \tag{26}$$

Assuming further that the convolved Doppler spectrum P_1 is Gaussian,

$$P_1 = A\,\exp\,(-f^2/2f_0^2) \tag{27}$$

with

$$f_0 = \frac{\sqrt{2}}{\pi} \cdot |\mathbf{V}|\,|\mathbf{k}|\delta\theta \gg \delta f_{\text{res}} \tag{28}$$

and

$$A = 4(2\pi)^{-1/2}\sigma_0^2 A_f^2 \alpha^2 f_0^{-1}\,, \tag{29}$$

we obtain approximately

$$\text{SNR} \approx 5 \cdot 10^2 \frac{f_0}{\delta f_{\text{res}}} \cdot \frac{1}{A_f} \cdot \hat{f} F(\hat{f}) S(\theta)\,. \tag{30}$$

Here $F(\hat{f})$ is the scalar spectrum of the long waves in $\text{m}^2\,\text{Hz}^{-1}$, A_f the area of the footprint in m^2 and $S(\theta)$ the angular spreading factor. Inserting a Philipps spectrum

$$E(\hat{f}) = \beta \cdot g^2 (2\pi)^{-4} \hat{f}^{-5} \tag{31}$$

with $\beta \approx 0.01$ and assuming a $\cos^4\theta$ spreading factor

$$S(\theta) = \frac{8}{3\pi}\cos^4\theta \quad \text{for } |\theta| \leq \frac{\pi}{2}, \qquad S(\theta) = 0 \quad \text{for } \frac{\pi}{2} < |\theta| < \pi\,, \tag{32}$$

we obtain as an estimate for the signal-to-noise ratio

$$\text{SNR} \approx 0.25 \hat{f}^{-4} A_f^{-1} f_0 \cdot (\delta f_{\text{res}})\,, \tag{33}$$

or, expressing A_f in terms of the footprint diameter $|\underline{x}_0|$, and assuming a Gaussian antenna pattern,

$$\text{SNR} = 0.1\left(\frac{\hat{\lambda}}{2|\underline{x}_0|}\right)^2 \frac{f_0}{\delta f_{\text{res}}} = 0.1\left(\frac{\hat{\lambda}}{2|\underline{x}_0|}\right)^2 \sqrt{2}\left|\frac{\mathbf{k}}{\Delta\mathbf{k}}\right|, \quad \text{where } \hat{\lambda} = \frac{2\pi}{|\hat{\underline{k}}|}\,. \tag{34}$$

Taking as example $|\mathbf{k}| = 2\pi/0.03\ \text{m}^{-1}$ (X-band), $\Delta\mathbf{k} = 2\pi/424\ \text{m}^{-1}$, $\theta = 45°$ corresponding to a water wavelength of $\hat{\lambda} = 300$ m and $2|\underline{x}_0| = 10$ km, we obtain

$$\text{SNR} = 1.8\,. \tag{35}$$

The same SNR is obtained, e.g., for a footprint diameter of 1 km and a water wavelength of 30 m, corresponding to a typical airplane application.

The SNR given by the formal definition can in fact be enhanced by perhaps an order of magnitude by substracting out the 'noise' background $P_1(f)$ which in the present case is easily detectable and measurable as it represents a spectrum which is much broader than the signal spectrum $P_2(f)$.

From Equation (33) or (34) it is evident that the SNR is larger for small footprint areas. However, small antenna footprints give poor spectral resolution (see Equation (19)). Experimentally, one should therefore aim for the smallest footprints which are still compatible with the desired spectral resolution.

The sampling time T_s required to resolve the Bragg peak with, say, 20 degrees of freedom (corresponding to 10 independent data pieces using a Bartlett spectral analysis technique) is of order $10\pi/|\underline{V}|\,|\Delta \underline{k}|\,\delta\theta$. However, more relevant for the application of the two-frequency technique is the sampling distance

$$D_s = T_s|\mathbf{V}| \approx \frac{10\pi}{|\Delta \underline{k}|\,\delta\theta}\,. \tag{36}$$

This is independent of the platform velocity, as anticipated, (see discussion of Equation (9)) and also of the footprint dimension. The quantity $2\Delta \underline{k} \cdot \delta\theta$ represents the horizontal wave number resolution $\delta \underline{k}_{\mathrm{res}}$. Equation (36) states simply that to measure a wave number spectrum with the resolution $\delta \underline{k}_{\mathrm{res}}$ with resonable statistical significance one needs a horizontal sampling distance an order of magnitude greater than the resolution wavelength.

This requirement follows from general sampling considerations and is independent of the particular technique used to sense the sea surface, or the speed with which the sea surface is sampled. Thus we conclude that theoretically the two-frequency technique is capable of yielding the maximal spectral resolution attainable with surface sampling schemes with an effective signal-to-noise ratio of order 10:1.

Appendix A: The Modulation Transfer Function

The modulation transfer function $R_{\underline{k}}$ is the sum of two terms

$$R_{\underline{k}} = R_{\underline{k}}^{\mathrm{tilt}} + R_{\underline{k}}^{\mathrm{hydr.}}\,. \tag{A1}$$

$R_{\underline{k}}^{\mathrm{tilt}}$ describes the modulation of the backscattering cross section due to the purely geometric effect that the facets are tilted by the carrier waves. Since the Bragg scattering cross section σ is given by

$$\sigma = T \cdot [E(-2\underline{k}) + E(+2\underline{k})]\,, \tag{A2}$$

where T is a scattering coefficient depending on the incidence angle, the 'tilt'-modulation of the cross sections is

$$\left(\frac{\delta\sigma}{\sigma}\right)_{\mathrm{tilt}} \doteq \frac{1}{\sigma_0}\frac{\partial\sigma}{\partial\underline{\hat{n}}}\bigg|_{\underline{\hat{n}}=0} \cdot \underline{\hat{n}}\,, \tag{A3}$$

where

$$\frac{1}{\sigma_0}\frac{\partial\sigma}{\partial\hat{\underline{n}}}=\frac{1}{E_m}\frac{\partial E_m}{\partial\hat{\underline{n}}}+\frac{1}{T}\frac{\partial T}{\partial\hat{\underline{n}}}, \tag{A4}$$

$$\hat{\underline{n}}=\frac{\partial\hat{\zeta}}{\partial\hat{\underline{n}}} \tag{A5}$$

and

$$E_m=E(-2\underline{k})+E(+2\underline{k}). \tag{A6}$$

Insertion of the Fourier representation (6) yields

$$R_{\underline{k}}^{\text{tilt}}=\frac{1}{\sigma_0}\frac{\partial\sigma}{\partial\hat{\underline{n}}}\bigg|_{\hat{\underline{n}}=0}\cdot i\hat{\underline{k}}. \tag{A7}$$

T can be calculated from electromagnetic theory and is given in the literature (e.g., Rice, 1951; Wright, 1968; Hasselmann and Schieler, 1972; Schieler, 1977).

The hydrodynamic modulation transfer function $R_{\underline{k}}^{\text{hydr.}}$ is calculated in Appendix B using WKB interaction theory. The modulation of the short-wave spectrum by the long waves is linearly related to the modulation of the radar cross section (see Equations (4) and (5)):

$$\left(\frac{\delta\sigma}{\sigma_0}\right)_{\text{hydr.}}=\frac{\delta E}{E_0}. \tag{A8}$$

Thus the dimensionless hydrodynamic modulation transfer function for radar backscattering required in Section 2 is identical with the transfer function for energy modulation.

Appendix B: Hydrodynamic Interactions

The hydrodynamic interaction between short and long surface waves has been treated by Keller and Wright (1975) in terms of the transport equation for the energy spectrum of short water waves in which the interactions are described by the radiation stress tensor of Longuet-Higgins and Stewart (1964).

Here we use a somewhat simpler representation of the WKB-type hydrodynamic interaction starting from the action balance rather than energy balance equation of the short waves. In the case of a vanishing source term, the equation describes the conservation of action density along a ray path in four-dimensional phase space. It can be shown quite generally that action density is a conserved quantity for waves propagating in a medium which varies on a much larger space and time scale than the waves. (see e.g., Whitham, 1965; Bretherton, 1970; Willebrand, 1975).

In solving the action balance equation we take the ratio of the group velocity of the short wave to the group velocity of the long wave and the long-wave slope as expansion parameters. Keller and Wright used a different expansion parameter, the ratio of the horizontal component of the orbital velocity of the long wave to the phase speed of the small wave. For short-wave radiation (e.g., X-band) and medium to high sea states this ratio is not always small.

In the two-scale approximation, the long waves are regarded as a slowly varying current in a vertically accelerated coordinate system on which the short waves propagate.

The evolution of the short-scale wave field is then governed by the action balance equation (radiation balance equation)

$$\mathscr{L}[N]=\frac{\mathrm{d}N}{\mathrm{d}t}=\left(\frac{\partial}{\partial t}+\dot{\underline{x}}\frac{\partial}{\partial \underline{x}}+\dot{\underline{k}}\frac{\partial}{\partial \underline{k}}\right)N=Q(\underline{k},\underline{x},t)\,, \tag{B1}$$

where $N(\underline{k}, \underline{x}, t)=E(\underline{k}, \underline{x}, t)/\omega'$ is the action spectrum, $\underline{k}$ the wave number, ω the frequency in a fixed coordinate system, ω' the intrinsic frequency of the wave components in a reference system moving with the local current $\hat{\underline{U}}(\underline{x}, t)$, and $Q=Q(\underline{k}, \underline{x}, t)$ a source function. The path velocities $\dot{\underline{x}}$ and $\dot{\underline{k}}$ are given by the ray equations

$$\dot{\underline{x}}=\frac{\partial\omega}{\partial\underline{k}} \tag{B2a}$$

$$\dot{\underline{k}}=-\frac{\partial\omega}{\partial\underline{k}}\,, \tag{B2b}$$

where $\omega(\underline{k}, \underline{x}, t)=\omega'(\underline{k})+\underline{k}\cdot\hat{\underline{U}}(\underline{x}, t)$ denotes the dispersion relation for deep water waves propagating in a variable current $\hat{\underline{U}}(\underline{x}, t)$ and an accelerated reference system. $\partial\omega/\partial\underline{k}$ is the group velocity in a fixed reference system, and $-\partial\omega/\partial\underline{x}$ the rate of change of the wave number (refraction) due to the large scale fluid motion. If the source function is zero, then Equation (B1) expresses in spectral form the conservation of action density of individual wave trains along ray paths in four-dimensional $\underline{x}$, $\underline{k}$-space.

The intrinsic frequency is given by

$$\omega'(\underline{k})=\left(\tilde{g}|\underline{k}|+\frac{\tau}{\rho}|\underline{k}|^3\right)^{1/2}\,, \tag{B3}$$

where τ is the surface tension, ρ the density of water and $\tilde{g}$ the sum of the acceleration of gravity g and the orbital acceleration (due to the fluid motion) projected in the direction of the facet normal.

Since the wave slopes $\partial\hat{\zeta}/\partial\underline{x}$ of the long waves are assumed to be small, we have to second order

$$\tilde{g}=g-\hat{\omega}^2\hat{\zeta}\,, \tag{B4}$$

where $\hat{\omega}$ is the radian frequency of the long waves.

The source function Q is the sum of 3 terms Q_i, Q_r, and Q_d. Q_i describes the energy input from the wind, Q_r the energy transfer within the wave field due to conservative resonant wave-wave interaction and Q_d the energy loss due to dissipative processes (Hasselmann, 1972; Hasselmann *et al.*, 1973).

Since the effect of the large-scale motion is small by assumption, we may apply a perturbation expansion of the Liouville operator

$$\mathscr{L} = \mathscr{L}_0 + \delta\mathscr{L} \tag{B5}$$

and the action spectrum

$$N = N_0 + \delta N . \tag{B6}$$

Furthermore, we assume that the source term Q vanishes to zeroth order and is given to first order by

$$Q' = \left.\frac{\delta Q}{\delta N}\right|_{N=N_0} [\delta N] , \tag{B7}$$

where $\delta Q/\delta N$ denotes the functional derivative of the source function Q.

In the absence of large waves the action spectrum satisfies the zeroth order radiation balance equation

$$\mathscr{L}_0[N_0] = 0 . \tag{B8}$$

To first order we have then

$$\mathscr{L}_0[\delta N] + \delta\mathscr{L}[N_0] = \left.\frac{\delta Q}{\delta N}\right|_{N=N_0} [\delta N] . \tag{B9}$$

We take the ratio of the group velocity of the short waves $\underline{v}_g$ to the group velocity of the long waves $\partial\hat{\omega}/\partial\underline{\hat{k}}$ as small and also $\hat{\zeta}|\underline{\hat{k}}| \ll 1$. The operator $\mathscr{L}$ can then be decomposed into

$$\mathscr{L}_0 = \frac{\partial}{\partial t} \tag{B10a}$$

and

$$\delta\mathscr{L} = \underline{v}_g \cdot \frac{\partial}{\partial \underline{x}} + \underline{\hat{U}} \cdot \frac{\partial}{\partial \underline{x}} - \frac{\partial}{\partial \underline{x}} (\underline{k} \cdot \underline{\hat{U}}) \frac{\partial}{\partial \underline{k}} . \tag{B10b}$$

The detailed structure of Q and therefore of $Q' = \delta Q/\delta N$ is not known. In particular, the wind dependence of Q' is not well established. This question can be resolved only by experiments and considerable effort is presently devoted to investigate this problem with modulation experiments in the ocean (e.g., JONSWAP 75, Plant *et al.*, 1977). We assume here for simplicity with Keller and Wright (1975) that Q' at the equilibrium state N_0 is a diagonal operator and make no assumption about its wind speed dependence:

$$\left.\frac{\delta Q}{\delta N}\right|_{N=N_0} [\delta N_0] = -\mu(\underline{k}, \underline{x}) \delta N . \tag{B11}$$

$\mu(\underline{k}, \underline{x})^{-1}$ is a characteristic relaxation time of the system which is determined by the input from the wind (Q_i), by damping processes (Q_d) and by conservative transfer processes within the wave field (Q_r).

The assumption that $\delta Q/\delta N$ is approximately a diagonal operator at $N = N_0$ means that near equilibrium the net effect of the perturbation forcing terms Q_i, Q_d and Q_r on the wave field can be described by a damping of each individual wave train, without a transfer of action from one wave train to another. Equation (B9) can be solved by using a Fourier representation of δN and $\hat{\underline{U}}$

$$\delta N = \frac{1}{\omega_0}\delta E = N_0 \int (R_{\hat{k}}^{\text{hydr.}} z_{\hat{k}} \, \mathrm{e}^{i(\hat{k}x - \hat{\omega}t)} + \text{c.c.})\,\mathrm{d}\hat{\underline{k}} \tag{B12}$$

$$\hat{\underline{U}} = \int \left(\frac{\hat{\underline{k}}}{|\hat{\underline{k}}|}\hat{\omega} z_{\hat{k}} \, \mathrm{e}^{i(\hat{k}x - \hat{\omega}t)} + \text{c.c.}\right)\mathrm{d}\hat{\underline{k}}, \tag{B13}$$

where $z_{\hat{k}}$ represents the Fourier transform of the long-wave field amplitude (see Equation (6)) and

$$\omega_0 = \left(g|\underline{k}| + \frac{\tau}{\rho}|\underline{k}|^3\right)^{1/2}, \tag{B14}$$

which is approximately equal to ω' because

$$\hat{\omega}^2 \hat{\zeta} \ll g\,.$$

In (B12) it is implied that $R_{\hat{k}}^{\text{hydr.}}$ is independent of $z_{\hat{k}}$, such that $R_{\hat{k}}^{\text{hydr.}}$ is a linear transfer function between the relative change in spectral energy of the short waves and the amplitude of the long waves.

Assuming that the zeroth order energy spectrum is homogeneous in $\underline{x}$- space, we obtain from equation (B9)

$$\begin{aligned} R_{\hat{k}}^{\text{hydr.}} &= \frac{\hat{\omega} - i\mu}{\hat{\omega}^2 + \mu^2}\frac{\hat{\omega}}{|\hat{\underline{k}}|}(\underline{k}\cdot\hat{\underline{k}})\left(\hat{\underline{k}}\frac{\partial N_0}{\partial \underline{k}}\right) \\ &= \frac{\hat{\omega} - i\mu}{\hat{\omega}^2 + \mu^2}\frac{\hat{\omega}}{|\hat{\underline{k}}|}\left[(\underline{k}\cdot\hat{\underline{k}})\left(\frac{1}{E_0}\hat{\underline{k}}\cdot\frac{\partial E_0}{\partial \underline{k}}\right) - \gamma\frac{\hat{\underline{k}}\cdot\underline{k}}{|\underline{k}|^2}\right] \end{aligned} \tag{B15}$$

with

$$\gamma = \frac{1}{2}\frac{1 + 3\dfrac{\tau}{\rho}\dfrac{|\underline{k}|^2}{\tilde{g}}}{1 + \dfrac{\tau}{\rho}\dfrac{|\underline{k}|^2}{\tilde{g}}} \tag{B16}$$

nd $E_0 = \omega_0 N_0$.

Choosing for the short-wave spectrum E a Phillips spectrum $E_0 \sim |\underline{k}|^{-4}$ we obtain from (B15) and (B16)

$$\frac{1}{|\underline{k}|} R_{\underline{k}}^{\text{hydr.}} = (4+\gamma)\frac{\hat{\omega}}{\mu}\frac{(\hat{\omega}/\mu)-i}{1+(\hat{\omega}/\mu)^2}, \tag{B17}$$

where $\gamma = \frac{1}{2}$ for gravity waves.

We regard μ in (B17) as an empirical complex parameter whose value must be determined by experiment.

In Figure 2 the absolute value and phase φ of $(1/|\hat{\underline{k}}|)R^{\text{hydr.}}$ are plotted as function of $y = \hat{\omega}/\mu$ for $\gamma = \frac{1}{2}$.

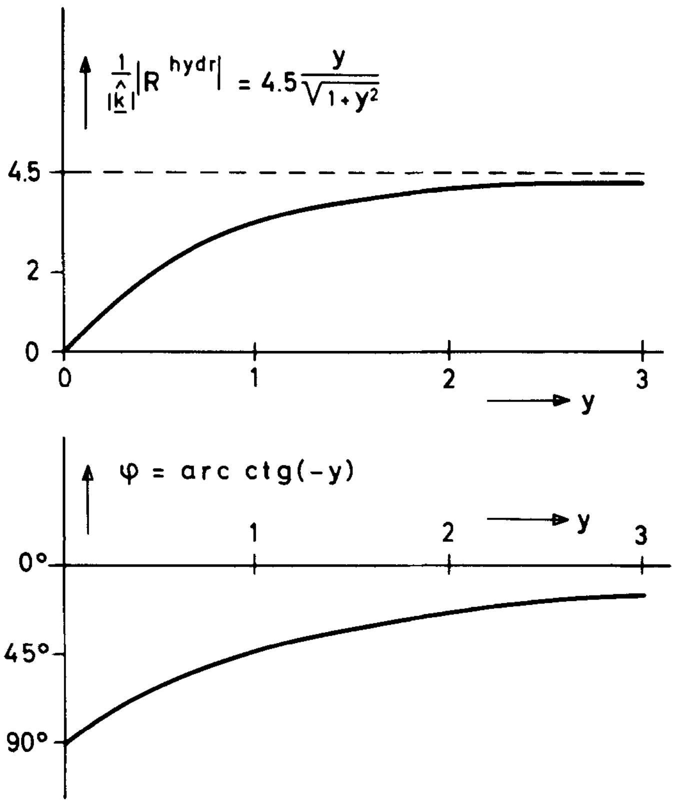

Fig. 2. Contribution to the modulation transfer function originating from the hydrodynamic interactions between short and long waves as a function of $y = 2\pi\hat{f}/\mu$. The first curve shows the modulus and the second, the phase (theoretical).

The hydrodynamic transfer function derived by Keller and Wright (1975) differs from our Equation (B15) because of the different expansions used. Their formula for $R_{\underline{k}}^{\text{hydr.}}$ generalized to a two-dimensional wave field may be obtained from our expression (B15) if the first factor

$$\frac{\hat{\omega} - i\mu}{\hat{\omega}^2 + \mu^2} \text{ is replaced by } \frac{(\hat{\omega} - \underline{v}_g \cdot \hat{\underline{k}}) - i\mu}{(\hat{\omega} - \underline{v}_g \cdot \hat{\underline{k}})^2 + \mu^2}. \tag{B18}^*$$

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* The parameter γ in their paper should be replaced by $\gamma + 1$.