On determining the directions of waves from a ship at sea

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(*Communicated by G. E. R. Deacon, F.R.S.—Received* 7 *September* 1955)

The paper describes a method by which a shipborne wave recorder can be used to determine directions and frequencies of component wave trains in fairly simple patterns of waves, from the Doppler shifts observed as the ship steams at constant speed along a succession of different courses. Measurements of the relationship between wave frequency and velocity made by the R.R.S. *Discovery II* in the North Atlantic Ocean agree with the classical relationship.

Introduction

In order to study the growth and propagation of sea waves it is necessary to determine the distribution of energy in the sea surface with regard to both frequency and direction. Moreover, it is desirable to make recordings not only near the coast but also in the open ocean, where the highest waves are generated. Accurate instrumental measurements of waves in deep water are considerably more difficult than in shallow water, but a new advance has recently been made by the design of a shipborne wave recorder (Tucker 1952). This instrument gives a continuous record of the elevation of the sea surface near a point on the ship's hull. By itself the recording does not give any indication of the direction of the waves if the ship is stationary. However, if records are taken with the ship under way there will be a Doppler shift in the frequency of each wave component. In this paper it will be shown that this effect can be used to determine the predominant directions of the wave energy, at least when there are one or two clearly distinguishable bands in the energy spectrum.

Theoretical

The representation of the surface of a confused sea produced by wave energy travelling in more than one direction has been treated theoretically in a number of papers, notably the following: Longuet-Higgins (1950, and in an extensive work as yet unpublished), Pierson (1952) and Barber (1954). Fundamentally, all these treatments agree in regarding the sea surface, over a reasonably large area and period of time, as the result of linear superposition of long-crested waves travelling independently of each other in a continuous set of directions.

We follow Longuet-Higgins (unpublished work) in using a representation which is an extension of a formula used by Rice (1945) to represent random noise fluctuations. If $\zeta(x, y, t)$ is the height of the sea surface at time t above a point with horizontal co-ordinates x and y in the mean level, then we assume

$$\zeta(x,y,t) = \sum_n c_n \cos(\sigma_n t - k_n \cos\theta_n x - k_n \sin\theta_n y + \epsilon_n), \tag{1}$$

where for each n, c_n is the amplitude of a long-crested wave component of period $2\pi/\sigma_n$, wavelength $2\pi/k_n = 2\pi/k(\sigma_n)$, travelling in a direction making an angle

θ_n to the x-axis. σ_n is densely distributed in the region $0 < \sigma_n < \infty$, and the amplitudes c_n are assumed mutually independent. ϵ_n are phase angles, randomly distributed with respect to n between 0 and 2π. This model is hydrodynamically valid for first-order wave motion, and certain statistical implications have been shown to apply quite well to sea waves. The quantity $\frac{1}{2}\Sigma c_n^2$ summed over all n for which

$$\sigma \leqslant \sigma_n \leqslant \sigma + \mathrm{d}\sigma \quad (\theta \leqslant \theta_n \leqslant \theta + \mathrm{d}\theta)$$

is proportional to the mean wave energy per unit area of surface contributed from the increment $\mathrm{d}\sigma$, $\mathrm{d}\theta$. Thus

$$\frac{1}{2} \sum_{\mathrm{d}\sigma,\, \mathrm{d}\theta} c_n^2 = E(\sigma, \theta)\, \mathrm{d}\sigma\, \mathrm{d}\theta, \tag{2}$$

and $E(\sigma, \theta)$ can be termed the 'two-dimensional energy spectrum' of the wave system.

Now it is clear that little information about $E(\sigma, \theta)$ can be obtained with the wave recorder on a stationary ship, since the recorder cannot thus distinguish between wave components from different directions. In fact, it measures

$$\zeta(0, 0, t) = \sum_n c_n \cos(\sigma_n t + \epsilon_n),$$

whose Fourier harmonics give an estimate only of

$$E(\sigma)\, \mathrm{d}\sigma = \mathrm{d}\sigma \int_0^{2\pi} E(\sigma, \theta)\, \mathrm{d}\theta.$$

If, however, we record waves with the ship moving with uniform velocity v in a direction ϕ, we obtain

$$\zeta(vt\cos\phi, vt\sin\phi, t) = \sum_n c_n \cos[(\sigma_n - k_n v \cos\overline{\theta_n - \phi})\, t + \epsilon_n], \tag{3}$$

each component (σ_n, θ_n) appearing with a 'Doppler shift' equal to $k_n v \cos(\theta_n - \phi)$. Calling $F(\omega)$ the energy spectrum measured from such a record, so that

$$F(\omega)\, \mathrm{d}\omega = \frac{1}{2} \sum_{\omega_n = \omega}^{\omega + \mathrm{d}\omega} c_n^2,$$

where
$$\omega_n = \omega_n(\phi) = \sigma_n - k_n v \cos(\theta_n - \phi), \tag{4}$$

then the relationship between $F(\omega)$ and $E(\sigma, \theta)$ will be

$$F(\omega)\, \mathrm{d}\omega = \int E(\sigma, \theta)\, \mathrm{d}\sigma\, \mathrm{d}\theta, \tag{5}$$

the integral being taken over the area between the curves in the (σ, θ) plane given by

$$\sigma - k(\sigma)\, v \cos(\theta - \phi) = \omega, \quad \omega + \mathrm{d}\omega. \tag{6}$$

The integral equation (5) could theoretically be solved for $E(\sigma, \theta)$ to any required order of approximation by measuring $F(\omega)$ for a sufficient number of values of ϕ, but in practice the statistical variations in the estimates of $F(\omega)$ are too great for such a method to be practicable.

The situation is simplified if $E(\sigma, \theta)$ contains some isolated peaks at values of σ and θ well separated, for in this case the values of $\omega(\sigma)$ corresponding to each peak will be observed to vary sinusoidally with respect to ϕ, according to equation (6). This sinusoidal variation will be about a mean value $\omega = \sigma$, and will have amplitude

$$k(\sigma)\, v = \sigma v / c(\sigma),$$

where $c(\sigma)$ is the phase velocity of a regular wave with period $2\pi/\sigma$.

Clearly, ω will be equal to σ only when $\phi = \theta \pm \frac{1}{2}\pi$ (wave component beam-on to the ship), and will be minimum when $\phi = \theta$ (ship steaming in same direction as wave component) and maximum when $\phi = \pi - \theta$ (ship steaming in opposite direction), as would be expected from elementary considerations.

Further, the energy density $F(\omega)$ corresponding to each peak value of $E(\sigma, \theta)$, should vary according to the relation

$$F(\omega) = E(\sigma) \left| \frac{\mathrm{d}\sigma}{\mathrm{d}\omega} \right|,$$

which in the case $k = \sigma^2/g$ takes the form

$$F(\omega) = E(\sigma) \left| 1 - \frac{2v\sigma}{g} \cos(\theta - \phi) \right|^{-1}. \tag{7}$$

This will be a minimum when $\phi = \theta$, and maximum when $\phi = \pi - \theta$, provided v is less than $g/2\sigma$, the group velocity of the wave component.

Consider the more general case, when $E(\sigma, \theta)$ has a wide range in σ, but is concentrated in a narrow band of θ; that is, $E(\sigma, \theta)\, \delta\theta = e(\sigma)$ in the interval $\theta, \theta + \delta\theta$, and zero for other values of θ. This may apply to a simple swell. Equations (5) and (6) reduce to

$$F(\omega)\, \mathrm{d}\omega = e(\sigma)\, \mathrm{d}\sigma,$$

where, for deep water,

$$\omega = \sigma - \frac{v}{g} \sigma^2 \cos(\theta - \phi).$$

If the rth moments of $e(\sigma)$ and $F(\omega)$ are m_r and M_r respectively, that is,

$$m_r = \int_0^\infty \sigma^r e(\sigma)\, \mathrm{d}\sigma, \quad M_r = \int_0^\infty \omega^r F(\omega)\, \mathrm{d}\omega,$$

then clearly

$$M_0 = m_0 \tag{8}$$

and

$$M_1 = m_1 - \frac{v}{g} m_2 \cos(\theta - \phi). \tag{9}$$

So the mean frequency M_1/M_0 of $F(\omega)$ will also vary sinusoidally with ϕ, about an average value m_1/m_0, and with an amplitude $(v/g)\, m_2/m_0$.

This argument can be extended to a wider interval of θ, in which case the quantities in (9) are mean values over that interval.

Experimental

R.R.S. *Discovery II*, installed with a shipborne wave recorder, was steered round regular dodecagonal circuits at constant speed in the North Atlantic Ocean, and waves were recorded for about 12 min on each of the twelve courses. The speed chosen was 7 knots, which is high enough to produce a measurable Doppler shift, and lower than the group velocities of components with σ less than $1{\cdot}36\,\mathrm{s}^{-1}$ (period greater than $4{\cdot}6$ s). This ensures that the relation of ω to σ is one-one for all σ less than $1{\cdot}36$, and that $F(\omega)$ will have no singularities. (Wave components with $\sigma > 1{\cdot}36$ are in any case greatly attenuated by the instrument.) Owing to the variations in the ship's resistance as its course changed relative to the waves, it was difficult to keep the speed to exactly 7 knots, and it occasionally varied by half a knot or so, but the actual speed was recorded by means of the Chernikeef log, and corrections to results duly made.

$F(\omega)$ was computed for each value of ϕ from the squares of Fourier harmonics of the record, obtained by means of the automatic analyzer, described by Darbyshire & Tucker (1953). The sum of squares in each group of five adjacent harmonics, corresponding to an interval $\delta\omega = 0{\cdot}05$, was taken as the estimate of $F(\omega)\,\delta\omega$ in the interval. The estimates of $F(\omega)$ so obtained for the dodecagon of 13 November 1954 are shown in figure 1. It is seen that quite a wide range of frequencies is present, but in this case one fairly narrow band of high-energy density stands out and can be

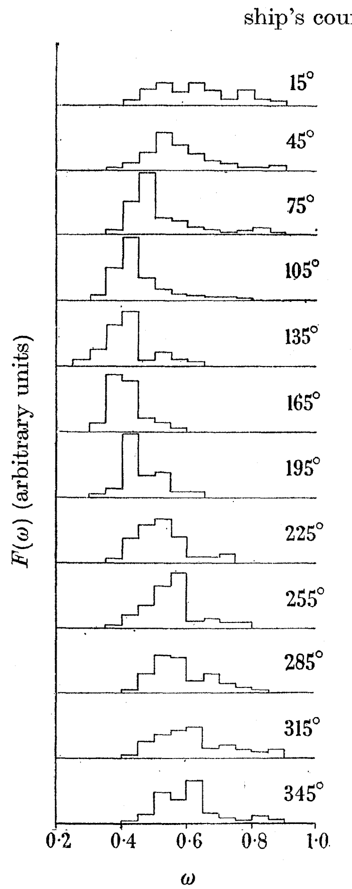
TABLE 1

σ (s^{-1})	$c(\sigma)$ (ft./s)	g/σ (ft./s)
0·785	42·9	41·1
0·422	78·7	76·4
0·560	60·0	57·6
0·502	63·7	64·2

identified for each value of ϕ. All trials gave results rather similar to this, the only exception being that of 21 May 1954, when two identifiable peaks were obtained. The mean values of ω for these bands are plotted against ϕ in figure 2, together with sine curves fitted to the data by the method of least squares. Curves (i) and (ii) show the movements of the two peaks from the dodecagon of 24 May 1954, and (iii) and (iv) are of single peaks from circuits made on 25 May 1954 and 13 November 1954 respectively.

The sine curves obviously fit the data quite closely. Each is defined by three independent parameters, phase, amplitude and mean ordinate. The phase indicates at once the direction of each component, $\phi = \theta$, and one observes that the two components (i) and (ii) present in the same sea are in different directions, namely, 283° and 5° respectively. The first is of higher frequency and corresponds to the direction of the prevailing wind, which had been roughly constant for some time, and so it almost certainly represents the waves recently generated in the area. Component (ii) is of much lower frequency and energy, and represents a superimposed swell travelling from a previous disturbance in the southern hemisphere. Components (iii) and (iv) have directions corresponding to local winds, and any swell there may have been present on those days was either of too low an amplitude or too high a frequency to be resolvable from the harmonic analyses.

The mean values and amplitudes of the sine curves provide independent estimates of σ and phase velocity $c(\sigma)$, respectively, and these are set out in table 1. $c(\sigma)$ is compared with g/σ, the theoretical phase velocity of a long-crested gravity wave of period $2\pi/\sigma$ in deep water. In each case the agreement is within 5 %. The variations in $F(\omega)$ tended to correspond roughly with equation (7), but owing to the statistical



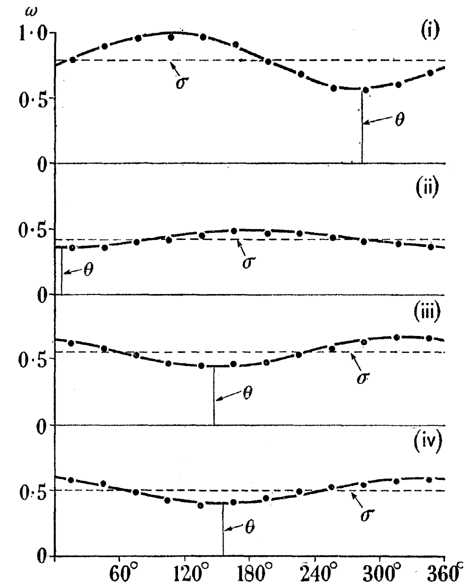


FIGURE 1

FIGURE 2

FIGURE 1. Estimates of energy spectra from 12 min records of waves taken with ship on twelve different courses.

FIGURE 2. Variation of frequency of encounter, ω, of principal wave components with the course of a ship steaming at 7 knots. (i) and (ii), isolated from a complex wave pattern, 21 May 1954; (iii) and (iv), single wave bands recorded on 25 May 1954 and 13 November 1954.

fluctuations inevitable in wave records of comparatively short duration, the correspondence was not remarkable.

Further work upon these lines is in progress, and, in particular, records of ship motion are being studied in relation to the waves.

REFERENCES

Barber, N. F. 1954 *Nature, Lond.*, **174**, 1048.
Darbyshire, J. & Tucker, M. J. 1953 *J. Sci. Instrum.* **30**, 212.
Longuet-Higgins, M. S. 1950 *Phil. Trans.* A, **243**, 1.
Pierson, W. J. 1952 Beach Erosion Board (Contract No. W49–055–eng-l) and Office of Naval Research (Contract No. N onr–285(03)) Report, prepared by Research Division, College of Engineering, N.Y. Univ., Dept. of Met.
Rice, S. O. 1944 *Bell. Syst. Tech. J.* **23**, 282.
Rice, S. O. 1945 *Bell. Syst. Tech. J.* **24**, 46.
Tucker, M. J. 1952 *Nature, Lond.*, **170**, 657.

The coherent scattering of γ-rays by K electrons in heavy atoms

III. The scattering of $0{\cdot}64\ mc^2$ γ-rays in mercury

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(*Communicated by R. E. Peierls, F.R.S.—Received* 19 *September* 1955)

The results of calculations for the coherent scattering of γ-rays of energy $0{\cdot}64mc^2$ are given in the form of scattering amplitudes as functions of angle. Values for the photo-electric effect at $0{\cdot}32mc^2$ and $0{\cdot}64mc^2$ are reported.

1. Introduction

A formalism for calculating the scattering of γ-rays by bound electrons has been developed in part I (Brown, Peierls & Woodward 1954). In part II (Brenner, Brown & Woodward 1954) this formalism has been applied to calculate the scattering of γ-rays of energy $0{\cdot}32\,mc^2$ by the K electrons of mercury. We have extended these calculations to γ-rays of energy $0{\cdot}64\,mc^2$ and report the results here.

The results for coherent γ-ray scattering at $0{\cdot}64\,mc^2$ are found to deviate from those given by the form-factor for a Dirac K electron (Franz 1935) qualitatively in the way predicted for light atoms by the calculation of Brown & Woodward (1952). The deviations are not very great, as can be seen from figure 1. However, from the arguments of Brown & Woodward, they might be expected to be four times greater at medium and large angles at $1{\cdot}28\,mc^2$, the next energy to be investigated. In fact these arguments depend on $Z\alpha$ being small, which is not the case with mercury; the largeness of $Z\alpha$ here appears to modify the quantitative conclusions, and we expect the relative