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The effects of randomness on the stability of two-dimensional surface wavetrains

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A simplified nonlinear spectral transport equation, for narrowband Gaussian random surface wavetrains, slowly varying in space and time, is derived from the weakly nonlinear equations of Davey & Stewartson. The stability of an initially homogeneous wave spectrum, to small oblique wave perturbations is studied for a range of spectral bandwidths, resulting in an integral equation for the amplification rate of the disturbance. It is shown for random deep water waves that instability of the wavetrain can exist, as in the corresponding deterministic Benjamin–Feir (B–F) problem, provided that the normalized spectral bandwidth σ/k_0 is less than twice the root mean square wave slope, multiplied by a function of the perturbation wave angle $\phi = \arctan(m/l)$. A further condition for instability is that the angle ϕ be less than 35.26°. It is demonstrated that the amplification rate, associated with the B–F type instability, diminishes and then vanishes as the correlation length scale of the random wave field (*ca.* $1/\sigma$) is reduced to the order of the characteristic length scale for modulational instability of the wave system.

1. Introduction

Studies of the evolution of nonlinear surface water waves have tended to treat the problem either from the deterministic point of view, with emphasis on the properties and stability of nonlinear wavetrains (Benney & Newell 1967; Benjamin & Feir 1967; Hasimoto & Ono 1972; Davey & Stewartson 1974) or from the random point of view, with emphasis on wave–wave energy transfer within a broad spectrum due to weak nonlinear couplings in a nearly homogeneous random ocean (Phillips 1960; Hasselman 1962, 1963; Watson & West 1975; Willebrand 1975). It should be noted that Longuet-Higgins (1976) has made a notable start in joining these two wave viewpoints together, in his study of nonlinear wave–wave interactions near the peak of a gravity wave spectrum.

In this paper we seek to provide a further bridge between the deterministic and random schools, by examining the stability properties of a weakly nonlinear random wavetrain. For this study, we assume the degree of randomness, or spectral spread σ about the carrier wavenumber k_0, is small.

We start the analysis by assuming that the Davey & Stewartson (D–S) equations (the two-dimensional version of the nonlinear Schrödinger equation of Hasimoto

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& Ono) governing the development of the complex wave amplitude $A(\boldsymbol{X}, \tau)$, remain valid when $A(\boldsymbol{X}, \tau)$ is a random function of $\boldsymbol{X}$. It is required by the D–S multiple scale analysis that spatial variations of the envelope function occur over distances L, which are large compared with the carrier wavelength, i.e. where $L = O(2\pi/k_0 \epsilon)$. The small parameter ϵ is identified with the characteristic wave slope, which for random wave systems is given by the r.m.s. wave slope $\epsilon = (k_0^2 \overline{a_0^2})^{\frac{1}{2}}$. $\overline{a_0^2}$ is the mean square amplitude of some given unperturbed spatially homogeneous wave field.

A transport equation for the ensemble averaged two point correlation function $\rho(X_1, X_2, \tau) = \langle A(\boldsymbol{X}+\frac{1}{2}\boldsymbol{r}, \tau) A^*(\boldsymbol{X}-\frac{1}{2}\boldsymbol{r}, \tau)\rangle$ is derived from the nonlinear D–S equations, allowing for slow variations of ρ in both $\boldsymbol{X}$ and τ. By specifying that the random surface wavetrain is described by a Gaussian random process, the nonlinear fourth-order correlation terms appearing in the derived correlation equation, are readily related to calculable second-order correlations, thus providing an approximate closure of the governing correlation equations. Upon Fourier transforming the correlation equation (with respect to $\boldsymbol{r}$) one obtains an equation describing the evolution of the wave power spectral density $F(\boldsymbol{X}, \boldsymbol{P}, \tau)$; F being the Fourier transform of the two-point correlation function. A similar procedure was used by Wigner (1932) in deriving a Boltzmann-like equation for the quantum-mechanical wave probability function using the Schrödinger equation as a starting point. Applications of this technique to the field of plasma physics and quantum mechanics have been made by Leaf (1968), Tappert (1971), and Hasegawa (1975).

The two-dimensional Schrödinger-like equations for water waves, developed by Davey & Stewartson, were employed by Longuet-Higgins (1976) to calculate the resonant transfer of energy between four waves in a narrow gravity-wave spectrum. The resonant interaction analysis provides a collision-like integral term to the spectral transport equation, which is third order in the spectral function. As in the studies of Watson & West (1975) and Willebrand (1975), we study in this paper the effects of the nonlinear terms which are second order in $F(\boldsymbol{X}, \boldsymbol{P}, \tau)$. These second-order terms only vanish when the spectrum is homogeneous in $\boldsymbol{X}$, or when the wave system is stable to small spatial perturbations.

The present derived spectral equation (further specialized for the case of deep water) is studied for its stability to small-amplitude long-wavelength perturbations, in the form of oblique spatial waves. An integral equation for the amplification rate Ω of the disturbances is obtained from a linearized version of the derived spectral equation, and is seen to depend upon the specific form of the unperturbed spectrum, $F_0(\boldsymbol{P})$.

In the present investigation, we examine the stability of a spectrum which is a very simple normal form, $F_0(\boldsymbol{P}) = (1/2\pi\sigma_\xi \sigma_\eta) \exp[-(P_\xi^2/2\sigma_\xi^2 + P_\eta^2/2\sigma_\eta^2)]$. A closed-form expression can then be found for Ω as a function of the wavenumbers l and m of the oblique-wave perturbations.

It is found that the wave spectrum is unstable (Ω has a positive imaginary part) to such long-wavelength disturbances, provided that the spectral bandwidth, σ_ξ/k_0, is less than twice the r.m.s. wave slope, $(k_0^2 \overline{a_0^2})^{\frac{1}{2}}$, multiplied by a function of the perturbation wave angle $\phi = \arctan(m/l)$. When $\sigma_\xi/k_0 \to 0$, the present results reduce to the form of the deterministic stability criterion of Davey–Stewartson–Hayes (or Benjamin & Feir for one-dimensional wavetrains). For oblique waves in deep water, the result is recovered that a random wavetrain is unstable provided that the perturbation wave is at an oblique angle less than 35.26° with respect to the direction of the carrier group velocity.

The present analysis shows the key result that the instability diminishes, from the deterministic-like limit at $\sigma_\xi = 0$, as the spectral bandwidth increases. The Benjamin–Feir type instability in fact totally vanishes, when the bandwidth of the spectrum satisfies the condition that

$$\frac{\sigma_\xi}{k_0} \geqslant 2\,(k_0^2\overline{a_0^2})^{\frac{1}{2}}\left[\frac{1-2(m/l)^2}{1+(2m/l)^2}\right] \qquad (\Omega_1 = 0).$$

2. The Davey–Stewartson equations

Davey & Stewartson (1974) used the method of multiple scales (as did Benney & Newell 1967) to develop equations governing the evolution of two-dimensional wavetrains (i.e. waves propagating in the two surface directions x and y). In this derivation the wave height $\zeta(x, y, t)$ of a weakly nonlinear progressive wave, above its undisturbed position, can be written in the form†

$$2\zeta(x,y,t) = \epsilon A(X,Y,\tau)\exp[\mathrm{i}(k_0 x-\omega_0 t)] + \text{c.c.} + O(\epsilon^2), \tag{2.1}$$

where
$$X = \epsilon(x-c_{g_0}t), \quad Y = \epsilon y, \quad \tau = \epsilon^2 t. \tag{2.2}$$

The complex amplitude $A(X, Y, \tau)$ is taken to be a slowly varying function of space and time relative to the carrier wavelength $2\pi/k_0$ and carrier period $2\pi/\omega_0$. X is the direction of the carrier group velocity $\boldsymbol{c}_{g_0}$, t is time and the carrier dispersion relation (for waves of depth h) is given by the linear relation

$$\omega_0^2 = gk_0\sigma, \quad \sigma = \tanh k_0 h. \tag{2.3}$$

The group velocity of the carrier wave is given by

$$c_{g_0} = \omega'(k_0) - (g/2\omega_0)\{\sigma + k_0 h(1-\sigma^2)\}. \tag{2.4}$$

When expansions for ζ and the fluid velocity potential are carried out to order ϵ^3 and are substituted in Laplace's equation, with appropriate surface wave and bottom boundary conditions, one obtains, after suppressing secular behaviour, the following pair of evolutionary equations for the complex amplitude $A(\boldsymbol{X},\tau)$,

$$\mathrm{i}\frac{\partial A}{\partial \tau} + \lambda\frac{\partial^2 A}{\partial X^2} + \mu\frac{\partial^2 A}{\partial Y^2} = \nu|A|^2 A + \nu_1 AQ, \tag{2.5}$$

$$\lambda_1\frac{\partial^2 Q}{\partial X^2} + \mu_1\frac{\partial^2 Q}{\partial Y^2} = \kappa_1\frac{\partial^2|A|^2}{\partial Y^2}. \tag{2.6}$$

† We have normalized the wave height differently than Davey & Stewartson so that the maximum wave height $\zeta_{\max} = |A| = a$.

The coefficients in (2.5) and (2.6) are

$$\left.\begin{aligned}
\lambda &= \tfrac{1}{2}\omega''(k_0) = -\frac{g}{4k_0\,\sigma\omega_0}[\{\sigma - k_0 h(1-\sigma^2)\} + 4k_0^2 h^2\sigma^2(1-\sigma^2)] \leqslant 0,\\
\nu &= \frac{\omega_0 k_0^2}{16\sigma^4}\left[9 - 10\sigma^2 + 9\sigma^4 - \frac{2\sigma^2}{(gh - c_{g_0}^2)}\{4c_p^2 + 4c_p c_{g_0}(1-\sigma^2) + gh(1-\sigma^2)^2\}\right],\\
\mu &= \omega'(k_0)/2k_0 = c_{g_0}/2k_0 > 0, \quad c_p = \omega_0/k_0,\\
\nu_1 &= k_0^4\{2c_p + c_{g_0}(1-\sigma^2)\}/c_{g_0}, \quad \lambda_1 = gh - c_{g_0}^2 \geqslant 0, \quad \mu_1 = gh,\\
\kappa_1 &= \frac{g^2 h c_{g_0}}{4k_0\,\sigma}\left\{\frac{2c_p + c_{g_0}(1-\sigma^2)}{gh - c_{g_0}^2}\right\}.
\end{aligned}\right\} \tag{2.7}$$

For the deterministic problem, the basic solution of (2.5) and (2.6), which represents a uniform wavetrain of fixed amplitude $|A| = a_0$ is the modified Stokes solution

$$A = a_0 \exp[-\mathrm{i}(\nu a_0^2 + \nu_1 Q_0)\tau], \quad Q = Q_0 = \text{constant}. \tag{2.8}$$

If one assumes an oblique-wave spatial perturbation (or modulation) of the Stokes solution given by

$$\left.\begin{aligned}
A &= a_0\{1 + \epsilon a(\boldsymbol{X},\tau)\}\exp[(-\mathrm{i}\nu a_0^2 + \nu_1 Q_0)\tau],\\
Q &= Q_0\{1 + \epsilon q(\boldsymbol{X},\tau)\},
\end{aligned}\right\} \tag{2.9}$$

where

$$\left.\begin{aligned}
a &= a_+ E + a_- E^{-1},\\
q &= q_+ E + q_- E^{-1},\\
E &\equiv \exp\{\mathrm{i}(lX + mY - \Omega\tau)\},
\end{aligned}\right\} \tag{2.10}$$

and then linearizes equations (2.5) and (2.6), the resulting dispersion relation for Ω, as a function of the wavenumbers l and m is given by

$$\left.\begin{aligned}
\Omega^2 &= (\lambda l^2 + \mu m^2)[2\tilde{\nu} a_0^2 + (\lambda l^2 + \mu m^2)],\\
\text{where}\quad \tilde{\nu} &= \nu + \nu_1\kappa_1 m^2/(\lambda_1 l^2 + \mu_1 m^2).
\end{aligned}\right\} \tag{2.11}$$

The above result was derived by Davey & Stewartson (1974) and by Hayes (1973) (using the Whitham analysis). Equation (2.11) shows that the wavetrain is *unstable* if the following criterion is satisfied:

$$(\lambda l^2 + \mu m^2)\tilde{\nu} < 0. \tag{2.12}$$

Note that λ is always negative, μ positive and ν changes from negative to positive as $k_0 h$ increases beyond $k_0 h = 1.363$ (as pointed out by Hasimoto & Ono 1972). Hayes has also shown that it is always possible to choose l and m so that the instability criterion (2.12) is satisfied. However, Hayes noted that the predicted instability is practically non-existent for shallow-water waves in the range $0 \leqslant k_0 h < 0.5$.

For deep-water waves ($k_0 h \to \infty$) the coefficients simplify,

$$\left.\begin{aligned} \lambda &= -\tfrac{1}{8}\sqrt{(g/k_0^3)}, \\ \mu &= \tfrac{1}{4}\sqrt{(g/k_0^3)}, \\ \tilde{\nu} \to \nu &= \tfrac{1}{2}\sqrt{(gk_0^5)}, \end{aligned}\right\} \quad k_0 h \to \infty, \tag{2.13}$$

and the wavetrain is unstable if $m/l < 1/\sqrt{2}$; i.e. if the angle ϕ of the oblique disturbance is less than 35.26° with respect to the direction of the wavetrain group velocity. For such shallow-angle oblique perturbations, the rate of amplification $\Omega = \Omega_i$ of the instability is given by

$$\left(\frac{\Omega_i}{\omega_0}\right)^2 = \tfrac{1}{8}k_0^2 a_0^2 \left[\left(\frac{l}{k_0}\right)^2 - 2\left(\frac{m}{k_0}\right)^2\right] - \frac{1}{64}\left[\left(\frac{l}{k_0}\right)^2 - 2\left(\frac{m}{k_0}\right)^2\right]^2. \tag{2.14}$$

The maximum rate of amplification is achieved when the wavenumbers of the disturbance satisfy the relation

$$\left(\frac{l}{k_0}\right)^2 - 2\left(\frac{m}{k_0}\right)^2 = 4k_0^2 a_0^2. \tag{2.15}$$

By defining the resultant wavenumber amplitude $\kappa = (l^2 + m^2)^{\frac{1}{2}}$ and the modulation wave direction $\phi = \arctan(m/l)$, the above expression can also be rewritten as

$$\frac{\kappa}{k_0}\left[\frac{1 - 2\tan^2\phi}{1 + \tan^2\phi}\right]^{\frac{1}{2}} = 2k_0 a_0 \quad \text{at} \quad \Omega_i = \Omega_{i,\max}. \tag{2.16}$$

For disturbances parallel to the group velocity, the corresponding one-dimensional result of Benjamin & Feir is given by setting $m = \tan\phi = 0$.

We now examine the corresponding random stability problem where the unperturbed wavetrain is a spatially homogeneous random function of $\boldsymbol{X}$ and we subject the wavetrain to long-wavelength oblique-wave perturbations.

3. The transport equation for the correlation and spectral functions

We assume that the D–S equations (2.5, 2.6) for the complex amplitude $A(\boldsymbol{X}, \tau)$ describe the evolution of the wavetrain when A is a random function of X and Y. For waves undergoing weak nonlinear interactions we seek an equation for the slow variation of the two-point space correlation function (the second-order statistical moment),

$$\rho(\boldsymbol{X}_1, \boldsymbol{X}_2, \tau) = \langle A(\boldsymbol{X}_1, \tau)\, A^*(\boldsymbol{X}_2, \tau)\rangle, \tag{3.1}$$

where superscript * denotes the complex conjugate.

In equation (3.1) the angle brackets denote an ensemble average. Thus we require, as in the derivation of 2.5 and 2.6, that variations in ρ (or its Fourier transform) occur over length scales of order $\epsilon^{-1}\lambda_0$ and time scales of order ϵ^{-2}/ω_0. One can show that the one-dimensional spectral width σ must be of order ϵk_0. Hence we are studying narrow-band processes, with carrier wavenumber k_0.

3.1. *Correlation equation*

To obtain the equation for the two-point correlation ρ from the envelope amplitude equation (2.5) we adopt the following procedure (Wigner 1932). We write equation (2.5) at the point $\boldsymbol{X}_1$ $(=X_1, Y_1)$, multiply it by $A^*(\boldsymbol{X}_2)$, add it to the equation for $A^*(\boldsymbol{X}_2)$, multiplied by $A(\boldsymbol{X}_1)$, and take ensemble averages. The resulting equation is

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial \tau}\langle A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_2)\rangle + \lambda\left[\frac{\partial^2}{\partial X_1^2}-\frac{\partial^2}{\partial X_2^2}\right]\langle A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_2)\rangle \\ +\mu\left[\frac{\partial^2}{\partial Y_1^2}-\frac{\partial^2}{\partial Y_2^2}\right]\langle A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_2)\rangle - \nu_1\langle A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_2)\rangle[Q(\boldsymbol{X}_1)-Q(\boldsymbol{X}_2)] \\ -\nu[\langle A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_1)A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_2)\rangle - \langle A(\boldsymbol{X}_2)A^*(\boldsymbol{X}_2)A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_2)\rangle] = 0. \quad (3.2) \end{aligned}$$

The derivatives with respect to X_1, Y_1, X_2 and Y_2 can be replaced by derivatives with respect to the average coordinates

$$X = \tfrac{1}{2}(X_1+X_2), \quad Y = \tfrac{1}{2}(Y_1+Y_2), \quad (3.3)$$

and with respect to the spatial separation coordinates

$$r_x = (X_1-X_2), \quad r_y = (Y_1-Y_2). \quad (3.4)$$

Thus from (3.2) we obtain

$$\begin{aligned} \mathrm{i}\frac{\partial\rho}{\partial\tau} + 2\lambda\frac{\partial^2\rho}{\partial X\,\partial r_x} + 2\mu\frac{\partial^2\rho}{\partial Y\,\partial r_y} - \nu_1\rho[Q(\boldsymbol{X}+\tfrac{1}{2}\boldsymbol{r}) - Q(\boldsymbol{X}-\tfrac{1}{2}\boldsymbol{r})] \\ -\nu[\langle A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_1)A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_2)\rangle - \langle A(\boldsymbol{X}_2)A^*(\boldsymbol{X}_2)A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_2)\rangle] = 0. \quad (3.5) \end{aligned}$$

As seen in (3.5) the evolutionary equation for the second-order correlation involves fourth-order correlation terms. To evaluate these terms we assume that $A(\boldsymbol{X},\tau)$ corresponds initially to a Gaussian random process and we further assume that the evolving random statistical amplitude field retains the same Gaussian statistical properties, as is made plausible by the results of Benney & Saffman (1966).

For Gaussian statistics, the fourth-order cumulant vanishes, allowing us to write the fourth-order correlation in terms of the products of pairs of second-order correlations, i.e.

$$\begin{aligned} \langle A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_1)A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_2)\rangle &= 2\langle A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_2)\rangle\langle A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_1)\rangle \\ &= 2\rho\overline{a^2}(\boldsymbol{X}_1), \end{aligned} \quad (3.6)$$

where $\overline{a^2}$ is the ensemble averaged mean square amplitude equivalent to

$$\langle A(\boldsymbol{X}_1)A^*(\boldsymbol{X}_1)\rangle = 2\overline{\zeta^2}.$$

A similar expression can be written for the final fourth-order correlation in (3.5). In the cumulant expansion, terms involving ensemble averages of the form $\langle A(\boldsymbol{X}_1)A(\boldsymbol{X}_1)\rangle$ vanish because of the required invariance of such a correlation to the addition of a random constant to the phases.

Under the Gaussian closure approximation then, equation (3.5) can be written

$$\mathrm{i}\frac{\partial\rho}{\partial\tau}+2\lambda\frac{\partial^2\rho}{\partial X\,\partial r_x}+2\mu\frac{\partial^2\rho}{\partial Y\,\partial r_y}-\nu_1\rho[Q(\boldsymbol{X}+\tfrac{1}{2}\boldsymbol{r})-Q(\boldsymbol{X}-\tfrac{1}{2}\boldsymbol{r})]$$
$$-2\nu\rho[\overline{a^2}(\boldsymbol{X}+\tfrac{1}{2}\boldsymbol{r})-\overline{a^2}(\boldsymbol{X}-\tfrac{1}{2}\boldsymbol{r})]=0, \qquad (3.7)$$

where $\overline{a^2}(\boldsymbol{X})=\rho(\boldsymbol{X}+\tfrac{1}{2}\boldsymbol{r},\boldsymbol{X}-\tfrac{1}{2}\boldsymbol{r},\tau)|_{r=0}$.

The above differential/difference equation can be converted into a differential equation with an infinite number of terms, by expanding the bracketed differences above in a double Taylor series about $r=0$,

$$[\overline{a^2}(X+\tfrac{1}{2}r_x,Y+\tfrac{1}{2}r_y)-\overline{a^2}(X-\tfrac{1}{2}r_x,Y-\tfrac{1}{2}r_y)]$$
$$=2\left[\left(\tfrac{1}{2}r_x\frac{\partial}{\partial X}+\tfrac{1}{2}r_y\frac{\partial}{\partial Y}\right)+\frac{1}{3!}\left(\tfrac{1}{2}r_x\frac{\partial}{\partial X}+\tfrac{1}{2}r_y\frac{\partial}{\partial Y}\right)^3+\ldots\right]\overline{a^2}(X,Y). \qquad (3.8)$$

In the following analysis, we retain all the terms in the above Taylor series expansion.

3.2. *Spectral transport equation*

The wave-envelope power spectral density $F(\boldsymbol{P},\boldsymbol{X},\tau)$ is defined as the Fourier transform of the two-point correlation function

$$F(\boldsymbol{P},\boldsymbol{X},\tau)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{r}}\rho(\boldsymbol{X}+\tfrac{1}{2}\boldsymbol{r},\boldsymbol{X}-\tfrac{1}{2}\boldsymbol{r},\tau)\,\mathrm{d}\boldsymbol{r}. \qquad (3.9)$$

The Fourier wavenumber $\boldsymbol{P}$, conjugate to the spatial separation $\boldsymbol{r}$, is taken to have cartesian coordinates P_x and P_y in the r_x, r_y directions.

The converse relation for the correlation coefficient is

$$\rho(\boldsymbol{X}+\tfrac{1}{2}\boldsymbol{r},X-\tfrac{1}{2}\boldsymbol{r},\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{r}}F(\boldsymbol{P},\boldsymbol{X},\tau)\,\mathrm{d}\boldsymbol{P}. \qquad (3.10)$$

When the spatial separation $\boldsymbol{r}=0$ we obtain the mean square wave amplitude (or height) as

$$\rho(\boldsymbol{X},\boldsymbol{X},\tau)=\overline{a^2}(\boldsymbol{X},\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\boldsymbol{P},\boldsymbol{X},\tau)\,\mathrm{d}\boldsymbol{P}. \qquad (3.11)$$

Note that for a narrow-band random process, the power spectrum $F(\boldsymbol{P})$ for the wave envelope can be simply related to the power spectrum $G(\boldsymbol{P})$ for the wave height ζ of equation (2.1). The spectral relation is given by the following expression

$$G(\boldsymbol{P})=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{r}}\langle\zeta(\boldsymbol{X}+\tfrac{1}{2}\boldsymbol{r})\,\zeta^*(\boldsymbol{X}-\tfrac{1}{2}\boldsymbol{r})\rangle\,\mathrm{d}\boldsymbol{r},$$
$$=\tfrac{1}{4}F(\boldsymbol{P}-k_0\boldsymbol{i}_1)+\tfrac{1}{4}F(\boldsymbol{P}+k_0\boldsymbol{i}_1), \qquad (3.12)$$

where $\boldsymbol{i}_1$ is the unit vector in the $\boldsymbol{X}$ direction.

For one-dimensional wavetrains, if we define the positive portion of the wave-height spectrum $\hat{G}(P)\equiv 2G(P)$ then one can readily show that

$$F(P)=2\hat{G}(P+k_0), \qquad (3.13)$$

that is, the one-dimensional wave-envelope spectrum is obtained directly from the wave-height spectrum $\hat{G}(P)$ by simply shifting the spectrum back towards the origin through a wavenumber translation k_0. Thus a wave-height spectrum $\hat{G}(P)$ which is symmetric about $k = k_0$ will yield an envelope spectrum $F(P)$ which is symmetric about the wavenumber origin, $P = 0$.

With the above spectral relation so defined, we proceed to derive the spectral transport equation for $F(\boldsymbol{P}, \boldsymbol{X}, \tau)$ by Fourier transforming the correlation transport equation (equation (3.7)), with the Taylor series expansion for the difference terms. The resulting equation for $F(\boldsymbol{P}, \boldsymbol{X}, \tau)$ is

$$\frac{\partial F}{\partial \tau} + 2\lambda P_x \frac{\partial F}{\partial X} + 2\mu P_y \frac{\partial F}{\partial Y} - \sin\left(\frac{1}{2}\frac{\partial}{\partial X}\frac{\partial}{\partial P_x} + \frac{1}{2}\frac{\partial}{\partial Y}\frac{\partial}{\partial P_y}\right)[4\nu\overline{a^2} + 2\nu_1 Q]F = 0, \quad (3.14)$$

where the sine operator is defined such that the spatial derivatives operate on either $\overline{a^2}(\boldsymbol{X}, \tau)$ or $Q(\boldsymbol{X}, \tau)$ and the wavenumber derivatives operate on $F(\boldsymbol{P}, \boldsymbol{X}, \tau)$, i.e.

$$\begin{aligned} &\sin\left(\frac{1}{2}\frac{\partial}{\partial X}\frac{\partial}{\partial P_x} + \frac{1}{2}\frac{\partial}{\partial Y}\frac{\partial}{\partial P_y}\right)\overline{a^2}F \\ &\quad = \left\{\frac{1}{2}\frac{\partial \overline{a^2}}{\partial X}\frac{\partial F}{\partial P_x} + \frac{1}{2}\frac{\partial \overline{a^2}}{\partial Y}\frac{\partial F}{\partial P_y}\right\} - \frac{1}{3!}\left(\frac{1}{2}\right)^3\left\{\frac{\partial^3 \overline{a^2}}{\partial X^3}\frac{\partial^3 F}{\partial P_x^3} + \frac{\partial^3 \overline{a^2}}{\partial Y^3}\frac{\partial^3 F}{\partial P_y^3}\right\} \\ &\quad\quad - \frac{1}{3!}\left(\frac{1}{2}\right)^3\left\{3\frac{\partial^3 \overline{a^2}}{\partial X^2 \partial Y}\frac{\partial^3 F}{\partial P_x^2 \partial P_y} + 3\frac{\partial^3 \overline{a^2}}{\partial X \partial Y^2}\frac{\partial^3 F}{\partial P_x \partial P_y^2}\right\} + \ldots. \end{aligned} \quad (3.15)$$

The companion equation for $Q(\boldsymbol{X}, \tau)$ is found by ensemble averaging equation (2.6), yielding

$$\lambda_1 \frac{\partial^2 Q}{\partial X^2} + \mu_1 \frac{\partial^2 Q}{\partial Y^2} = \kappa_1 \frac{\partial^2 \overline{a^2}}{\partial Y^2}, \quad (3.16)$$

where

$$\overline{a^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\boldsymbol{P}, \boldsymbol{X}, \tau)\,\mathrm{d}\boldsymbol{P}.$$

When the spatial length-scale over which the spectrum varies (*ca.* $1/\kappa$) is very large compared with the correlation scale (of the order of $1/\sigma$ where σ is the 1D spectral bandwidth), i.e. when κ/σ is small, then the third- and higher-order derivatives in equation (3.15) can be neglected compared to the first-derivative terms. In this long-wavelength limit ($\kappa/\sigma \ll 1$) equation (3.14) becomes

$$\frac{\partial F}{\partial \tau} + 2\lambda P_x \frac{\partial F}{\partial X} + 2\mu P_y \frac{\partial F}{\partial Y} - \left[2\nu\frac{\partial \overline{a^2}}{\partial X} + \nu_1 \frac{\partial Q}{\partial X}\right]\frac{\partial F}{\partial P_x} - \left[2\nu\frac{\partial \overline{a^2}}{\partial Y} + \nu_1 \frac{\partial Q}{\partial Y}\right]\frac{\partial F}{\partial P_y} = 0. \quad (3.17)$$

In this form, the spectral transport equation is similar in structure to the wave-propagation equations of geometric optics, with an important nonlinear refraction term typically given by

$$\frac{\partial \overline{a^2}}{\partial X}\frac{\partial F}{\partial P_x} = \frac{\partial}{\partial X}\left[\iint_{-\infty}^{\infty} F\,\mathrm{d}\boldsymbol{P}\right]\frac{\partial F}{\partial P_x}. \quad (3.18)$$

Watson & West (1975) and Willebrand (1975) have obtained similar nonlinear refraction terms in their spectral equations for the evolution of broad-band wave spectra. It is not clear that their more complex wavenumber coefficients are easily reducible to the present simpler form when the spectrum is narrow banded.

As noted above, the first-derivative nonlinear term (equation (3.18)) is only the first term in a formal infinite series expansion in powers of κ/σ. For a narrow spectrum, centred on a carrier wavenumber k_0, $\sigma/k_0 = O(\epsilon)$, it will be shown that all the terms in the series expansion are required if one is to recover the Benjamin–Feir type stability properties of the corresponding deterministic system, i.e. a wave system where the bandwidth approaches zero.

A spectral equation of the form given by equation (3.17) has also been studied by Hasegawa (1975) for problems related to the dynamics of random optical waves in strongly dispersive dielectric media. Tappert (1971) derived a 'collisionless'-wave kinetic equation for non-random dispersive-wave systems which included a more general sine operator series, similar to that given in equation (3.14).

Almost all of the applications of the Boltzmann or spectral-type transport equations have utilized the long-wavelength approximations $\kappa/\sigma \ll 1$ (except for some applications in quantum mechanics; Wigner 1932; Snider 1960). It will be shown, however, in the following section, when we consider the stability of initially homogeneous narrow-band random wavetrains, that it is necessary to retain all the nonlinear terms in the sine-operator expansion (or retain the basic nonlinear difference term in the correlation equation (3.7)) in order to properly recover the complete Benjamin–Feir type instability solution.

4. Stability analysis

The nonlinear spectral transport equation (3.14)–(3.16) has as one basic solution,

$$F = F_0(\boldsymbol{P}), \quad Q = Q_0 = \text{constant}. \tag{4.1}$$

This spectral solution, independent of $\boldsymbol{X}$ and τ, is the random counterpart of the uniform amplitude Stokes wavetrain of deterministic theory. $F_0(\boldsymbol{P})$ represents a homogeneous background 'ocean', with Gaussian random properties in $\boldsymbol{X}$, which is statistically uniform in space and time $(\overline{a^2} = \overline{a_0^2} = \text{constant})$. Q_0 is an arbitrary uniform surface current also independent of X and τ. We will examine the stability of this homogeneous solution, to small-amplitude oblique-wave spatial modulations.

4.1. *The linearized spectral equation*

We assume a perturbed solution of equation (3.14) of the form

$$F(\boldsymbol{P}, \boldsymbol{X}, \tau) = F_0(\boldsymbol{P}) + \epsilon F_1(\boldsymbol{P}, \boldsymbol{X}, \tau), \tag{4.2}$$

$$Q = Q_0 + \epsilon Q_1(\boldsymbol{X}, \tau), \tag{4.3}$$

where

$$F_1(\boldsymbol{P}, \boldsymbol{X}, \tau) = f_1(\boldsymbol{P}) \exp\{\mathrm{i}(lX + mY - \Omega\tau)\}, \tag{4.2a}$$

$$Q_1(\boldsymbol{X}, \tau) = q_1 \exp\{\mathrm{i}(lX + mY - \Omega\tau)\}, \tag{4.3a}$$

and seek to determine under what conditions, if any, does Ω have a positive imaginary part (Ω_1) corresponding to unstable growth. Note that the mean square amplitude $\overline{a^2}$, related to the spectrum by equation (3.11), is given by

$$\overline{a^2}(\boldsymbol{X},\tau) = \overline{a_0^2} + \epsilon \overline{a_1^2}(\boldsymbol{X},\tau), \tag{4.4}$$

where

$$\overline{a_0^2} = \iint F_0(\boldsymbol{P})\,\mathrm{d}\boldsymbol{P}, \tag{4.4a}$$

$$\overline{a_1^2} = \iint F_1\,\mathrm{d}\boldsymbol{P} = \left[\iint f_1\,\mathrm{d}\boldsymbol{P}\right]\exp\{\mathrm{i}(lX+mY-\Omega\tau)\}. \tag{4.4b}$$

We substitute equations (4.2)–(4.4) into the spectral transport equations (3.14)–(3.16) and linearize the resulting expressions by dropping terms of order ϵ^2, thereby obtaining

$$\frac{\partial F_1}{\partial \tau} + 2\lambda P_x \frac{\partial F_1}{\partial X} + 2\mu P_y \frac{\partial F_1}{\partial Y} - \sin\left(\frac{1}{2}\frac{\partial}{\partial X}\frac{\partial}{\partial P_x} + \frac{1}{2}\frac{\partial}{\partial Y}\frac{\partial}{\partial P_y}\right)(4\nu\overline{a_1^2} + 2\nu_1 Q_1)F_0 = 0, \tag{4.5}$$

$$\lambda_1 \frac{\partial^2 Q_1}{\partial X^2} + \mu_1 \frac{\partial^2 Q_1}{\partial Y^2} = \kappa_1 \frac{\partial^2 \overline{a_1^2}}{\partial Y^2}. \tag{4.6}$$

4.2. *The stability eigenvalue relation*

Replacing F_1, $\overline{a_1^2}$ and Q_1 by their assumed oblique-wave representations, ((4.2*a*), (4.3*a*), (4.4*b*)), equations (4.5) and (4.6) become

$$f_1(\Omega - 2\lambda P_x l - 2\mu P_y m) + \left(4\nu\iint f_1\,\mathrm{d}\boldsymbol{P} + 2\nu_1 q_1\right)\sinh\left(\tfrac{1}{2}l\frac{\partial}{\partial P_x} + \tfrac{1}{2}m\frac{\partial}{\partial P_y}\right)F_0 = 0, \tag{4.7}$$

$$q_1 = \kappa_1 m^2 \iint f_1\,\mathrm{d}\boldsymbol{P} \Big/ (\lambda_1 l^2 + \mu_1 m^2). \tag{4.8}$$

We recognize the sinh operator expansion as an infinite Taylor series expansion for the following difference expression:

$$\sinh\left(\tfrac{1}{2}l\frac{\partial}{\partial P_x} + \tfrac{1}{2}m\frac{\partial}{\partial P_y}\right)F_0(\boldsymbol{P}) = \tfrac{1}{2}[F_0(\boldsymbol{P}+\tfrac{1}{2}\boldsymbol{\kappa}) - F_0(\boldsymbol{P}-\tfrac{1}{2}\boldsymbol{\kappa})]. \tag{4.9}$$

If we divide equation (4.7) by $(\Omega - 2\lambda P_x l - 2\mu P_y m)$, substitute equation (4.8), and integrate the resulting expression over all $\boldsymbol{P}$, we obtain the integral stability eigenvalue relation†

$$1 + 2\tilde{\tilde{\nu}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F_0(\boldsymbol{P}+\frac{1}{2}\boldsymbol{\kappa}) - F_0(\boldsymbol{P}-\frac{1}{2}\boldsymbol{\kappa})}{(\Omega - 2\lambda P_x l - 2\mu P_y m)}\,\mathrm{d}\boldsymbol{P} = 0, \tag{4.10}$$

where

$$\tilde{\tilde{\nu}} = \nu + \nu_1\kappa_1 m^2/2(\lambda_1 l^2 + \mu_1 m^2).$$

It is possible to reduce the above double integral over $\boldsymbol{P}$ to a single integral (over one component of $\boldsymbol{P}$) after making the following simple vector transformations.

† Equation (4.10) can also be obtained directly from the correlation equation (3.7) by first linearizing and then Fourier transforming the resulting expression.

Let us define a wave vector $\hat{\boldsymbol{\kappa}}$ by

$$\hat{\boldsymbol{\kappa}} = l\boldsymbol{i}_1 + (\mu/\lambda)\, m\boldsymbol{i}_2, \tag{4.11}$$

where $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$ are respectively unit vectors in the X and Y directions. The denominator of the double integral in equation (4.10) can then be written as

$$\Omega - 2\lambda \boldsymbol{P}\cdot\hat{\boldsymbol{\kappa}} = \Omega - 2\lambda P_\xi |\hat{\kappa}|, \tag{4.12}$$

where
$$|\hat{\kappa}| = [l^2 + (\mu/\lambda)^2 m^2]^{\frac{1}{2}}. \tag{4.12a}$$

Note that the only component of $\boldsymbol{P}$ appearing in the denominator is that in the direction of the unit vector $\boldsymbol{i}_\xi = \hat{\boldsymbol{\kappa}}/|\hat{\kappa}|$, i.e.

$$P_\xi = \boldsymbol{P}\cdot\boldsymbol{i}_\xi = [P_x l + (\mu/\lambda)\, P_y m]/|\hat{\kappa}|. \tag{4.13}$$

The component of $\boldsymbol{P}$ in the direction normal to $\hat{\boldsymbol{\kappa}}$ is given by

$$P_\eta = \boldsymbol{P}\ (\boldsymbol{i}_3 \times \hat{\boldsymbol{\kappa}})/|\hat{\kappa}| = [P_y l - (\mu/\lambda)\, P_x m]/|\hat{\kappa}|. \tag{4.14}$$

P_η appears solely in the numerator of the double integral in equation (4.10). Hence, integration of $F_0(\boldsymbol{P} \pm \frac{1}{2}\boldsymbol{\kappa})$ over P_η reduces the double integral to a single integral over P_ξ with the wavenumber component

$$\kappa_\xi = \boldsymbol{\kappa}\,.\,\boldsymbol{i}_\xi = [l^2 + (\mu/\lambda)\, m^2]/|\hat{\kappa}| \tag{4.15}$$

as a parameter. Note $\boldsymbol{\kappa} \equiv l\boldsymbol{i}_1 + m\boldsymbol{i}_2$.

We may thus rewrite (4.10) as the following single integral eigenvalue equation for Ω

$$1 + 2\tilde{\tilde{\nu}} \int_{-\infty}^{\infty} \frac{[\tilde{F}_0(P_\xi + \frac{1}{2}\kappa_\xi) - \tilde{F}_0(P_\xi - \frac{1}{2}\kappa_\xi)]}{(\Omega - 2\lambda P_\xi \hat{\kappa})}\, \mathrm{d}P_\xi = 0, \tag{4.16}$$

where
$$\tilde{F}_0(P_\xi) = \int_{-\infty}^{\infty} F_0(\boldsymbol{P})\, \mathrm{d}P_\eta. \tag{4.16a}$$

We note the similarity of equation (4.16) to the integral dispersion relation appearing in the eigensolution treatment of the linearized Vlasov equation for ionized plasmas (cf. Eckar 1972). As in the Vlasov problem the definition and evaluation of an integral of the type given by equation (4.16) presents special problems because of the pole at $P_\xi = \Omega/2\lambda\hat{\kappa}$. The general procedure adopted by Landau (1946) is to solve the time dependent equation for F_1 as an initial value problem, by using Laplace transforms with respect to τ. If we applied this technique to solve equations (4.5) and (4.6), we would find, applying the inverse Laplace transform definition, the following solution for $\overline{a^2}$,

$$\overline{a^2}(\boldsymbol{X}, \tau)\, \mathrm{e}^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} = \frac{1}{2\pi \mathrm{i}} \int_{\sigma^* - \mathrm{i}\infty}^{\sigma^* + \mathrm{i}\infty} \mathrm{d}S \mathrm{e}^{S\tau} \left\{ \frac{\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{i}\tilde{f}_1(P_\xi, 0)\, \mathrm{d}P_\xi}{(\mathrm{i}S - 2\lambda P_\xi)}}{D(\boldsymbol{\kappa}, S)} \right\}, \tag{4.17}$$

where
$$D(\boldsymbol{\kappa}, S) = 1 + 2\tilde{\tilde{\nu}} \int_{-\infty}^{\infty} \frac{[\tilde{F}_0(P_\xi + \frac{1}{2}\kappa_\xi) - \tilde{F}_0(P_\xi - \frac{1}{2}\kappa_\xi)]}{(\mathrm{i}S - 2\lambda P_\xi \hat{\kappa})}\, \mathrm{d}P_\xi,$$

and
$$\tilde{f}_1(P_\xi, 0) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{x}} \int_{-\infty}^{\infty} F_1(\boldsymbol{P}, 0)\, \mathrm{d}P_\eta. \tag{4.18}$$

Integration is along a line parallel to the imaginary S axis and to the right of all singularities of the integrand (assuming such a line exists). We may use the calculus of residues to evaluate (4.17). For $\tilde{F}_0$ and $\tilde{f}_1(\tau = 0)$ analytic functions of P, the only singularities in the S plane are where the function $D(\kappa, S)$ vanishes; that is, where our previous equation (4.16) is satisfied with Ω replaced by iS.

The integral in equation (4.16) is thus, on the basis of the existence of the Laplace transform, defined for $\mathrm{Re}(S) \equiv S_r > 0$ or $\mathrm{Im}(\Omega) \equiv \Omega_i > 0$, that is, for wave instability. The integration in the P_ξ plane is along a path above the pole† $P_\xi = -\Omega/2|\lambda|\hat{\kappa}$. For the case of wave stability, $[\Omega_i < 0, S_r < 0]$ the pole in the integral of (4.16) is above the real axis, and we must replace it by its analytical continuation into the negative half of the S plane. Hence the path of integration in the P_ξ plane must stay above the pole, which contributes an additional term to $D(\kappa, S)$ equal to 2πi times the residue at $P_\xi = -\Omega/2|\lambda|\hat{\kappa}$. Thus for stability we replace (4.16) by‡

$$1 + 2\tilde{\tilde{\nu}}\left\{\int_{-\infty}^{\infty} \frac{[\tilde{F}_0(P_\xi + \frac{1}{2}\kappa_\xi) - \tilde{F}_0(P_\xi - \frac{1}{2}\kappa_\xi)]}{(\Omega - 2\lambda P_\xi \hat{\kappa})} \mathrm{d}P_\xi + 2\pi\mathrm{i}\left[\tilde{F}_0\left(-\frac{\Omega}{2|\lambda|\hat{\kappa}} + \tfrac{1}{2}\kappa_\xi\right) - \tilde{F}_0\left(-\frac{\Omega}{2|\lambda|\hat{\kappa}} - \tfrac{1}{2}\kappa_\xi\right)\right]\right\} = 0. \tag{4.19}$$

Our primary interest in this paper is to examine wave instability, which we know from the deterministic solution of the Davey–Stewartson equations occurs for deep-water waves (in the Benjamin–Feir sense) when the oblique wave angle $\phi = \arctan(m/l)$ is less than 35.26°. Hence we can use the eigenvalue equation, 4.16, to determine Ω_i as a function of $\boldsymbol{\kappa}$ with the non-dimensional spectral bandwidth, σ_ξ/k_0 as a parameter.

4.3. *Deep-water stability eigenvalue relation*

For the deep-water limit $k_0 h \to \infty$, the coefficient λ and μ reduce to the form given by equation (2.13), the ratio of the two coefficients being $\lambda/\mu = -\frac{1}{2}$. In addition, the coefficient $\tilde{\tilde{\nu}} \to \nu = \frac{1}{2}\surd(gk_0^5) = \frac{1}{2}\omega_0 k_0^2$. Since 2λ is a negative quantity, we will for convenience define

$$\alpha \equiv -2\lambda = \tfrac{1}{4}\surd(g/k_0^3) = \tfrac{1}{4}\omega_0/k_0^2. \tag{4.20}$$

The stability eigenvalue equation (4.16) may then be rewritten in the form (for the deep-water limit)

$$1 + 2\nu \int_{-\infty}^{\infty} \frac{[\tilde{F}_0(P_\xi + \frac{1}{2}\kappa_\xi) - \tilde{F}_0(P_\xi - \frac{1}{2}\kappa_\xi)]}{(\Omega + \alpha\hat{\kappa} P_\xi)} \mathrm{d}P_\xi = 0. \tag{4.21}$$

To evaluate equation (4.21) we separate Ω into its real and imaginary parts $(\Omega = \Omega_r + \mathrm{i}\Omega_i)$. Dividing the integral above into its real and imaginary parts (with κ_ξ and $\hat{\kappa}$ real) we obtain the following two integral equations for Ω_r and Ω_i,

$$\Omega_i \int_{-\infty}^{\infty} \frac{[\tilde{F}_0(P_\xi + \frac{1}{2}\kappa_\xi) - \tilde{F}_0(P_\xi - \frac{1}{2}\kappa_\xi)]}{[(\Omega_r + \alpha\hat{\kappa} P_\xi)^2 + \Omega_i^2]} \mathrm{d}P_\xi = 0, \tag{4.22}$$

$$1 + 2\nu \int_{-\infty}^{\infty} \frac{[\Omega_r + \alpha\hat{\kappa} P_\xi][\tilde{F}_0(P_\xi + \frac{1}{2}\kappa_\xi) - \tilde{F}_0(P_\xi - \frac{1}{2}\kappa_\xi)]}{[(\Omega_r + \alpha\hat{\kappa} P_\xi)^2 + \Omega_i^2]} \mathrm{d}P_\xi = 0. \tag{4.23}$$

† Note that $\lambda < 0$ (cf. equation (2.7)).

‡ For the case $\Omega_i = 0$ only half the second term in (4.19) is taken, i.e. πi multiplied into the term in the square brackets. The Cauchy principal value is taken for the integral.

If $\tilde{F}_0$ is an even function of P_ξ (that is, the envelope spectrum is symmetric about $P_\xi = 0$) we see that equation (4.22) can be satisfied identically if $\Omega_r = 0$. This simplifies the evaluation of Ω_i to the solution of a single equation, either equation (4.23) with $\Omega_r = 0$, or equation (4.21) with $\Omega = i\Omega_i$.

The evaluation of equation (4.21) for a general non-symmetric initial spectrum $\tilde{F}_0(P_\xi)$ will require special numerical treatment and will be considered in a future addition to this present paper.

A particularly simple spectrum, which is an even function of $\boldsymbol{P}$ and which facilitates the solution of equation (4.21), is the two-dimensional normal spectrum

$$F_0(\boldsymbol{P}) = \frac{\overline{a_0^2}}{2\pi\sigma_\xi\sigma_\eta}\exp\left[-\left(\frac{P_\xi^2}{2\sigma_\xi^2}+\frac{P_\eta^2}{2\sigma_\eta^2}\right)\right]. \tag{4.24}$$

Note that σ_ξ, σ_η are the basic bandwidths of the undisturbed spectrum, respectively in the P_ξ and P_η coordinate directions. One should consider the normal spectrum, given by (4.24) as only a useful mathematical approximation to the spectrum of some physically realizable random process, but one useful for illustrating the basic nature of the stability solution.

From equation (4.16) we first carry out the integration over P_η to obtain $\tilde{F}_0(P_\xi)$, i.e.

$$\tilde{F}_0(P_\xi) = \int_{-\infty}^{\infty} F_0(\boldsymbol{P})\,\mathrm{d}P_\eta = \frac{\overline{a_0^2}}{\sqrt{(2\pi)}\,\sigma_\xi}\exp\left[-\frac{P_\xi^2}{2\sigma_\xi^2}\right]. \tag{4.25}$$

Substituting the reduced one-dimensional spectrum into equation (4.21) (with $\Omega = i\Omega_i$) we obtain, after the transformation of variables $(P_\xi \pm \frac{1}{2}\kappa_\xi)/\sqrt{2}\,\sigma_\xi = -t$, a specialized form of the stability eigenvalue equation,

$$1-\sqrt{\frac{2}{\pi}}\left(\frac{\nu\overline{a_0^2}}{\alpha\hat{\kappa}\sigma_\xi}\right)\left[\int_{-\infty}^{\infty}\frac{e^{-t^2}\,\mathrm{d}t}{z-t}-\int_{-\infty}^{\infty}\frac{e^{-t^2}\,\mathrm{d}t}{-z^*-t}\right] = 0, \tag{4.26}$$

where
$$z \equiv \tilde{\kappa}_\xi + iS \equiv \frac{\kappa_\xi}{2\sqrt{2}\,\sigma_\xi}+\frac{i(\Omega_i/\omega_0)\,2\sqrt{2}}{(\sigma_\xi/k_0)\,(\hat{\kappa}/k_0)}.$$

The first integral appearing in equation (4.26) is related to the complex integral function $w(z)$ defined by Abramowitz & Stegun (1964),

$$w(z) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}\,\mathrm{d}t}{z-t} = \frac{2iz}{\pi}\int_0^{\infty}\frac{e^{-t^2}\,\mathrm{d}t}{z^2-t^2}, \tag{4.27}$$

for $\mathrm{Im}\, z > 0$. Another useful integral form for $w(z)$ is given by

$$w(z) = e^{-z^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}\,\mathrm{d}t\right) = e^{-z^2}\,\mathrm{erfc}\,(-iz). \tag{4.27a}$$

Rewriting equation (4.26) in terms of the w function, we have

$$1+(i\sqrt{\pi}/2N\hat{\kappa})\,[w(z)-w(-z^*)] = 0 \tag{4.28}$$

where
$$\hat{\kappa} = \frac{\hat{\kappa}/k_0}{2\sqrt{2}\,(\sigma_\xi/k_0)},\quad N = \frac{(\sigma_\xi/k_0)^2}{2k_0^2\,\overline{a_0^2}}. \tag{4.28a}$$

The function $w(z)$ has the following symmetry property

$$w(-z^*) = [w(z)]^*,$$

so that

$$[w(z) - w(-z^*)] = 2\mathrm{i}\,\mathrm{Im}\,\{w(z)\}.$$

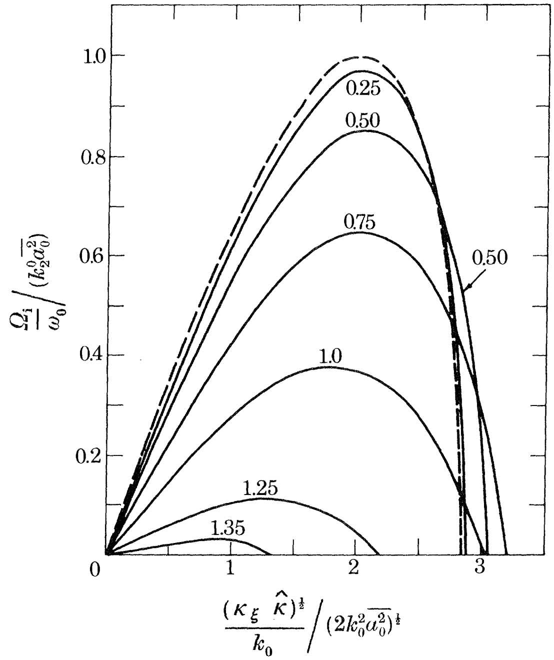

Figure 1. Amplification rate of the instability. Ω_i as a function of the effective modulation wavenumber, for various values of the effective bandwidth parameter.

$$---, \; \frac{\sigma_\xi}{k_0}\left(\frac{\kappa}{\kappa_\xi}\right)^{\frac{1}{2}} \Big/ (2k_0^2\overline{a_0^2})^{\frac{1}{2}} = 0.$$

Hence, the stability eigenvalue equation for deep water waves takes the final form

$$\boxed{(\sqrt{\pi}/\tilde{\tilde{\kappa}})\,\mathrm{Im}\,\{w(z)\} = N,} \tag{4.29}$$

where

$$z = \tilde{\kappa}_\xi + \mathrm{i}S, \quad \tilde{\kappa}_\xi = \frac{[(l/k_0)^2 - 2(m/k_0)^2]}{2\sqrt{2}\,[(l/k_0)^2 + (2m/k_0)^2]^{\frac{1}{2}}\,(\sigma_\xi/k_0)}, \tag{4.29a}$$

$$S = \frac{2\sqrt{2}\,(\Omega_i/\omega_0)}{[(l/k_0)^2 + (2m/k_0)^2]^{\frac{1}{2}}\,(\sigma_\xi/k_0)}, \quad \tilde{\tilde{\kappa}} = \frac{[(l/k_0)^2 + (2m/k_0)^2]^{\frac{1}{2}}}{2\sqrt{2}\,(\sigma_\xi/k_0)}. \tag{4.29b}$$

Abramowitz & Stegun (1964) tabulate the real and imaginary parts of $w(z)$ as a function of the real and imaginary part of z. Given a value of the spectral band- σ_ξ/k_0, the mean square wave slope $k_0^2\overline{a_0^2}$ and the two normalized modulation wavenumbers (l/k_0) and (m/k_0), one can interpolate in the $w(z)$ tables to determine S, and hence the amplification rate $\Omega_i = \Omega_i(\boldsymbol{\kappa}, \epsilon, \sigma_\xi)$. These calculations can then be used to construct a 'universal' plot (shown in figure 1) of the non-dimensional amplification rate $(\Omega_i/\omega_0)/2k_0^2\overline{a_0^2}$ as a function of the effective disturbance wavenumber $[(l/k_0)^2-2(m/k_0)^2]^{\frac{1}{2}}\,(2k_0^2\overline{a_0^2})^{\frac{1}{2}}$ for given values of the effective bandwidth

$$\frac{(\sigma_\xi/k_0)}{(2k_0^2\,\overline{a_0^2})^{\frac{1}{2}}}\left[\frac{1+(2m/l)^2}{1-2(m/l)^2}\right]^{\frac{1}{2}}.$$

4.4. *The limit of vanishing bandwidth*

When the bandwidth becomes vanishingly small, $\sigma_\xi/k_0 \to 0$, we obtain a special stability curve (in figure 1) similar in all features to that produced by Benjamin & Feir for deterministic wavetrains. The analytical form of the stability curve, for $\sigma_\xi \to 0$, is readily obtained from equation (4.29) by taking the asymptotic expansion for $w(z)$ as $z\to\infty$, i.e.

$$w(z) \simeq (\mathrm{i}/\sqrt{\pi})\,z^{-1}[1+\tfrac{1}{2}z^{-2}+\tfrac{3}{4}z^{-4}+\ldots]. \tag{4.30}$$

Hence, to order z^{-2},

$$\operatorname{Im}\{w(z)\} \simeq (1/\sqrt{\pi})\,\tilde{\kappa}_\xi/(\tilde{\kappa}_\xi^2+S^2). \tag{4.31}$$

Substituting (4.31) into the eigenvalue equation (4.29), yields the following expression for the amplification rate of a vanishingly small narrow-band process,

$$\left(\frac{\Omega_i}{\omega_0}\right)^2 = \tfrac{1}{8}(k_0^2\,2\overline{a_0^2})\left[\left(\frac{\kappa_\xi}{k_0}\right)\left(\frac{\hat{\kappa}}{k_0}\right)\right]-\frac{1}{64}\left(\frac{\kappa_\xi}{k_0}\right)^2\left(\frac{\hat{\kappa}}{k_0}\right)^2. \tag{4.32}$$

Note that from equation (4.15)

$$\left(\frac{\kappa_\xi}{k_0}\right)\left(\frac{\hat{\kappa}}{k_0}\right) = \left(\frac{l}{k_0}\right)^2-2\left(\frac{m}{k_0}\right)^2.$$

A comparison of equation (4.32) with the results of the stability analysis for deterministic deep-water waves, equation (2.14), shows that the amplification rates predicted for the Gaussian random process with vanishingly small bandwidth, are identical to those for the deterministic problem, if one makes the identification

$$\underset{\text{(Gaussian random process)}}{2\overline{a_0^2}} = \underset{\text{(deterministic wavetrain)}}{a_0^2}. \tag{4.33}$$

It should be kept in mind that $\overline{a_0^2}$ is the mean square wave amplitude for an approximately Rayleigh distributed random amplitude $a(X)$ (Cartwright & Longuet-Higgins 1956), and that a uniform wavetrain is not a special case of the Gaussian random process, except in the trivial limit $\overline{a_0^2} = a_0^2 = 0$.

With the above identification (equation (4.33)) in mind we see, as shown in § 2, that the maximum rate of amplification is

$$(\Omega_i/\omega_0)_{\max} = k_0^2\overline{a_0^2}$$

when the effective wavenumber is

$$[(l/k_0)^2-2(m/k_0)^2]^{\frac{1}{2}} = (8k_0^2\overline{a_0^2})^{\frac{1}{2}}.$$

The limiting solution given by equation (4.32) is plotted as the dashed curve in figure 1. Thus a wavetrain with Gaussian random amplitude and phase will be unstable, just like a Stokes deterministic wavetrain, as the bandwidth of the spectrum tends to zero. But what of the behaviour of the random wavetrain for small but finite σ_ξ?

4.5. *Solution for finite bandwidth*

The amplification rate of the instability for *finite* σ_ξ given by the eigenvalue relation of equation (4.29), is shown plotted in figure 1. For increasing bandwidth, we note a decrease of the initial slope of the amplification curve, $\Omega_i/(l^2-2m^2)^{\frac{1}{2}}$ and a decrease in the maximum amplification rate. For the most part, the region of instability gets smaller, finally vanishing at a particular value of the bandwidth parameter. To fix this value we look at the solution for Ω_i/κ in the limit of small but finite wavenumber κ.

Low wavenumber limit

An expression for the initial slope $\mathrm{d}\Omega_i/\mathrm{d}\kappa|_{\kappa\to 0}$, or the rate of change of Ω_i with the effective wavenumber of the disturbance $(l^2-2m^2)^{\frac{1}{2}} = \kappa(1-2\tan^2\phi)/(1+\tan^2\phi)$ for a fixed disturbance angle ϕ and fixed bandwidth, can be obtained from equation (4.29) by finding the solution $S = S(\tilde{\kappa}_\xi)$ for small wavenumbers $\kappa_\xi/k_0 \ll 1$. Expanding the w function for small $\tilde{\kappa}_\xi$ we obtain the following equation for $S = (2\surd 2\,\Omega_i/\omega_0)/[(\sigma_\xi/k_0)(\hat{\kappa}/k_\xi)]$ in terms of the complementary error function, in the low-wavenumber limit

$$2(1-\surd\pi\, S e^{S^2}\operatorname{erfc} S) = \frac{(\sigma_\xi/k_0)^2}{2k_0^2\overline{a_0^2}}\left[\frac{1+(2m/l)^2}{1-2(m/l)^2}\right]. \tag{4.34}$$

The above approximate solution can also be obtained from the long-wavelength limit approximation, equation (3.17), for the spectral transport equation. In this approximation, unlike the full solution, equation (4.29), Ω_i is unbounded for large κ. Thus we see the need to retain all terms in the sine operator expansion in equations (3.14)–(3.15).

Figure 2 shows a plot of the amplification rate slope

$$2\left(\frac{\Omega_i}{\omega_0}\right)\Big/(k_0^2\overline{a_0^2})^{\frac{1}{2}}\left[\frac{1-2(m/l^2)^2}{1+(m/l)^2}\right]^{\frac{1}{2}}\frac{\kappa}{k_0}$$

(in the low-wavenumber limit) as a function of the bandwidth parameter

$$\frac{(\sigma_\xi/k_0)}{(k_0^2\overline{a_0^2})^{\frac{1}{2}}}\left[\frac{1+(2m/l)^2}{1-2(m/l)^2}\right]^{\frac{1}{2}}.$$

When the bandwidth is very small, $\sigma_\xi\to 0$, we recover the low-wavenumber approximation to the analogue Benjamin–Feir instability–stability equation,

$$\left(\frac{\Omega_i}{\omega_0}\right)^2 = \frac{1}{8}\left(\frac{\kappa}{k_0}\right)^2\left[\frac{1-2(m/l)^2}{1+2(m/l)^2}\right](2k_0^2\overline{a_0^2}). \tag{4.35}$$

Note that equation (4.35) yields only the first term appearing on the right side of the eigenvalue equation (4.32), and corresponds to the small-wavenumber case where $\kappa/k_0 \ll 8(k_0^2\overline{a_0^2})^{\frac{1}{2}}$.

The low-wavenumber solution (equation (4.34)) is quite useful though in determining the effect of increasing spectral bandwidth on the stability boundaries of the present solution. As is evident from figures 1 and 2, the initial slope of the amplification curve diminishes with increasing bandwidth. By setting the amplification parameter $S = 0$ in equation (4.34) one sees that a condition for *stability* is that

$$\Omega_i/\kappa \overset{\text{(stable)}}{\leqslant} 0 \quad \text{for} \quad \frac{\sigma_\xi}{k_0} \geqslant 2(k_0^2\overline{a_0^2})^{\frac{1}{2}}\left[\frac{1-2(m/l)^2}{1+(2m/l)^2}\right]. \tag{4.36}$$

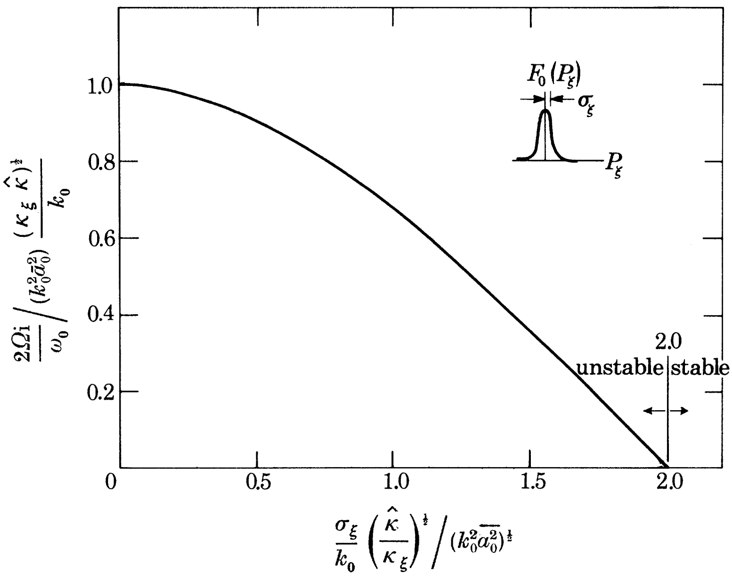

FIGURE 2. Effective amplification rate slope as a function of the spectral bandwidth. Low modulation wavenumber limit, $\kappa/k_0 \ll (k_0^2\overline{a_0^2})^{\frac{1}{2}}$.

Thus the amplification rate vanishes, regardless of the wavelength of the disturbance, if the bandwidth, σ_ξ/k_0, is large enough, of the order of twice the r.m.s. wave slope. For one-dimensional wavetrains the criterion becomes

$$\sigma_\xi/k_0 \geqslant 2(k_0^2\overline{a_0^2})^{\frac{1}{2}} \quad (m/l = 0).$$

For two-dimensional wave fields we see that the deep-water random wavetrain will always be *stable*, regardless of how small the bandwidth is, if

$$\tan\phi = m/l \geqslant 1/\sqrt{2} \quad \text{(stable)}. \tag{4.37}$$

i.e. if $\phi \geqslant 35.26°$.

For the case of unstable random wavetrains we can also say that increasing wave obliqueness (i.e. m/l increasing from zero) produces an effective increase in the bandwidth of the spectrum in the ratio $(\sigma_\xi/k_0) : [1-2(m/l)^2]^{\frac{1}{2}}$.

5. Discussion

The instability criterion for deterministic deep-water wavetrains requires that the modulation wavenumbers lie in the range

$$0 < \left[\left(\frac{l}{k_0}\right)^2 - 2\left(\frac{m}{k_0}\right)^2\right]^{\frac{1}{2}} \leqslant 2\sqrt{2}\, k_0 a_0, \quad \Omega_1 > 0, \quad \underset{\text{(deterministic)}}{\text{instability,}} \tag{5.1}$$

or

$$0 < \frac{\kappa}{k_0}\left[\frac{1-2(m/l)^2}{1+(m/l)^2}\right]^{\frac{1}{2}} \leqslant 2\sqrt{2}\, k_0 a_0.$$

As Benjamin & Feir have noted, it is always possible, for a given wave amplitude, to find in any wave generation process, a short enough perturbation wavenumber κ (or long enough modulational wavelength) such that the above criterion is always satisfied. Thus Benjamin & Feir concluded that one-dimensional deep-water Stokes waves are always unstable to spatial modulations.

For the current study of the stability properties of a Gaussian random deep-water wavetrain, we have found that instability occurs if the following two criteria are satisfied. The first criterion, for the modulational wavenumber, is quite similar to equation (5.1) found for deterministic wavetrains, i.e.

$$0 < \frac{\kappa}{k_0}\left[\frac{1-2(m/l)^2}{1+(m/l)^2}\right]^{\frac{1}{2}} \leqslant (k_0^2\overline{a_0^2})^{\frac{1}{2}}\, G\left(\sigma_\xi, \frac{m}{l}\right), \tag{5.2}$$

where

$$0 < G(\sigma_\xi, m/l) < 4.53.$$

Note from figure 1 that a small region of increasing instability (over the equivalent B–F limit) develops for small σ_ξ/k_0. However when $\sigma_\xi/k_0 > 1.56\sqrt{(k_0^2\overline{a_0^2})}\,(\hat{\kappa}/\kappa_\xi)^{\frac{1}{2}}$ the range of unstable wavenumbers is less than for an equivalent B–F unstable wavetrain and eventually vanishes as the bandwidth increases. A new additional instability criterion, based on the spectral bandwidth, is that

$$\frac{\sigma_\xi}{k_0}\left[\frac{1+(2m/l)^2}{1-2(m/l)^2}\right]^{\frac{1}{2}} < 2(k_0^2\overline{a_0^2})^{\frac{1}{2}}. \tag{5.3}$$

For fixed values of κ/k_0, wave angle $\phi = \arctan m/l$, and bandwidth σ_ξ/k_0, equations (5.2) and (5.3) state that the wave amplitude (or mean square wave slope) has to exceed a certain value (set by both equations) before any instability or exponential growth can occur.

As an illustration of this effect, let us look at the amplification rate for a uni-directional case, i.e. where $m = 0$ and where we set nominal values, $\kappa/k_0 = 0.2$ and $\sigma_\xi/k_0 = 0.2$. Figure 3 shows a plot of normalized amplification rate Ω_1/ω_0 as a function of the normalized amplitude (or slope) $(k_0^2\overline{a_0^2})^{\frac{1}{2}}$ for two values of the bandwidth. The dashed line corresponds to the result for the quasi-deterministic B–F instability limit. For this case, $\sigma_\xi = 0$, equation (5.3) is automatically satisfied and the parameter $G = 4.0$.

When $\sigma_\xi = 0$, we note from figure 3 (the dashed curve) that the exponential growth rate stays at zero until $(k_0^2\overline{a_0^2})^{\frac{1}{2}} = 0.05$, as given by the wavenumber criterion of equation (5.2). When the bandwidth is non-zero $(\sigma_\xi/k_0 = 0.2)$ Ω_i stays at zero until equation (5.3) is satisfied, that is until $(k_0^2\overline{a_0^2})^{\frac{1}{2}} \geqslant 0.10$.

We see by the above example, that the effect of increasing randomness is to delay the onset of instability and to reduce the amplification rates of the modulation, once instability is initiated.

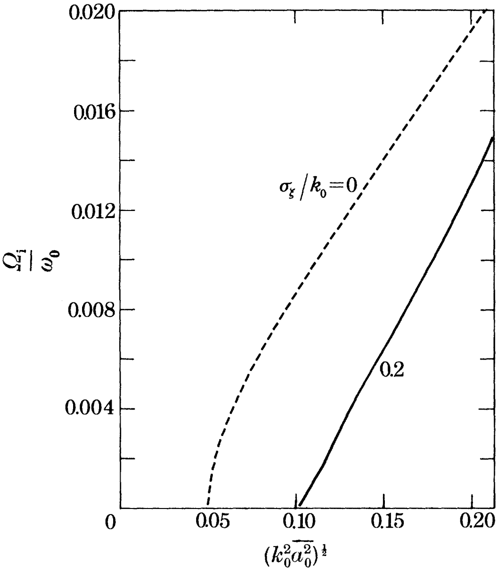

FIGURE 3. Amplification rate of the instability with wave amplitude (or r.m.s. wave slope). Fixed disturbance wavenumber $(\kappa/k_0 = 0.2)$. Perturbation wave vector aligned with carrier group velocity vector $(m/l = 0)$.

When we consider the instability problem in two dimensions $(m/l \neq 0)$, finite bandwidth effects are also seen to play an important rôle in reducing the B–F type instability mechanism.

For the two-dimensional problem, first consider the quasi-deterministic case $\sigma_\xi = 0$. Figure 1 indicates that when $\kappa/k_0 < (8k_0^2\overline{a_0^2})^{\frac{1}{2}}$ the maximum rate of amplification in the l, m plane is found for modulational waves with zero vertical wavenumber, $m = 0$. For larger values of κ/k_0 (above $(8k_0^2\overline{a_0^2})^{\frac{1}{2}}$) the maximum rate of amplification is found for those modulational waves for which

$$\frac{m}{l} = \pm\frac{1}{\sqrt{2}}\left[\frac{(\kappa/k_0)-2(2k_0^2\overline{a_0^2})^{\frac{1}{2}}}{(\kappa/k_0)+(2k_0^2\overline{a_0^2})^{\frac{1}{2}}}\right]^{\frac{1}{2}}, \quad \sigma_\xi = 0. \tag{5.4}$$

Thus for quasi-deterministic wavetrains with large modulational wavenumbers, κ/k_0, the maximum rate of amplification tends to be found in the l, m plane where the wave angle of the modulation $m/l \to 1/\sqrt{2}$ or $\phi \to 35.26°$.

With the added feature of a finite-bandwidth random-wave process, we see from equation (5.3) that as m/l approaches $1/\sqrt{2}$ that the term $(\sigma_\xi/k_0)\,[1-2(m/l)^2]^{-\frac{1}{2}}$ grows larger (i.e. the effective bandwidth increases). This indicates that it is much more difficult to satisfy equation (5.3), and hence to achieve any instability at all. Thus the wave randomness acts to reduce or eliminate the exponential wave growth of instabilities at wave angles approaching 35.26°.

Another figure of merit to aid in interpreting the effect of wavetrain randomness on the B–F instability process, is the ratio of the modulational length scale for maximum amplification (*ca.* $2\pi/\kappa_{max}$) and the correlation length scale, $L_{corr} \sim 1/\sigma_\xi$, for the random process, i.e.

$$\frac{L_{modulation}}{L_{correlation}} \sim \frac{\sigma_\xi}{\kappa_{max}} \sim \frac{\sigma_\xi/k_0}{(k_0^2\overline{a_0^2})^{\frac{1}{2}}}. \tag{5.5}$$

We note that a quasi-deterministic wave system ($\sigma_\xi = 0$) has, by definition, an infinite correlation length. Also by virtue of the D–S scaling, the modulation length scale for the B–F type instability is of the order of $(2\pi/k_0)/(k_0^2\overline{a_0^2})^{\frac{1}{2}}$. As the bandwidth of the wavetrain spectrum increases, the correlation length scale is reduced (in inverse proportions), hence the ratio (equation (5.5)) of the modulation to correlation length scales increases. At the same time, the maximum amplification rate of the instability diminishes as shown in figure 1. When the correlation length scale is reduced to the order of the modulational length scale (or when L_{mod} increases to the order of L_{corr}) the instability diminishes to zero, vanishing when equation (5.3) is satisfied. Thus decorrelation of the wave system, or alternatively decorrelation of the phases of the wave envelope, leads to stabilization of the wavetrain. One might say that it is the phase-mixing of the random composite wave system which leads to a weakening in those wavetrain links which are necessary to support the basic B–F instability mechanism, thus leading to stability for a wave system with a finite spectral bandwidth.

Finally, one may ask whether actual ocean-wave spectra are stable, or unstable, according to the bandwidth instability criterion given by equation (5.3). In this context it is useful to examine the frequency spectra measurements of Hasselmann *et al.* (1973), which provide details of the development of ocean wave spectra in the North Sea with increasing fetch, under conditions of nearly steady off-shore winds. These spectral measurements indicate a narrowing of the spectral bandwidth with increasing fetch, for distances ranging from 10 to 80 km.

To evaluate the stability of ocean spectra, one needs to estimate first the parameter $(k_0\overline{a_0^2})^{\frac{1}{2}}$, and second the spectral bandwidth. The wave-amplitude parameter, $(k_0\overline{a_0^2})^{\frac{1}{2}}$, can be shown, on the basis of a similarity spectrum, to be equal to a constant, independent of fetch. Based on data correlations given in Phillips (1977), $(k_0\overline{a_0^2})^{\frac{1}{2}} \approx 0.066$.

As shown by Fox (1976), the width of a narrow-band spectrum in frequency space $[\sigma_\omega/\omega_0]$ is half the width in wavenumber space $[\sigma_\xi/k_0]$. Hence the one-dimensional instability criterion, according to equation (5.3), becomes

$$\sigma_\omega/\omega_0 \leqslant (k_0^2\overline{a_0^2})^{\frac{1}{2}} = 0.066 \quad \text{(for instability)}.$$

The following estimated values of the frequency bandwidth are found for the two non-symmetrical JONSWAP (Joint North Sea Wave Project) spectra examined by Fox,

$$\text{JONSWAP spectrum JN5: } \sigma_\omega/\omega_0 \approx 0.075,$$

$$\text{JONSWAP spectrum R3C: } \sigma_\omega/\omega_0 \approx 0.082,$$

where σ_ω/ω_0 is based on the width of the spectral peak at half its maximum intensity.

These empirical estimates indicate that the JN5 and R3C spectra are just *stable*, according to the present criterion,† and are close to the neutral stability condition.

On the basis of this brief empirical observation of the stability of ocean wave spectra, one may conjecture that perhaps a quasi-stationary limit exists for the narrowing of any initially broad-band wave spectrum, and that this limit is given by the present bandwidth stability criterion. To prove such a hypothesis would require one to show that the bandwidth of an initially 'narrow' spectrum will eventually broaden, either as a result of the instability process itself, or as the result of the weaker nonlinear wave–wave interaction process described by Longuet-Higgins (1976).

The calculation of the long-time evolution of an initially unstable spectrum is beyond the scope of the present stability analysis, and would probably require an efficient numerical algorithm for integrating equation (3.14). Such numerical methods have already been developed for integrating the 'similar looking' Vlasov equation of plasma physics and hence could be readily applied to the integration of equation (3.14).

A direct application of our derived spectral transport equation to actual ocean-wave dynamics is, of course, formally outside the bounds of our weakly nonlinear analysis. It is evident that we have not considered in the analysis such phenomena as wind energy input to the waves, or wave breaking. However, equation (3.14) can readily be amended to include suitable source and sink energy terms, which model these wave processes, hence leading to a useful transport equation for studying the evolution of non-homogeneous ocean-wave spectra.

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† To formally obtain the stability criterion for an asymmetric spectrum (such as JN5) one would have to solve (perhaps by numerical means) the integral equation (4.16), for some given non-symmetric spectrum $F_0(\boldsymbol{P})$.

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